Master–Slave Synchronization of Fourth-Order Lü Chaotic Oscillators via Linear Output Feedback

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***Abstract*—We solve the problem of master–slave synchronization of fourth-order Lü's hyperchaotic systems via feedback control. We use only *one* control input that enters in the slave system. We show that this simple feedback suffices to synchronize both systems exponentially fast. We provide a proof of stability and convergence (hence, that synchronization takes place) via Lyapunov's second stability method. We provide some numeric simulations that illustrate our findings.**

***Index Terms*—Chaos, control, synchronization.**

I. Introduction and Problem Formulation

SYNCHRONIZATION of dynamical systems is a multifacet subject of research depending on the context. It may be hierarchy based (unidirectional), such as "master–slave" synchronization in which, typically but not necessarily, *two* systems are synchronized in a way that the "slave" system mimics the motion of the "master" system [8], [6], [16]. Synchronization may involve several systems synchronizing *without* a prescribed hierarchy (bidirectional) as is the case of synchronization of *networks* of systems [20], [24], often happening *naturally*, for instance, in certain biological systems. Another intensive area of research to emphasize within bidirectional synchronization is the study of the *consensus* paradigm; see the excellent text [22].

Controlled synchronization pertains to the case when synchronization is induced by a control action, which may take different forms, depending on the strategy. It may result from feedback control, induced-delay synchronization [6], observer-based synchronization [12], and impulsive control [9], to mention a few. From a control viewpoint, the problem may be recasted as a tracking control problem for the slave system where the time-varying reference is given by the slave system [14].

Master–slave synchronization has many applications in technology such as in secured telecommunication [3], [7]. Specifically, motivated by the vulnerability to attacks of certain architectures, the use of chaotic systems for data scrambling has thoroughly been investigated. It is shown in [2] and [13] that some schemes in which information is simply added as an input to the (chaotic) transmitter may be unrobust *vis-a-vis* of attacks. See also [1]. In [2], the fundamental property of identifiability is studied in detail.

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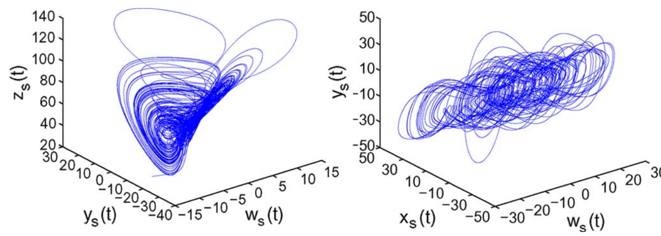


Fig. 1. Attractor of a hyperchaotic system without inputs, i.e., with $u_x = u_y = u_w = u_z = 0$.

On the other hand, a number of chaotic systems continue to appear in the literature: beyond the celebrated Lorenz and Rössler systems, we shall mention the Chen system [23], the Colpitts circuit [21], and different versions of the so-called Lü and Chen system [15], [17].

In this brief, we address a problem of controlled master–slave synchronization of a chaotic system largely studied in the literature—the so-called Lü oscillator. There are at least two models of Lü chaotic systems: one is of third order [18], [25], and the other is of fourth order [5]. We propose a solution to master–slave synchronization for the generalized fourth-order Lü system given by [5]

$$\begin{aligned} \dot{x}_s &= a(y_s - x_s) + u_x && \text{(1a)}\\ \dot{y}_s &= bx_s - kx_s z_s + w_s + u_y && \text{(1b)}\\ \dot{z}_s &= -cz_s + hx_s^2 + u_z && \text{(1c)}\\ \dot{w}_s &= -dx_s + u_w && \text{(1d)} \end{aligned}$$

where a, b, c, d, k, and h are constant parameters,[1] and u_x, u_y, u_z, and u_w are control inputs. For the following values of these physical parameters, the *unforced* system, that is, with all control inputs set to zero, exhibits a chaotic behavior:

$$\begin{aligned} &a = 10 \quad b = 40 \quad c = 2.5\\ &d = 10.6 \quad k = 1 \quad h = 4. \end{aligned} \tag{2}$$

The attractors for the case when the system is unforced (i.e., with all controls set to zero) are shown in Fig. 1. The initial conditions are set to

$$\begin{aligned} x_s(0) &= -4 & y_s(0) &= -8 && \text{(3a)}\\ w_s(0) &= 12 & z_s(0) &= -6. && \text{(3b)} \end{aligned}$$

In the sequel, we refer to system (1) as the slave system. Correspondingly, we introduce the master system, which has no control inputs, as

$$\begin{aligned} \dot{x}_m &= a(y_m - x_m) && \text{(4a)}\\ \dot{y}_m &= bx_m - kx_m z_m + w_m && \text{(4b)}\\ \dot{z}_m &= -cz_m + hx_m^2 && \text{(4c)}\\ \dot{w}_m &= -dx_m. && \text{(4d)} \end{aligned}$$

[1]For simplicity, we consider these parameters to be positive.

The synchronization problem that we address consists in designing a controller (u_x, u_y, u_z, u_w) such that the trajectories of the slave system asymptotically follow those of the master, i.e., we wish to make that

$$\lim_{t\to\infty} |x_s - x_m| = 0 \quad \lim_{t\to\infty} |y_s - y_m| = 0 \tag{5a}$$
$$\lim_{t\to\infty} |w_s - w_m| = 0 \quad \lim_{t\to\infty} |z_s - z_m| = 0 \tag{5b}$$

for all initial conditions in a set $D \subseteq \mathbb{R}^4$.

We assume that the master system operates in a chaotic regime; hence, defining for a continuous function

$$|f|_\infty := \sup_{t\geq 0} |f(t)| \tag{6}$$

there exist constants β_{xm}, β_{ym}, β_{zm}, and β_{wm} such that

$$\begin{aligned} |x_m|_\infty \leq \beta_{xm} \quad |y_m|_\infty \leq \beta_{ym} \\ |w_m|_\infty \leq \beta_{wm} \quad |z_m|_\infty \leq \beta_{zm}. \end{aligned} \tag{7}$$

Synchronization of third-order Lü systems

$$\dot{x}_s = \beta x_s - y_s z_s + c + u_1 \tag{8a}$$
$$\dot{y}_s = -a y_s + x_s z_s + u_2 \tag{8b}$$
$$\dot{z}_s = -b z_s + x_s y_s + u_3 \tag{8c}$$

has been addressed in a few recent papers, for instance, in [18], via nonlinear control and assuming that three control inputs are available. The results of the latter were improved in [25], where it is shown that, with the choice of only two controls, the master–slave synchronization of two identical systems (8) may be achieved. See also [4]. In the recent note [11], we solved the master–slave synchronization for the third-order system with *one* input and partial-state measurement. Here, we solve the synchronization problem stated above for the fourth-order system (1) via output feedback linear control. As in [11], we use only one control input to achieve the full-state synchronization objective but measurement of *one* variable only. However, the rationale and, hence, the method of proof are fundamentally different from those used in [11] since the third- and the fourth-order systems are dynamically very different.

The rest of this paper is organized as follows. In Section II, we present our main proposition and proof, which is based on the standard Lyapunov stability theory. In Section II-A, we present some simulations that illustrate our theoretical findings, and we wrap up this brief with some concluding remarks.

II. Main Result

For the system (1), we assume that only *one* control input is available, which corresponds to u_y, i.e., $u_x = u_w = u_z = 0$. We also assume that only the variables x_s and x_m are measured. With this under consideration, let us introduce the synchronization error coordinates

$$e_x := x_s - x_m \qquad e_y := y_s - y_m \tag{9a}$$
$$e_w := w_s - w_m \qquad e_z := z_s - z_m. \tag{9b}$$

The error dynamics equations are obtained by subtracting (4) from (1) and using $u_x = u_w = u_z = 0$ to obtain

$$\dot{e}_x = a(e_y - e_x) \tag{10a}$$
$$\dot{e}_y = b e_x - k(x_s z_s - x_m z_m) + e_w + u_y \tag{10b}$$
$$\dot{e}_z = -c e_z + h e_x^2 + 2h x_m e_x \tag{10c}$$
$$\dot{e}_w = -d e_x. \tag{10d}$$

The synchronization problem boils down to stabilizing the system (10) to the origin $e = 0$, where $e := [e_x, e_y, e_w, e_z]^\top$ via the feedback control u_y. We are ready to present our main result.

Proposition 1: Let $T > 0$ and consider the system (1) in closed loop with

$$u_y(t, e_x) = \begin{cases} -k_1 e_x & \forall\, t \geq T \\ 0 & \forall\, t \in [0, T] \end{cases}$$
$$u_x = 0 \qquad u_z = 0 \qquad u_w = 0.$$

Then, we have the following.

1) For any fixed $\delta > 0$ and $T > 0$ and any initial conditions such that $|e(0)| \leq \delta$, one can find a control gain $k_1(\delta, T)$ that is sufficiently large such that the origin of (10) is exponentially stable.
2) On the other hand, for any given k_1 independent of the initial conditions and for all $e(0) \in \mathbb{R}^4$, the system is globally exponentially stable provided that T is taken to be sufficiently large.

Proof: Adding $-k(x_s e_z - x_s e_z)$ to (10b), we obtain

$$\begin{aligned} \dot{e}_y = b e_x - k x_s z_s - k x_s e_z + k x_s e_z \\ + k x_s z_m - k e_x z_m + e_w + u_y. \end{aligned} \tag{11}$$

Then, defining $\mathbf{b}(t) = b - k z_m(t)$, the closed-loop equations (for $t \geq T$) are

$$\dot{e}_x = a(e_y - e_x) \tag{12a}$$
$$\dot{e}_y = [\mathbf{b}(t) - k_1]\, e_x + e_w - k x_s(t) e_z \tag{12b}$$
$$\dot{e}_z = -c e_z + h e_x^2 + 2h x_m(t) e_x \tag{12c}$$
$$\dot{e}_w = -d e_x. \tag{12d}$$

We stress that the system above is a nonlinear time-varying system with state $\xi^\top := [e_x, e_y, e_w, e_z]$ and depends on time through the functions $x_m(\cdot)$, $x_s(\cdot)$, and $\mathbf{b}(\cdot)$.

By assumption, the master system operates in the chaotic regime; hence, all master signals $(\cdot)_m$ are bounded. Furthermore, let us *temporarily* assume that the trajectory of the slave system in closed loop,[2] i.e., $x_s(t)$ is bounded for all t (this will be relaxed and proved at the end). Then, there exists β_{xs} such that

$$\sup_{t\geq 0} |x_s(t)| \leq \beta_{xs}. \tag{13}$$

To show Lyapunov stability, we introduce the functions

$$V_1(\xi) := \frac{1}{2}\left(\alpha_1 e_x^2 + \alpha_3 e_y^2 + \alpha_2 e_w^2\right) \quad V_3(\xi) := \frac{\alpha_4}{2} e_z^2$$
$$V_2(\xi) := -\varepsilon_1 e_x e_y - \varepsilon_2 e_w e_y, \qquad \varepsilon_1, \varepsilon_2, \alpha_1, \alpha_2, \alpha_3 > 0.$$

The total time derivative of V_1 along the closed-loop trajectories yields

$$\begin{aligned} \dot{V}_1(\xi) = -\alpha_1 a e_x^2 + \alpha_1 a e_y e_x + \alpha_3(\mathbf{b} - k_1) e_x e_y \\ - \alpha_3 k x_s(t) e_y e_z + \alpha_3 e_w e_y - \alpha_2 d e_w e_x. \end{aligned} \tag{14}$$

The total time derivative of V_2 along the closed-loop trajectories yields

$$\begin{aligned} \dot{V}_2(\xi) = -[\varepsilon_1 e_x + \varepsilon_2 e_w]\,[(\mathbf{b} - k_1) e_x - k x_s(t) e_z + e_w] \\ - \varepsilon_1 [a(e_y - e_x)]\, e_y + \varepsilon_2 d e_x e_y. \end{aligned} \tag{15}$$

Let $k_1' > 0$ be arbitrarily chosen and define

$$\alpha_1 := \frac{k_1'}{a}. \tag{16}$$

[2]That is under the proposed feedback.

Substituting the latter in (14) and adding (15) to it, we obtain

$$\begin{aligned}\dot V_1(\xi)+\dot V_2(\xi)\le &-k_1'e_x^2+k_1'e_xe_y+\alpha_3(\mathbf{b}-k_1)e_xe_y\\&+\alpha_3e_we_y-\alpha_2de_we_x-\varepsilon_1\left[a(e_y-e_x)\right]e_y\\&-[\varepsilon_1e_x+\varepsilon_2e_w][(\mathbf{b}-k_1)e_x-kx_se_z+e_w]\\&+\varepsilon_2de_xe_y-\alpha_3kx_se_ye_z. \end{aligned}\tag{17}$$

Now, recalling that $\mathbf{b}:=b-kz_m(t)$, we see that the factors of e_xe_y in the expression above are

$$\alpha_3\left(b-kz_m(t)-k_1\right)+\varepsilon_1a+\varepsilon_2d+k_1'$$

which is equal to $-\alpha_3kz_m(t)$ if we define

$$k_1:=\frac{1}{\alpha_3}\left(k_1'+\varepsilon_1a+\varepsilon_2d\right)+b. \tag{18}$$

For $|z_m|_\infty\le\beta_{zm}$ and $|x_s|_\infty\le\beta_{xs}$, we have

$$\begin{aligned}\dot V_1(\xi)+\dot V_2(\xi)\le &-k_1'e_x^2+\alpha_3e_we_y-\alpha_2de_we_x-\varepsilon_1ae_y^2\\&-[\varepsilon_1e_x+\varepsilon_2e_w][(\mathbf{b}-k_1)e_x-kx_se_z+e_w]\\&+\alpha_3k\beta_{zm}|e_xe_y|+\alpha_3k\beta_{xs}|e_ze_y|. \end{aligned}\tag{19}$$

Developing and rearranging terms, we obtain

$$\begin{aligned}\dot V_1(\xi)+\dot V_2(\xi)\le &\, e_xe_w(-\varepsilon_1-\varepsilon_2\mathbf{b}+\varepsilon_2k_1-\alpha_2d)-\varepsilon_2e_w^2\\&-\left[k_1'+\varepsilon_1(\mathbf{b}-k_1)\right]e_x^2+\alpha_3e_we_y\\&+kx_s(\varepsilon_1e_x+\varepsilon_2e_w)e_z-\varepsilon_1ae_y^2\\&+\alpha_3k\beta_{zm}|e_xe_y|+\alpha_3k\beta_{xs}|e_ze_y|.\end{aligned}$$

Next, we define

$$\alpha_2:=\frac{1}{d}[-\varepsilon_1-\varepsilon_2b+\varepsilon_2k_1] \tag{20}$$

and substitute it in the factor of e_xe_w in (20) to obtain

$$\begin{aligned}\dot V_1(\xi)+\dot V_2(\xi)\le &\,\varepsilon_2k\beta_{zm}|e_xe_w|-\varepsilon_2e_w^2+\alpha_3k\beta_{zm}|e_xe_y|\\&-\left[k_1'+\varepsilon_1(\mathbf{b}-k_1)\right]e_x^2+\alpha_3e_we_y\\&+k\beta_{xs}\left(\varepsilon_1|e_xe_z|+\varepsilon_2|e_we_z|\right)-\varepsilon_1ae_y^2\\&+\alpha_3k\beta_{xs}|e_ze_y|. \end{aligned}\tag{21}$$

Now, if

$$\frac{1}{2}k_1'+\varepsilon_1\left[b-kz_m(t)-k_1\right]\ge 0 \tag{22}$$

we obtain

$$\begin{aligned}\dot V_1(\xi)+\dot V_2(\xi)\le &-\frac{1}{4}\begin{bmatrix}|e_x|\\|e_w|\end{bmatrix}^\top\begin{bmatrix}\frac{1}{2}k_1' & 2\varepsilon_2k\beta_{zm}\\2\varepsilon_2k\beta_{zm} & \varepsilon_2\end{bmatrix}\begin{bmatrix}|e_x|\\|e_w|\end{bmatrix}\\&-\frac{1}{4}\begin{bmatrix}|e_y|\\|e_w|\end{bmatrix}^\top\begin{bmatrix}\varepsilon_1a & 2\alpha_3\\2\alpha_3 & \varepsilon_2\end{bmatrix}\begin{bmatrix}|e_y|\\|e_w|\end{bmatrix}\\&-\frac{1}{4}\begin{bmatrix}|e_x|\\|e_y|\end{bmatrix}^\top\begin{bmatrix}\frac{1}{2}k_1' & 2\alpha_3k\beta_{zm}\\2\alpha_3k\beta_{zm} & \varepsilon_1a\end{bmatrix}\begin{bmatrix}|e_x|\\|e_y|\end{bmatrix}\\&+k\beta_{xs}|e_z|\left[\varepsilon_2|e_w|+\varepsilon_1|e_x|+\alpha_3|e_y|\right]\\&-\frac{1}{4}\left(k_1'e_x^2+2\varepsilon_2e_w^2+2\varepsilon_1ae_y^2\right). \end{aligned}\tag{23}$$

In view of (18), (22) holds if

$$\frac{1}{2}k_1'-\frac{\varepsilon_1}{\alpha_3}\left(k_1'+\varepsilon_1a+\varepsilon_2d\right)\ge\varepsilon_1k\beta_{zm}$$

which, in turn, holds if

$$k_1'\ge\frac{\varepsilon_1\left(\frac{1}{\alpha_3}(\varepsilon_1a+\varepsilon_2d)+k\beta_{zm}\right)}{\frac{1}{2}-\frac{\varepsilon_1}{\alpha_3}}. \tag{24}$$

The matrices in (23) are positive semidefinite if

$$k_1'\ge 8\varepsilon_2(k\beta_{zm})^2 \tag{25a}$$
$$\alpha_3\le\frac{1}{2}\sqrt{a\varepsilon_1\varepsilon_2}. \tag{25b}$$

Hence, under (24) and (25), the total time derivative of the Lyapunov function $V_{12}:=V_1+V_2$ satisfies

$$\begin{aligned}\dot V_{12}(\xi)\le &\,k\beta_{xs}|e_z|\left[\varepsilon_2|e_w|+\varepsilon_1|e_x|+\alpha_3|e_y|\right]\\&-\frac{1}{4}\left(k_1'e_x^2+\varepsilon_2e_w^2+\varepsilon_1ae_y^2\right). \end{aligned}\tag{26}$$

Next, consider the total derivative of V_3 along the trajectories of (12c); it satisfies (with $|x_m|_\infty\le\beta_{xm}=\beta_{xs}=\beta_x$)

$$\dot V_3(\xi)\le-\alpha_4ce_z^2+\alpha_4he_x^2e_z+2h\alpha_4\beta_{xm}|e_xe_z|.$$

Hence, the time derivative of $V(\xi):=V_1(\xi)+V_2(\xi)+V_3(\xi)$ satisfies, for all[3] $|e_x|\le\beta_{xs}$

$$\begin{aligned}\dot V(\xi)\le &\,\beta_{xs}|e_z|\left[(\varepsilon_1k+3\alpha_4h)|e_x|+\alpha_3k|e_y|+\varepsilon_2k|e_w|\right]\\&-\alpha_4ce_z^2-\frac{1}{4}\left(k_1'e_x^2+\varepsilon_2e_w^2+\varepsilon_1ae_y^2\right). \end{aligned}\tag{27}$$

Rearranging terms, we obtain

$$\begin{aligned}\dot V\le &-\frac{1}{4}\begin{bmatrix}|e_z|\\|e_w|\end{bmatrix}^\top\begin{bmatrix}\alpha_4c & 2\varepsilon_2k\beta_{xs}\\2\varepsilon_2k\beta_{xs} & \frac{1}{2}\varepsilon_2\end{bmatrix}\begin{bmatrix}|e_z|\\|e_w|\end{bmatrix}\\&-\frac{1}{4}\begin{bmatrix}|e_y|\\|e_z|\end{bmatrix}^\top\begin{bmatrix}\frac{1}{2}\varepsilon_1a & 2\alpha_3k\beta_{xs}\\2\alpha_3k\beta_{xs} & \alpha_4c\end{bmatrix}\begin{bmatrix}|e_y|\\|e_z|\end{bmatrix}\\&-\frac{1}{4}\begin{bmatrix}|e_x|\\|e_z|\end{bmatrix}^\top\begin{bmatrix}\frac{1}{2}k_1' & 2(\varepsilon_1k+3\alpha_4h)\beta_{xs}\\2(\varepsilon_1k+3\alpha_4h)\beta_{xs} & \alpha_4c\end{bmatrix}\\&\times\begin{bmatrix}|e_x|\\|e_z|\end{bmatrix}-\frac{\alpha_4c}{4}e_z^2-\frac{1}{8}\left(k_1'e_x^2+\varepsilon_2e_w^2+\varepsilon_1ae_y^2\right). \end{aligned}\tag{28}$$

The matrices in (28) are positive semidefinite, respectively, if

$$\alpha_4c\ge 8(k\beta_{xs})^2\varepsilon_2 \tag{29a}$$
$$\alpha_4c\varepsilon_1a\ge 8(\alpha_3k\beta_{xs})^2 \tag{29b}$$
$$k_1'\alpha_4c\ge 8(\varepsilon_1k+3\alpha_4h)^2\beta_{xs}^2. \tag{29c}$$

In summary, $\dot V$ is negative definite and satisfies

$$\dot V(\xi)\le-\frac{\alpha_4c}{4}e_z^2-\frac{1}{8}\left(k_1'e_x^2+\varepsilon_2e_w^2+\varepsilon_1ae_y^2\right) \tag{30}$$

if (24), (25), and (29) hold with (16), (18), and (20).

We now investigate the positivity of V

$$\begin{aligned}V_{12}(\xi)\ge &\,\frac{1}{2}\begin{bmatrix}|e_x|\\|e_y|\end{bmatrix}^\top\begin{bmatrix}\alpha_1 & -\varepsilon_1\\-\varepsilon_1 & \alpha_3/2\end{bmatrix}\begin{bmatrix}|e_x|\\|e_y|\end{bmatrix}\\&\times\frac{1}{2}\begin{bmatrix}|e_x|\\|e_w|\end{bmatrix}^\top\begin{bmatrix}\alpha_3/2 & -\varepsilon_2\\-\varepsilon_2 & \alpha_2\end{bmatrix}\begin{bmatrix}|e_x|\\|e_w|\end{bmatrix} \end{aligned}\tag{31}$$

in which both matrices are positive definite, respectively, if

$$\alpha_3\alpha_1>2\varepsilon_1^2\quad\alpha_3\alpha_2>2\varepsilon_2^2. \tag{32}$$

Conditions (25) and (32) impose upper and lower bounds on the parameters ε_1, ε_2, α_3, etc. However, except for the control gain k_1, all parameters appear only in the Lyapunov function; hence, considerable freedom is left to choose them. One possible way is the following.

1) Pick $k_1>0$ as a "large" number.
2) Given β_{zm}, β_{xs}, and the system's parameters a, b, c, d, and h, pick ε_1 and ε_2 as relatively small and choose α_3

[3]Recall that we temporarily assume that $|x_s|\le\beta_{xs}$; hence, there exists β_x such that $|x_m-x_s|\le\beta_x$; without loss of generality, we can pick $\beta_x=\beta_{xm}=\beta_{xs}$ by redefining the constants if necessary.

satisfying (25b) and the second inequality in (32), i.e.,

$$\frac{2\varepsilon_2^2}{\alpha_2} < \alpha_3 \leq \frac{1}{2}\sqrt{a\varepsilon_1\varepsilon_2}.$$

3) In view of (18), we have

$$k_1' = \alpha_3(k_1 - b) - \varepsilon_1 a - \varepsilon_2 d$$

which is positive for large values of k_1.

4) Verify that ε_1 and ε_2 satisfy (24) and (32)—these hold for sufficiently small ε_1 and ε_2 and large k_1 (respectively, k_1').

The first inequality in (32) is equivalent to [in view of (16)]

$$\varepsilon_1 < \sqrt{\frac{\alpha_3 k_1'}{2a}}$$

which also holds for large values of k_1'. Finally, the inequalities in (29) hold under the conditions described above and for small α_3; in particular, one may pick $\alpha_3 \propto (1/k\beta_{xs})$ to satisfy (29b) and (29c) for ε_2. Thus, we conclude global exponential stability of the origin of (12); in particular, the synchronization objective (5) is attained.

It is left to show that the trajectories of the slave system are bounded under feedback. We invoke the following. 1) Since the systems are assumed to operate in chaotic mode without feedback, their trajectories converge to a compact invariant set. Let $r > 0$ and let the closed ball $\bar{B}_r$ strictly contain such a compact set; let $\infty > T^* \geq 0$ be the smallest number such that $\xi(t) \in B_r \, \forall \, t \geq T^*$. 2) The previous development shows that V is a positive definite Lyapunov function with a negative definite derivative for any values of the states contained in a compact set.[4] Hence, there always exist control gains such that the system under feedback is exponentially stable at the origin for any finite initial conditions. 3) It may be shown as in [10] that the system under feedback is forward complete, that is, if there exists a set of initial conditions and gains such that, together, they generate solutions that tend to infinity, these solutions may unboundedly grow only in infinite time. From this, it follows that, for each $\delta > 0$ and $T \geq 0$, there exists $M(\delta, T)$ such that

$$|\xi(0)| \leq \delta \Longrightarrow |\xi(t)| \leq M(\delta, T), \qquad \forall \, t \in [0, T]$$

where M is in general a nondecreasing function of its arguments. Since, by assumption, the system operates in open loop for all $t \in [0, T]$ for any $T > 0$, $|\xi(t)| \leq \max\{M(\delta, T), r\}$ for all $t \in [0, T)$ and any $T > 0$. That is, the solutions are bounded. Note that the bound $\max\{M(\delta, T), r\}$ is independent of the gains, and for $T \geq T^*$, we can safely assume that $\max\{M(\delta, T), r\} = r$ and β_{xs} depends only on r; hence, point 2 of the proposition follows. If T and δ are given and $\max\{M(\delta, T), r\} = M$, then β_{xs} depends on T and δ and so does k_1; hence, point 1 of the proposition follows. In either case, k_1 does not depend on the initial conditions $\xi(0)$, nor is it required that $|\xi(0)| \leq r$. ■

A. Simulation Results

We have tested the controller in simulations using parameters from the literature: the system parameters are set as in (2), and the initial conditions are set as in (3), so the hyperchaotic fourth-order system is in the chaotic regime. The system is left to freely evolve under the chaotic regime for 10 s, that is, $T = 10$ in Proposition 1. Then, the control action starts at this moment. All the conditions in the proof—(24), (25), (29), and (32)—hold for $k_1 = 5e7$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.005$, $\beta_{xs} = 30$, $\beta_{zm} = 130$, $\alpha_3 = 0.01/\beta_{xs}$, and $\alpha_4 = 0.01$.

Some simulation results are shown in Fig. 2.

We ran another series of simulations with $k_1 = 1e4$ for which the conditions of Proposition 1 do not necessarily hold. However, the simulation results show that the system is exponentially stable. This demonstrates that the conditions of Proposition 1 are by no means necessary and may only be regarded as one tuning scheme among many others.

The numeric simulations results for this second run are shown in Figs. 3–6. In Figs. 3(a), 4(a), 5(a), and 6(a), we show the systems' trajectories for both the master and the slave; from Figs. 3(b), 4(b), 5(b), and 6(b), one can appreciate the exponential convergence of the synchronization errors, notably the short transient after $T = 50$ s when the control is turned on.

[4]Indeed, if there exists $\beta_\xi > 0$ such that $|\xi| \leq \beta_\xi$, there also exists $\beta_{xs} > 0$ such that $|x_s| \leq \beta_{xs}$; this is also the case for all other state variables.

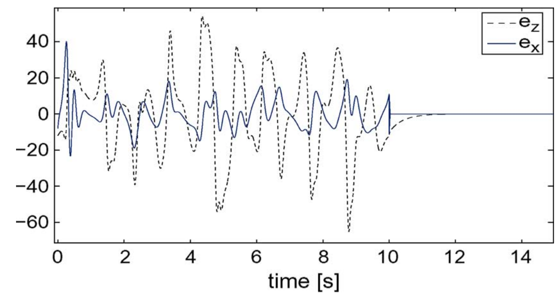
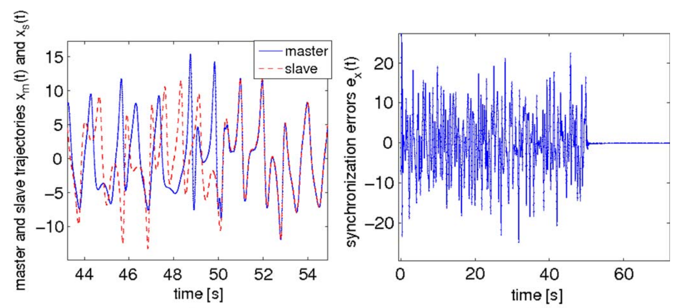
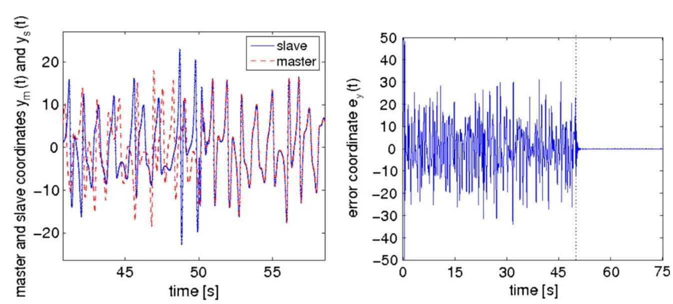


Fig. 2. Synchronization error trajectories for the x- and the z-coordinates. Control action starts at $t = 10$ s.



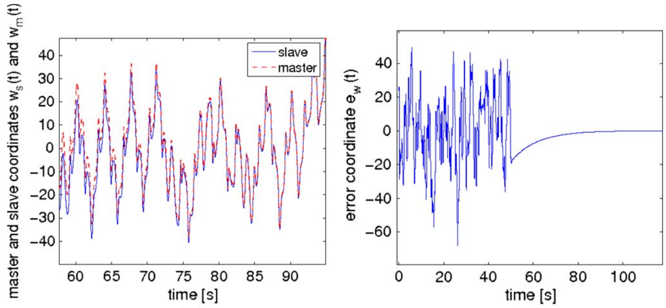
(a) (b)

Fig. 3. (a) Master and slave system's integral curves $x_m(t)$ and $x_s(t)$. (b) Synchronization error trajectories for the x-coordinates. Control action starts at $t = 50$ s.



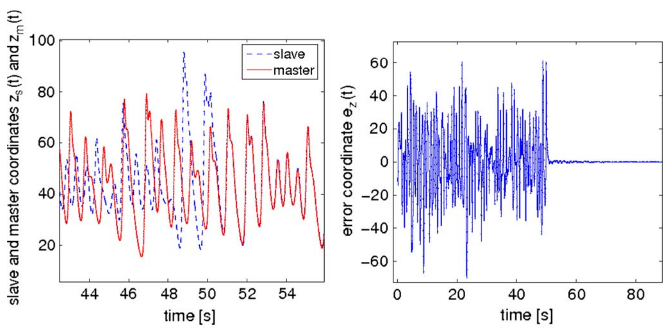
(a) (b)

Fig. 4. (a) Master and slave system's integral curves $y_m(t)$ and $y_s(t)$. (b) Synchronization error trajectories $e_y(t)$. Control action starts at $t = 50$ s.



(a) (b)

Fig. 5. (a) Master and slave system's integral curves $w_m(t)$ and $w_s(t)$. (b) Synchronization error trajectories $e_w(t)$. Control action starts at $t = 50$ s.



(a) (b)

Fig. 6. (a) Master and slave system's integral curves $z_m(t)$ and $z_s(t)$. (b) Synchronization error trajectories $e_z(t)$. Control action starts at $t = 50$ s.

III. Conclusion

We have shown that synchronization of Lü chaotic systems is possible via *one* control input and measuring only one variable. Our control law is a simple proportional feedback that is activated after a certain amount of time (one switch). The controller achieves exponential synchronization.

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