MIMO RADAR MODELING
THROUGH RANDOM VANDERMONDE MATRICES

G. Alfano[†], *C. Chiasserini*[†], *M. Debbah*[‡], *A. Nordio*[†], *O. Ryan*[*]

[†] DELEN, Politecnico di Torino
C.so Duca degli Abruzzi 24,
I-10129, Torino
alfano,nordio,chiasserini@tlc.polito.it

[‡] Chaire on Flexible Radio, Supélec
3, rue Joliot-Curie,
F-91192, Gif sur Yvette
merouane.debbah@supelec.fr

[*]CMA, University of Oslo,
Blindern,
NO-0316, Oslo
oyvindry@ifi.uio.no

ABSTRACT

A MIMO radar system is conveniently modeled via random matrices, and its optimal design strongly relies on spectral properties of the matrix exploited to build the model itself. We offer a way to model a High Resolution Radar (HRR) detection in the MIMO case, based on recent results on the asymptotic spectral analysis of random Vandermonde matrices with entries lying on the unit circle. Achievability of compact-form expressions for the design parameters sought for in the multiple-transmitter-multiple-receiver case is investigated, and together with the results of such a starting analysis, some open mathematical questions that arise from the new model formulation are listed and discussed.

1. INTRODUCTION

The seminal idea of Woodward and Davies of applying information theory to radar systems analysis [1, 2] has been only rarely exploited in radar waveform design [3], [4, and references therein]; more precisely, joint information- and estimation-theoretic criteria are adopted in several works by the same authors (see, e.g. [4, 5]) to optimally shape the transmitted waveform from a multiple-antenna equipped radar transmitter which aims at detecting an extended target at high resolution. The receiver too is assumed to be equipped with multiple antennas, and the model applies to both mono- or bi-static scenario.

The main finding of the abovementioned works is the dependence of the parameters of interest in the waveform design strategy on the spectrum of the Gram matrix associated to the (random) matrix modeling the input-output relationship between the transmitted waveform from each sensor and the target echo[1] at any receive antenna. For sake of simplicity, we will refer hereinafter to the input-output matrix as the *channel matrix* as in any linear framework for MIMO wireless communication analysis.

Stationarity assumptions in [4] yields to a Toeplitz [6] structure for the blocks of the channel matrix modeling the signal reflection on the target from a single transmit antenna to a single receive antenna. This feature of the channel matrix, when taking into account also the unknown reflection angle from each of the elementary scatterers, allows, as noticed in [7], to model the Gram matrix associated to the channel matrix as random Vandermonde. Spectral properties of such kind of matrices have been only recently investigated in some works [2] [7, 9, 10] under different assumptions on the entries distribution (especially on their dependence) and on the matrix *aspect-ratio* convergence properties.

While the works appeared on the line of [4] have shed light on the connection between large system properties and waveform design, we aim with this contribution at investigating the performance-side impact of the Vandermonde modeling for MIMO radar, as well as at stressing some open problems in Vandermonde matrices spectrum characterization arising from the MIMO detection setting itself and that have not yet been encountered when dealing with random Vandermonde determinants in other applications like finance, cognitive radio, security and, in general, wireless communications issues (for a list of references on each subject, please refer to [9], where a comprehensive analysis of the Vandermonde modeling state-of-the-art is provided).

The paper is structured as follows: Section 2, contains some essential background material on (large) random Vandermonde matrices spectral properties. Section 3 reports the model for a MIMO HRR system in terms of Vandermonde matrices, while Section 4 discusses the available results on the MIMO HRR system matrices and open issues. Conclusions are given in Section 5.

2. MATHEMATICAL BACKGROUND

This section addresses some essential definitions in random matrix theory that will be useful in the following, as well as the exploited notation. Throughout the paper, matrices are denoted by uppercase boldface letters, vectors by lowercase boldface; $\mathbb{E}[\cdot]$ denotes the statistical expectation, $(\cdot)^\dagger$ indicates the conjugate transpose operator, $|\cdot|$ and $\mathrm{Tr}(\cdot)$, respectively, the determinant and the trace of a square matrix, and $||\cdot||$ for the euclidean norm.

Definition 2.1 *Let us consider an $N \times N$ Hermitian matrix $\mathbf{A}$. The averaged empirical cumulative distribution function of the eigenvalues (also referred to as the averaged empirical spectral distribution (ESD)) of $\mathbf{A}$ is defined as*

$$\mathsf{F}^N_{\mathbf{A}}(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[1\{\lambda_i(\mathbf{A}) \leq \lambda\}],$$

where $\lambda_1(\mathbf{A}), \ldots, \lambda_N(\mathbf{A})$ are the eigenvalues of $\mathbf{A}$ and $1\{\cdot\}$ is the indicator function. If $\mathsf{F}^N_{\mathbf{A}}(\cdot)$ converges as $N \to \infty$, then

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[1]Since we are under high resolution assumptions, we refer as radar echo actually the superposition of the incoming echoes from each resolved scatterer in the extended target.

[2]It is worth to note that in [9, Appendix F] a useful connection between Vandermonde moments evaluation and the analysis on random Toeplitz carried on in [8] has been pointed out too.

the corresponding limit (asymptotic ESD, AESD) is denoted by $\mathsf{F}_{\mathbf{A}}(\cdot)$. The corresponding asymptotic probability density function is denoted by $f_{\mathbf{A}}(\cdot)$.

Definition 2.2 *The aspect-ratio of a $N \times L$ random matrix is the number $\beta = \lim_{K,L\to+\infty} \frac{K}{L}$, provided the limit is finite.*

Definition 2.3 *[11] The η-transform of the random matrix $\mathbf{A}$ is defined as*

$$\eta_{\mathbf{A}}(\gamma) = \mathbb{E}\left[\frac{1}{1+\gamma\lambda}\right] \tag{1}$$

where γ is a scalar and λ is a random variable distributed as the asymptotic eigenvalues of $\mathbf{A}$.

By denoting with $\mathbb{E}[\lambda^p]$ the p-th asymptotic moment of $\mathbf{A}$, $\eta_{\mathbf{A}}(\gamma)$ can be regarded as a generating function for the asymptotic moments of $\mathbf{A}$, i.e. [11],

$$\eta_{\mathbf{A}}(\gamma) = \sum_{p=0}^{\infty}(-\gamma)^p\mathbb{E}[\lambda^p], \tag{2}$$

whenever the moments of $\mathbf{A}$ exist and the series in (2) converges.

The η-transform, introduced in [11], is intimately related to other integral transforms defined over the spectrum of a random matrix. For a detailed discussion on the links between such transforms and the R and $\mathscr{S}$ transforms usually exploited in free probability, the reader is referred to [11, Ch. 2]. Recently, in [13] some more general results appeared on the connection between the η-transform and other functions of the asymptotic moments of a random matrix under quite generic assumptions.

The random matrix we will mainly concerned with throughout the paper are the Vandermonde ones, that are classified as follows:

Definition 2.4 *We define Vandermonde matrix an $N \times L$ matrix $\mathbf{V}$ whose $(k,\ell)-th$ entry can be expressed as x_ℓ^{k-1} [14].*

Definition 2.5 *An $N^d \times L$ matrix $\mathbf{V}$, with complex exponential entries lying on the unit circle, is a d-fold Vandermonde matrix if its generic entry can be written as*

$$\mathbf{V}_{\nu(\ell),q} = \frac{1}{\sqrt{N^d}}\mathrm{e}^{-2\pi\mathrm{i}\ell^{\mathsf{T}}f(\mathbf{x}_q)}. \tag{3}$$

where $\ell = [\ell_1,\ldots,\ell_d]^{\mathsf{T}}$ is a vector of integers, $\ell_m = 0,\ldots,N-1$, $m = 1,\ldots,d$, the function f is evaluated component-wise on the random vectors $\mathbf{x}_q$, and the function

$$\nu(\ell) = \sum_{m=1}^{d} N^{m-1}\ell_m, \tag{4}$$

maps the vector ℓ onto a scalar index. According to [9], we will refer to the quantity $2\pi f(\mathbf{x}_q)$ as the phase distributions of the Vandermonde matrix at hand. Notice that for a d-fold Vandermonde matrix the phase is a vector while for $d = 1$ the phase is a scalar quantity.

Notice that the assumption of the entries lying on the unit circle is crucial to the asymptotic convergence of the spectrum and will be retained through the paper. When $d = 1$, we will refer to (3) as a random Vandermonde matrix. Notice further that, due to the previous definitions, we will have $\beta = \frac{L}{N}$ for a Vandermonde matrix and $\beta = \frac{L}{N^d}$ for the d-fold version.

The application we are interested in leads us to consider the function $f = sin(\mathbf{x}_q)$.

3. SYSTEM MODEL

HRR is often exploited in military applications for detecting extended targets. To address the problem of detection through HRR, several mathematical tools have been proposed in the literature. One of the simplest, but effective, approaches is to resort to random matrix theory and exploit a linear model involving Vandermonde matrices. Recall that the echo from an extended target which is resolvable in L elementary scatterers can be viewed as the impulse response of a linear filter of length ν, where $L = \nu+1$. Let us assume that the transmitted signal has length N. Then, by considering a (mono or bistatic) scenario for HRR detection, the received signal can be written as [4, and refs. therein]

$$\mathbf{r} = \mathbf{V}\mathbf{P}^{1/2}\mathbf{s}+\mathbf{n} \tag{5}$$

where $\mathbb{E}[\mathsf{tr}\{\mathbf{P}\}] = 1$ and $\mathbf{V}$ is a $N \times L$ Vandermonde matrix with generic entry[3] $(\mathbf{V})_{n,q} = \exp(-2\pi\mathrm{i}n\sin\theta_q)/\sqrt{N}$, θ_q is the angle of arrival of the echo reflected on the q-th elementary scatterer constituting the target, and $\mathbf{s}$ is the target impulse response of length L. The diagonal matrix $\mathbf{P}$ accounts for the (eventually) different power levels of the echoes coming from each of the L scatterers constituting the overall extended target, which is usually assumed to have a Gaussian vector impulse response [4]. When dealing with the MIMO case, the channel matrix turns out to be a d-fold Vandermonde, where d is the number of antennas deployed at the receiver. This way, the generic entry of $\mathbf{V}$ will depend, as in the multiuser multiantenna case with line of sight contribution [7], [9, Sec. V], on the vector of the arrival angles on each of the d receiving antennas, for each echo reflected from the q-th elementary scatterer.

For the linear model in (5), the Mean Square Error (MSE) on the LMMSE estimate of $\mathbf{s}$, normalized over the size L of $\mathbf{s}$, is obtained as [15]

$$\mathrm{MSE} = \mathsf{tr}\left\{(\gamma\mathbf{V}\mathbf{P}\mathbf{V}^\dagger+\mathbf{I})^{-1}\right\} \tag{6}$$

For large systems, we also define the *asymptotic* MSE as

$$\mathrm{MSE}_\infty = \lim_{\substack{M,N\to+\infty \\ \beta}} \mathrm{MSE} = \mathbb{E}\left[\frac{1}{1+\gamma\lambda}\right] \tag{7}$$

where λ is a random variable distributed as the asymptotic eigenvalues of $\mathbf{V}\mathbf{P}\mathbf{V}^\dagger$ [10].

From (1) and (7), we can note that the expression of MSE_∞ can be written through the η-transform as

$$\mathrm{MSE}_\infty = \eta_{\mathbf{H}\mathbf{H}^\dagger}(\gamma) \tag{8}$$

In the following, particular emphasis will be on the asymptotic moments, from which the η-transform evaluation can be carried out.

[3] In this case, $n = 0,\ldots,N-1$, and $q = 0,\ldots,\nu$

4. MODEL ANALYSIS

Mirroring the MMSE design criterion of [4], we aim at providing compact expressions for the MSE in the target impulse response parameter estimation, based on recent findings on large random Vandermonde matrices spectra. Throughout the Section, $\mathbb{E}[\lambda_d^p]$ will denote the p-th moment of a d-fold random matrix; as a consequence, we will denote as $\mathbb{E}[\lambda_\infty^p]$ the p-th asymptotic moment.

4.1 Performance discussion

Notice first that, if we were detecting the target over a very narrow angle spread, $\sin(\theta_q) \approx \theta_q$, then, asymptotically in the number of the receive antennas, results in ([10, 7]) apply, respectively, for the case of $\mathbf{P} = \mathbf{I}$ and $\mathbf{P} \neq \mathbf{I}$. Specifically, when d grows large, the p-th asymptotic moments of $\mathbf{V}\mathbf{V}^\dagger$ coincides with that of the Marcenko-Pastur law [10, Lemma 6.1] and is given by

$$\mathbb{E}[\lambda_\infty^p] = \sum_{k=1}^{p} T(p,k)\beta^{p-k} \tag{9}$$

while those of $\mathbf{VPV}$ with $\mathbf{P} \neq \mathbf{I}$ are given by the well-known Bai-Yin formula [11, and references therein], namely[4]

$$\mathbb{E}[\lambda_\infty^p] \rightarrow \sum_{k=1}^{p} \beta^k \sum_{\substack{p_1+\cdots+p_k=p \\ p_1\leq\cdots\leq p_k}} \frac{p!}{(p-k+1)!f(p_1,\ldots,p_k)} \mathbb{E}[\mathsf{P}^{p_1}]\cdots\mathbb{E}[\mathsf{P}^{p_k}], \tag{10}$$

as proven in [7, Theorem 4.2][5]. In (10), P is a nonnegative random variable whose distribution is given by the asymptotic spectral distribution of $\mathbf{P}$. As a consequence, the sought-for η-transform is given by that of the Marcenko-Pastur law [11, Formula 2.52] in the case of equal-power echoes, and by [11, Theorem 2.39] when the scattered echoes have different powers.

It is worth to stress that, as numerically verified already in [10, 7], the speed of convergence toward the asymptotic moments expression with respect to d is quite fast; indeed, already $d > 3$ does offer a satisfactorily agreement between theoretical and simulated curves, thus the model may offer good prediction properties in a real-world scenario even for a moderate number of radar receive antennas.

The case of widely spread, tough not uniformly distributed over the interval $(0, 2\pi)$, detection angle θ_i, for each elementary scatterer, would lead to a different model whose spectral behavior has not yet been analyzed as d grows large.

Indeed, in some settings the phases distribution turns out to be not uniform; however, it is quite likely it still is continuous.

Under these more general assumptions on the phase distributions, the expression of $\mathbb{E}[\lambda_1^p]$ is known and is given by [9, Formula (12)], while no results are yet available for $d > 1$.

[4]Herein, following [11], we suppose a vector of k integers $p_1,\ldots,p_k$ is partitioned into n equivalence classes under the equivalence relation $a = b$, and the cardinalities of the equivalence classes are given by $f_1,\ldots,f_n$, then we can define the following function:

$$f(p_1,\ldots,p_k) \triangleq f_1!\cdots f_n!.$$

For example, $f(1,1,4,2,1,2) = 3!\cdot 2!\cdot 1!$

[5]For the details of the proof, the reader is referred to [12].

4.2 Open issues

Interestingly, the d-fold Vandermonde matrix with uniformly distributed phases has an asymptotic law whose moments converge to those of the Marcenko-Pastur one, and this issue has not yet been investigated in a generic case.

The approach we propose to investigate the MSE behavior in such a more general setting is to go through some very recent results on Vandermonde matrices with generic (continuous) phase distribution and exploit the inherent links between the asymptotic Vandermonde moments and those of classical spectrum law in random matrix theory.

Notice that the moments of the spectrum of a Gram matrix of the type $\mathbf{V}\mathbf{V}^\dagger$ when $\mathbf{V}$ has continuous, non-uniform phase distributions, are bounded from below by the corresponding moments of a Gram matrix with uniform phase distributions [9, Proposition 5] whose values, in turn, lay between the Marcenko-Pastur and the Poisson distribution moments [9].

Below, we propose then two alternative roadmaps in order to provide expressions for the MSE in HRR MIMO radar detection for a generic phase distribution of the channel matrix entries:

1. *Series truncation* We intend to rely on inequalities relationships between moments; the steps to be followed would be
   - Consider a generic continuous phase distribution;
   - Exploit [9, Corollary 2 and Proposition 5] to lower bound the moments;
   - Provide an upper bound to (at least some) of the moments of the non-uniform phases matrix at hand;
   - Compare the truncated series of the η-transform obtained in the two cases with the outcomes of a numerical simulation to perform a sensitivity analysis of the bounds to the system parameters and to the order of the highest estimated moment.
2. *Asymptotic analysis and support estimation*
   - Borrow the proof strategy from [10] in order to provide as compact as possible expressions for the p-th moment of a random, d-fold Vandermonde matrix with non-uniformly distributed phases, as d grows large.
   - Investigate the compactness of the support of the limiting spectral distribution for such a $\mathbf{V}$ so as to exploit any further available result linking moments and integral transforms.

5. CONCLUSION

A model for MIMO HRR detection, based on recent results on the asymptotic spectral analysis of random Vandermonde matrices, has been proposed and discussed. Some compact-form expressions for the design parameters are provided, and together with the results of such a starting analysis, open mathematical questions that arise from a MSE analysis strategy are listed and discussed.

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