Asymptotic independence in the spectrum of the Gaussian unitary ensemble

Pascal Bianchi, Merouane Debbah, Jamal Najim

Asymptotic Independence in the Spectrum of the Gaussian Unitary Ensemble

P. Bianchi, M. Debbah and J. Najim
6 novembre 2008

Abstract

Consider a $n \times n$ matrix from the Gaussian Unitary Ensemble (GUE). Given a finite collection of bounded disjoint real Borel sets $(\Delta_{i,n},\ 1 \le i \le p)$, properly rescaled, and eventually included in any neighbourhood of the support of Wigner's semi-circle law, we prove that the related counting measures $(\mathcal{N}_n(\Delta_{i,n}),\ 1 \le i \le p)$, where $\mathcal{N}_n(\Delta)$ represents the number of eigenvalues within Δ, are asymptotically independent as the size n goes to infinity, p being fixed.

As a consequence, we prove that the largest and smallest eigenvalues, properly centered and rescaled, are asymptotically independent ; we finally describe the fluctuations of the condition number of a matrix from the GUE.

I. Introduction and main result

Denote by $\mathcal{H}_n$ the set of $n \times n$ random Hermitian matrices endowed with the probability measure

$$P_n(d\mathbf{M}) := Z_n^{-1} \exp\left\{-\frac{n}{2}\operatorname{Tr}\left(\mathbf{M}^2\right)\right\} d\mathbf{M}\ ,$$

where Z_n is the normalization constant and where

$$d\mathbf{M} = \prod_{i=1}^{n} d\,M_{ii} \prod_{1 \le i < j \le n} \Re\left[d\,M_{ij}\right] \prod_{1 \le i < j \le n} \Im\left[d\,M_{ij}\right]$$

for every $\mathbf{M} = (M_{ij})_{1 \le i,j \le n}$ in $\mathcal{H}_n$ ($\Re[z]$ being the real part of $z \in \mathbb{C}$ and $\Im[z]$ its imaginary part). This set is known as the Gaussian Unitary Ensemble (GUE) and corresponds to the case where a $n \times n$ hermitian matrix $\mathbf{M}$ has independent, complex, zero mean, Gaussian distributed entries with variance $\mathbb{E}|M_{ij}|^2 = \frac{1}{n}$ above the diagonal while the diagonal entries are independent real Gaussian with the same variance. Much is known about the spectrum of $\mathbf{M}$. Denote by $\lambda_1^{(n)}, \lambda_2^{(n)}, \cdots, \lambda_n^{(n)}$ the eigenvalues of $\mathbf{M}$ (all distinct with probability one), then :

- The joint probability density function of the (unordered) eigenvalues $(\lambda_1^{(n)}, \cdots, \lambda_n^{(n)})$ is given by :

  $$p_n(x_1, \cdots, x_n) = C_n e^{-\frac{\sum x_i^2}{2}} \prod_{j<k} |x_j - x_k|^2\ ,$$

  where C_n is the normalization constant.
- [9] The empirical distribution of the eigenvalues $\frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i^{(n)}}$ (δ_x stands for the Dirac measure at point x) converges toward Wigner's semi-circle law whose density is :

  $$\frac{1}{2\pi}\mathbf{1}_{(-2,2)}(x)\sqrt{4 - x^2}\ .$$
- [1] The largest eigenvalue $\lambda_{\max}^{(n)}$ (resp. the smallest eigenvalue $\lambda_{\min}^{(n)}$) almost surely converges to 2 (resp. -2), the right-end (resp. left-end) point of the support of the semi-circle law as $n \to \infty$.

This work was partially supported by the "Agence Nationale de la Recherche" via the program "Masses de données et connaissances", project SESAME and by the "GDR ISIS" via the program "jeunes chercheurs".
P. Bianchi and M. Debbah are with SUPELEC, France. {pascal.bianchi,merouane.debbah}@supelec.fr ,
M. Debbah holds the Alcatel-Lucent Chair on flexible radio,
J. Najim is with CNRS and Télécom ParisTech, Paris, France. najim@enst.fr

– [6] The centered and rescaled quantity $n^{\frac{2}{3}}\left(\lambda_{\max}^{(n)}-2\right)$ converges in distribution toward Tracy-Widom distribution function F_{GUE}^{+} which can be defined in the following way :

$$F_{GUE}^{+}(s)=\exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right)\ ,$$

where q solves the Painlevé II differential equation :

$$q''(x)=xq(x)+2q^3(x)\ ,$$
$$q(x)\sim \mathrm{Ai}(x)\quad \text{as}\quad x\to\infty$$

and $\mathrm{Ai}(x)$ denotes the Airy function. In particular, F_{GUE}^{+} is continuous. Similarly, $n^{\frac{2}{3}}\left(\lambda_{\min}^{(n)}+2\right)\xrightarrow{\mathcal{D}} F_{GUE}^{-}$ where

$$F_{GUE}^{-}(s)=1-F_{GUE}^{+}(-s)\ .$$

If Δ is a Borel set in $\mathbb{R}$, denote by :

$$\mathcal{N}_n(\Delta)=\#\left\{\lambda_i^{(n)}\in\Delta\right\},$$

i.e. the number of eigenvalues in the set Δ. The following theorem is the main result of the article.

Theorem 1: Let $\mathbf{M}$ be a $n\times n$ matrix from the GUE with eigenvalues $(\lambda_1^{(n)},\cdots,\lambda_n^{(n)})$. Let $p\geq 2$ be an integer and let $(\mu_1,\cdots,\mu_p)\in\mathbb{R}^p$ be such that $-2=\mu_1<\mu_2<\cdots<\mu_p=2$. Denote by $\boldsymbol{\Delta}=(\Delta_1,\cdots,\Delta_p)$ a collection of p bounded Borel sets in $\mathbb{R}$ and consider $\boldsymbol{\Delta}_n=(\Delta_{1,n},\cdots,\Delta_{p,n})$ defined by the following scalings :

$$\begin{array}{llll}(edge) & \Delta_{1,n} := -2+\dfrac{\Delta_1}{n^{2/3}}\ , & \Delta_{p,n} := 2+\dfrac{\Delta_p}{n^{2/3}}\ , \\ (bulke) & \Delta_{i,n} := \mu_i+\dfrac{\Delta_i}{n}\ , & 2\leq i\leq p-1\ .\end{array}$$

Let $(\ell_1,\cdots,\ell_p)\in\mathbb{N}^p$, then :

$$\lim_{n\to\infty}\left(\mathbb{P}\left(\mathcal{N}_n(\Delta_{1,n})=\ell_1,\cdots,\mathcal{N}_n(\Delta_{p,n})=\ell_p\right)-\prod_{k=1}^{p}\mathbb{P}\left(\mathcal{N}_n(\Delta_{k,n})=\ell_k\right)\right)=0\ .$$

Proof of Theorem 1 is postponed to Sections III. In Section II, we state and prove the asymptotic independence of the random variables $n^{\frac{2}{3}}\left(\lambda_{\min}^{(n)}+2\right)$ and $n^{\frac{2}{3}}\left(\lambda_{\max}^{(n)}-2\right)$, where $\lambda_{\min}^{(n)}$ and $\lambda_{\max}^{(n)}$ are the smallest and largest eigenvalues of $\boldsymbol{M}$. We then describe the asymptotic fluctuations of the ratio $\frac{\lambda_{\max}^{(n)}}{\lambda_{\min}^{(n)}}$.

Acknowlegment

The authors are grateful to Walid Hachem for fruitful discussions and many helpful remarks; the authors wish also to thank Eric Amar for useful references related to complex analysis.

II. Asymptotic independence of extreme eigenvalues

In this section, we prove that the random variables $n^{\frac{2}{3}}\left(\lambda_{\max}^{(n)}-2\right)$ and $n^{\frac{2}{3}}\left(\lambda_{\min}^{(n)}+2\right)$ are asymptotically independent as the size of matrix $\boldsymbol{M}$ goes to infinity. We then apply this result to describe the fluctuations of $\frac{\lambda_{\max}^{(n)}}{\lambda_{\min}^{(n)}}$. In the sequel, we drop the upperscript $^{(n)}$ to lighten the notations.

A. Asymptotic independence

Specifying $p=2$, $\mu_1=-2$, $\mu_2=2$ and getting rid of the boundedness condition over Δ_1 and Δ_2 in Theorem 1 yields the following :

Corollary 1: Let $\boldsymbol{M}$ be a $n\times n$ matrix from the GUE. Denote by $\lambda_{\min}$ and $\lambda_{\max}$ its smallest and largest eigenvalues, then the following holds true :

$$\mathbb{P}\left(n^{\frac{2}{3}}\left(\lambda_{\min}+2\right)<x\ ,\ n^{\frac{2}{3}}\left(\lambda_{\max}-2\right)<y\right)$$
$$-\,\mathbb{P}\left(n^{\frac{2}{3}}\left(\lambda_{\min}+2\right)<x\right)\mathbb{P}\left(n^{\frac{2}{3}}\left(\lambda_{\max}-2\right)<y\right)\xrightarrow[n\to\infty]{}0\ .$$

Otherwise stated,

$$\left(n^{\frac{2}{3}}(\lambda_{\min}+2), n^{\frac{2}{3}}(\lambda_{\max}-2)\right) \xrightarrow[n\to\infty]{\mathcal{D}} (\lambda_-, \lambda_+),$$

where λ_- and λ_+ are independent random variables with distribution functions F^-_{GUE} and F^+_{GUE}.

Proof: Denote by $(\lambda_{(i)})$ the ordered eigenvalues of $\boldsymbol{M}$: $\lambda_{\min} = \lambda_{(1)} \le \lambda_{(2)} \le \cdots \le \lambda_{(n)} = \lambda_{\max}$.

Let $(x, y) \in \mathbb{R}^2$ and take $\alpha \ge \max(|x|, |y|)$. Let $\Delta_1 = (-\alpha, x)$ and $\Delta_2 = (y, \alpha)$ so that

$$\Delta_{1,n} \;=\; \left(-2 - \frac{\alpha}{n^{\frac{2}{3}}}, -2 + \frac{x}{n^{\frac{2}{3}}}\right) \quad \text{and} \quad \Delta_{2,n} \;=\; \left(2 + \frac{y}{n^{\frac{2}{3}}}, 2 + \frac{\alpha}{n^{\frac{2}{3}}}\right) \; .$$

We have :

$$\begin{aligned} \{\mathcal{N}(\Delta_{1,n}) = 0\} \;&=\; \left\{n^{\frac{2}{3}}(\lambda_{\min}+2) > x\right\} \cup \left\{\exists i \in \{1, \cdots, n\};\ \lambda_{(i)} \le -2 - \frac{\alpha}{n^{\frac{2}{3}}}\ ,\ \lambda_{(i+1)} \ge -2 + \frac{x}{n^{\frac{2}{3}}}\right\} \\ &:= \left\{n^{\frac{2}{3}}(\lambda_{\min}+2) > x\right\} \cup \ \{\,\Pi(-\alpha, x)\,\}\ , \end{aligned} \tag{1}$$

with the convention that if $i = n$, the condition simply becomes $\lambda_{\max} \le -2 - \alpha n^{-\frac{2}{3}}$. Note that both sets in the right-hand side of the equation are disjoint. Similarly :

$$\begin{aligned} \{\mathcal{N}(\Delta_{2,n}) = 0\} \;&=\; \left\{n^{\frac{2}{3}}(\lambda_{\max}-2) < y\right\} \cup \left\{\exists i \in \{1, \cdots, n\};\ \lambda_{(i-1)} \le 2 + \frac{y}{n^{\frac{2}{3}}}\ ,\ \lambda_{(i)} \ge 2 + \frac{\alpha}{n^{\frac{2}{3}}}\right\}\ , \\ &:= \left\{n^{\frac{2}{3}}(\lambda_{\max}-2) < y\right\} \cup \{\,\tilde{\Pi}(y, \alpha)\,\}\ , \end{aligned} \tag{2}$$

with the convention that if $i = 1$, the condition simply becomes $\lambda_{\min} \ge 2 + \alpha n^{-\frac{2}{3}}$. Gathering the two previous equalities enables to write $\{\mathcal{N}(\Delta_{1,n}) = 0, \mathcal{N}(\Delta_{2,n}) = 0\}$ as the following union of disjoint events :

$$\begin{aligned} &\{\mathcal{N}(\Delta_{1,n}) = 0\ ,\ \mathcal{N}(\Delta_{2,n}) = 0\} \\ &\quad = \left\{\Pi(-\alpha, x)\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2) < y\right\} \cup \left\{\Pi(-\alpha, x)\ ,\ \tilde{\Pi}(y, \alpha)\right\} \cup \left\{n^{\frac{2}{3}}(\lambda_{\min}+2) > x\ , \tilde{\Pi}(y, \alpha)\right\} \\ &\qquad \cup \left\{n^{\frac{2}{3}}(\lambda_{\min}+2) > x\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2) < y\right\}\ . \end{aligned} \tag{3}$$

Define :

$$\begin{aligned} u_n \;&:=\; \mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2) > x\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2) < y\right\} - \mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2) > x\right\}\mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\max}-2) < y\right\} \\ &=\; \mathbb{P}\left\{\mathcal{N}(\Delta_{1,n}) = 0\ ,\ \mathcal{N}(\Delta_{2,n}) = 0\right\} - \mathbb{P}\left\{\mathcal{N}(\Delta_{1,n}) = 0\right\}\mathbb{P}\left\{\mathcal{N}(\Delta_{2,n}) = 0\right\} + \epsilon_n(\alpha)\ , \end{aligned} \tag{4}$$

where, by equations (1), (2) and (3),

$$\begin{aligned} \epsilon_n(\alpha) := -\mathbb{P}&\left\{\Pi(-\alpha, x)\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2) < y\right\} - \mathbb{P}\left\{\Pi(-\alpha, x)\ ,\ \tilde{\Pi}(y, \alpha)\right\} \\ &- \mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2) > x\ , \tilde{\Pi}(y, \alpha)\right\} + \mathbb{P}\left\{\mathcal{N}(\Delta_{1,n}) = 0\right\}\mathbb{P}\left\{\tilde{\Pi}(y, \alpha)\right\} \\ &+ \mathbb{P}\left\{\Pi(-\alpha, x)\right\}\mathbb{P}\left\{\mathcal{N}(\Delta_{2,n}) = 0\right\} - \mathbb{P}\left\{\Pi(-\alpha, x)\right\}\mathbb{P}\left\{\tilde{\Pi}(y, \alpha)\right\}\ . \end{aligned}$$

Using the triangular inequality, we obtain :

$$|\epsilon_n(\alpha)| \le 6 \max\left(\mathbb{P}\left\{\Pi(-\alpha, x)\right\}, \mathbb{P}\left\{\tilde{\Pi}(y, \alpha)\right\}\right)$$

As $\{\,\Pi(-\alpha, x)\,\} \subset \{n^{\frac{2}{3}}(\lambda_{\min}+2) < -\alpha\}$, we have

$$\mathbb{P}\{\Pi(-\alpha, x)\ \} \le \mathbb{P}\{n^{\frac{2}{3}}(\lambda_{\min}+2) < -\alpha\} \xrightarrow[n\to\infty]{} F^-_{GUE}(-\alpha) \xrightarrow[\alpha\to\infty]{} 0\ .$$

We can apply the same arguments to $\{\,\tilde{\Pi}(y, \alpha)\,\} \subset \{n^{\frac{2}{3}}(\lambda_{\max}-2) > \alpha\}$. We thus obtain :

$$\lim_{\alpha\to\infty} \limsup_{n\to\infty} |\epsilon_n(\alpha)| = 0\ . \tag{5}$$

The difference $\mathbb{P}\left\{\mathcal{N}(\Delta_{1,n})=0\ ,\ \mathcal{N}(\Delta_{2,n})=0\right\}-\mathbb{P}\left\{\mathcal{N}(\Delta_{1,n})=0\right\}\mathbb{P}\left\{\mathcal{N}(\Delta_{2,n})=0\right\}$ in the righthand side of (4) converges to zero as $n\to\infty$ by Theorem 1. We therefore obtain :

$$\limsup_{n\to\infty}|u_n| = \limsup_{n\to\infty}|\epsilon_n(\alpha)|\ .$$

The lefthand side of the above equation is a constant w.r.t. α while the second term (whose behaviour for small α is unknown) converges to zero as $\alpha\to\infty$ by (5). Thus, $\lim_{n\to\infty}u_n=0$. The mere definition of u_n together with Tracy and Widom fluctuation results yields :

$$\lim_{n\to\infty}\mathbb{P}\left\{n^{\frac{2}{3}}(\lambda_{\min}+2)>x\ ,\ n^{\frac{2}{3}}(\lambda_{\max}-2)<y\right\}=\left(1-F^-_{GUE}(x)\right)\times F^+_{GUE}(y)\ .$$

This completes the proof of Corollary 1.

■

B. Application : Fluctuations of the condition number in the GUE

As a simple consequence of Corollary 1, we can easily describe the fluctuations of the condition number $\frac{\lambda_{\max}}{\lambda_{\min}}$.

Corollary 2: Let $\boldsymbol{M}$ be a $n\times n$ matrix from the GUE. Denote by $\lambda_{\min}$ and $\lambda_{\max}$ its smallest and largest eigenvalues, then :

$$n^{\frac{2}{3}}\left(\frac{\lambda_{\max}}{\lambda_{\min}}+1\right)\xrightarrow[n\to\infty]{\mathcal{D}}-\frac{1}{2}(\lambda_-+\lambda_+)\ ,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution, λ_- and λ_+ are independent random variable with respective distribution F^-_{GUE} and F^+_{GUE}.

Proof: The proof is a mere application of Slutsky's lemma (see for instance [8, Lemma 2.8]). Write :

$$\begin{aligned} n^{\frac{2}{3}}\left(\frac{\lambda_{\max}}{\lambda_{\min}}+1\right) &= -\frac{1}{2}\left[n^{\frac{2}{3}}(\lambda_{\max}-2)+n^{\frac{2}{3}}(\lambda_{\min}+2)\right] \\ &\quad +\frac{\lambda_{\min}+2}{2\lambda_{\min}}\left[n^{\frac{2}{3}}(\lambda_{\max}-2)+n^{\frac{2}{3}}(\lambda_{\min}+2)\right]\ . \end{aligned} \tag{6}$$

Now, $\frac{\lambda_{\min}+2}{2\lambda_{\min}}$ goes almost surely to zero as $n\to\infty$, and $n^{\frac{2}{3}}(\lambda_{\max}-2)+n^{\frac{2}{3}}(\lambda_{\min}+2)$ converges in distribution to the convolution of F^-_{GUE} and F^+_{GUE} by Corollary 1. Thus, Slutsky's lemma implies that

$$\frac{\lambda_{\min}+2}{2\lambda_{\min}}\left[n^{\frac{2}{3}}(\lambda_{\max}-2)+n^{\frac{2}{3}}(\lambda_{\min}+2)\right]\xrightarrow[n\to\infty]{\mathcal{D}}0.$$

Another application of Slutsky's lemma yields the convergence (in distribution) of the right-hand side of (6) to the limit of $-\frac{1}{2}\left[n^{\frac{2}{3}}(\lambda_{\max}-2)+n^{\frac{2}{3}}(\lambda_{\min}+2)\right]$, that is $-\frac{1}{2}(X+Y)$ with X and Y independent and distributed according to F^-_{GUE} and F^+_{GUE}. Proof of Corollary 2 is completed. ■

III. Proof of Theorem 1

A. Useful results

1) Kernels: Let $\{H_k(x)\}_{k\geq 0}$ be the classical Hermite polynomials $H_k(x):=e^{x^2}\left(-\frac{d}{dx}\right)^k e^{-x^2}$ and consider the function $\psi_k^{(n)}(x)$ defined for $0\leq k\leq n-1$ by :

$$\psi_k^{(n)}(x):=\left(\frac{n}{2}\right)^{\frac{1}{4}}\frac{e^{-\frac{nx^2}{4}}}{(2^k k!\sqrt{\pi})^{\frac{1}{2}}}H_k\left(\sqrt{\frac{n}{2}}x\right)\ .$$

Denote by $K_n(x,y)$ the following Kernel on $\mathbb{R}^2$:

$$K_n(x,y) := \sum_{k=0}^{n-1}\psi_k^{(n)}(x)\psi_k^{(n)}(y) \tag{7}$$

$$= \frac{\psi_n^{(n)}(x)\psi_{n-1}^{(n)}(y)-\psi_n^{(n)}(y)\psi_{n-1}^{(n)}(x)}{x-y} \tag{8}$$

Equation (8) is obtained from (7) by the Christoffel-Darboux formula. We recall the two well-known asymptotic results

Proposition 1: a) *Bulk of the spectrum.* Let $\mu \in (-2,2)$.

$$\forall (x,y) \in \mathbb{R}^2,\ \lim_{n\to\infty} \frac{1}{n} K_n \left(\mu + \frac{x}{n}, \mu + \frac{y}{n}\right) = \frac{\sin \pi\rho(\mu)(x-y)}{\pi(x-y)}\ , \tag{9}$$

where $\rho(\mu) = \frac{\sqrt{4-\mu^2}}{2\pi}$. Furthermore, the convergence (9) is uniform on every compact set of $\mathbb{R}^2$.

b) *Edge of the spectrum.*

$$\forall (x,y) \in \mathbb{R}^2,\ \lim_{n\to\infty} \frac{1}{n^{2/3}} K_n \left(2 + \frac{x}{n^{2/3}}, 2 + \frac{y}{n^{2/3}}\right) = \frac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x-y}\ , \tag{10}$$

where $Ai(x)$ is the Airy function. Furthermore, the convergence (10) is uniform on every compact set of $\mathbb{R}^2$. We will need as well the following result on the asymptotic behavior of functions $\psi_k^{(n)}$.

Proposition 2: Let $\mu \in (-2,2)$, let $k=0$ or $k=1$ and denote by K a compact set of $\mathbb{R}$.

a) *Bulk of the spectrum.* There exists a constant C such that for large n,

$$\sup_{x\in K} \left|\psi_{n-k}^{(n)}\left(\mu + \frac{x}{n}\right)\right| \le C\ . \tag{11}$$

b) *Edge of the spectrum.* There exists a constant C such that for large n,

$$\sup_{x\in K} \left|\psi_{n-k}^{(n)}\left(\pm 2 \pm \frac{x}{n^{2/3}}\right)\right| \le n^{1/6} C\ . \tag{12}$$

The proof of these results can be found in [3, Chapter 7].

2) Determinantal representations, Fredholm determinants: There are determinantal representations using kernel $K_n(x,y)$ for the joint density p_n of the eigenvalues $(\lambda_i^{(n)}; 1 \le i \le n)$, and for its marginals (see for instance [2, Chapter 6] :

$$p_n(x_1,\cdots,x_n) = \frac{1}{n!} \det\left\{K_n(x_i,x_j)\right\}_{1\le i,j\le n}\ , \tag{13}$$

$$\int_{\mathbb{R}^{n-m}} p_n(x_1,\cdots,x_n) dx_{m+1}\cdots dx_n = \frac{(n-m)!}{n!} \det\left\{K_n(x_i,x_j)\right\}_{1\le i,j\le m} \quad (m \le n)\ . \tag{14}$$

Definition 1: Consider a linear operator S defined for any bounded integrable function $f:\mathbb{R}\to\mathbb{R}$ by

$$Sf : x \mapsto \int_{\mathbb{R}} S(x,y) f(y) dy\ ,$$

where $S(x,y)$ is a bounded integrable Kernel on $\mathbb{R}^2 \to \mathbb{R}$ with compact support. The Fredholm determinant $D(z)$ associated with operator S is defined as follows :

$$\det(1 - zS) := 1 + \sum_{k=1}^{\infty} \frac{(-z)^k}{k!} \int_{\mathbb{R}^k} \det\left\{S(x_i,x_j)\right\}_{1\le i,j\le k} dx_1\cdots dx_k\ , \tag{15}$$

for each $z \in \mathbb{C}$, i.e. it is an entire function. Its logarithmic derivative has the simple expression :

$$\frac{D'(z)}{D(z)} = -\sum_{k=0}^{\infty} T(k+1) z^k\ , \tag{16}$$

where

$$T(k) = \int_{\mathbb{R}^k} S(x_1,x_2)S(x_2,x_3)\cdots S(x_k,x_1)\, dx_1\cdots dx_k\ . \tag{17}$$

For details related to Fredholm determinants, see for instance [5], [7].

The following kernel will be of constant use in the sequel :

$$S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta}) = \sum_{i=1}^{p} \lambda_i \mathbf{1}_{\Delta_i}(x) K_n(x,y), \tag{18}$$

where $\boldsymbol{\lambda} = (\lambda_1,\cdots,\lambda_p) \in \mathbb{R}^p$ or $\boldsymbol{\lambda} \in \mathbb{C}^p$, depending on the need.

Remark 1: Kernel $K_n(x,y)$ is unbounded and one cannot consider its Fredholm determinant without caution. Kernel $S_n(x,y)$ is bounded in x since the kernel is zero if x is outside the compact set $\cup_{i=1}^p \Delta_i$, but a priori unbounded in y. In all the forthcoming computations, one may replace S_n with the bounded kernel $\tilde{S}_n(x,y) = \sum_{i,\ell=1}^p \lambda_i \mathbf{1}_{\Delta_i}(x)\mathbf{1}_{\Delta_\ell}(y)K_n(x,y)$ and get exactly the same results. For notational convenience, we keep on working with S_n.

Proposition 3: Let $p \geq 1$ be a fixed integer ; let $\boldsymbol{\ell} = (\ell_1, \cdots, \ell_p) \in \mathbb{N}^p$ and denote by $\boldsymbol{\Delta} = (\Delta_1, \cdots, \Delta_p)$, where every Δ_i is a bounded Borel set. Assume that the Δ_i's are pairwise disjoint. Then the following identity holds true :

$$\mathbb{P}\left\{\mathcal{N}(\Delta_1) = \ell_1, \cdots, \mathcal{N}(\Delta_p) = \ell_p\right\} = \frac{1}{\ell_1! \cdots \ell_p!}\left(-\frac{\partial}{\partial \lambda_1}\right)^{\ell_1} \cdots \left(-\frac{\partial}{\partial \lambda_p}\right)^{\ell_p} \det\left(1 - S_n(\boldsymbol{\lambda}, \boldsymbol{\Delta})\right)\Bigg|_{\lambda_1 = \cdots = \lambda_p = 1}, \quad (19)$$

where $S_n(\boldsymbol{\lambda}, \boldsymbol{\Delta})$ is the operator associated to the kernel defined in (18).

Proof of Proposition 3 is postponed to Appendix A.

3) Useful estimates for kernel $S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta})$ and its iterations: Consider $\boldsymbol{\mu}$, $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_n$ as in Theorem 1. Assume moreover that n is large enough so that the Borel sets $(\Delta_{i,n}; 1 \leq i \leq p)$ are pairwise disjoint. For $i \in \{1, \cdots, p\}$, define κ_i as

$$\kappa_i = \begin{cases} 1 & \text{if } -2 < \mu_i < 2 \\ \frac{2}{3} & \text{if } \mu_i = \pm 2 \end{cases}, \quad (20)$$

i.e. $\kappa_1 = \kappa_p = \frac{2}{3}$ and $\kappa_i = 1$ for $1 < i < p$.

Let $\boldsymbol{\lambda} \in \mathbb{C}^p$. With a slight abuse of notation, denote by $S_n(x,y;\boldsymbol{\lambda})$ the kernel :

$$S_n(x,y;\boldsymbol{\lambda}) := S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta}_n) \ . \quad (21)$$

For $1 \leq m, \ell \leq p$, define :

$$\mathcal{M}_{m\times\ell,n}(\boldsymbol{\Lambda}) \ := \ \sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \ \sup_{(x,y)\in\Delta_{m,n}\times\Delta_{\ell,n}} |S_n(x,y;\boldsymbol{\lambda})| \ , \quad (22)$$

where $S_n(x,y;\boldsymbol{\lambda})$ is given by (21).

Proposition 4: Let $\boldsymbol{\Lambda} \subset \mathbb{C}^p$ be a compact set. There exist two constants $R = R(\boldsymbol{\Lambda}) > 0$ and $C = C(\boldsymbol{\Lambda}) > 0$, independent from n, such that for n large enough,

$$\begin{cases} \mathcal{M}_{m\times m,n}(\boldsymbol{\Lambda}) & \leq & R^{-1} n^{\kappa_m}, & 1 \leq m \leq p \\ \mathcal{M}_{m\times\ell,n}(\boldsymbol{\Lambda}) & \leq & C n^{1-\frac{\kappa_m+\kappa_\ell}{2}}, & 1 \leq m,\ell \leq p, \ m \neq \ell \end{cases} . \quad (23)$$

Proposition 4 is proved in Appendix B.

Consider the iterated kernel $|S_n|^{(k)}(x,y;\boldsymbol{\lambda})$ defined by :

$$\begin{cases} |S_n|^{(1)}(x,y;\boldsymbol{\lambda}) = |S_n(x,y;\boldsymbol{\lambda})| \\ |S_n|^{(k)}(x,y;\boldsymbol{\lambda}) = \int_{\mathbb{R}^{k-1}} |S_n(x,u;\boldsymbol{\lambda})| \times |S_n|^{(k-1)}(u,y;\boldsymbol{\lambda})\, du \quad k \geq 2 \end{cases}, \quad (24)$$

where $S_n(x,y;\boldsymbol{\lambda})$ is given by (21). The next estimates will be stated with $\boldsymbol{\lambda} \in \mathbb{C}^p$ fixed. Note that $|S_n|^{(k)}$ is nonnegative and writes :

$$\int_{\mathbb{R}^{k-1}} |S_n(x,u_1;\boldsymbol{\lambda}) S_n(u_1,u_2;\boldsymbol{\lambda}) \cdots S_n(u_{k-1},y;\boldsymbol{\lambda})| du_1 \cdots du_{k-1} \ .$$

As previously, define for $1 \leq m, \ell \leq p$:

$$\mathcal{M}^{(k)}_{m\times\ell,n}(\boldsymbol{\lambda}) := \sup_{(x,y)\in\Delta_{m,n}\times\Delta_{\ell,n}} |S_n|^{(k)}(x,y;\boldsymbol{\lambda})$$

The following estimates hold true :

Proposition 5: Consider the compact set $\boldsymbol{\Lambda} = \{\boldsymbol{\lambda}\}$ and the associated constants $R = R(\boldsymbol{\lambda})$ and $C = C(\boldsymbol{\lambda})$ as given by Prop. 4. Let $\beta > 0$ be such that $\beta > R^{-1}$ and consider $\epsilon \in (0, \frac{1}{3})$. There exists an integer $N_0 = N_0(\beta, \epsilon)$ such that for every $n \geq N_0$ and for every $k \geq 1$,

$$\begin{cases} \mathcal{M}^{(k)}_{m\times m,n}(\boldsymbol{\lambda}) & \leq & \beta^k n^{\kappa_m}, & 1 \leq m \leq p \\ \mathcal{M}^{(k)}_{m\times\ell,n}(\boldsymbol{\lambda}) & \leq & C\beta^{k-1}\, n^{\left(1+\epsilon-\frac{\kappa_m+\kappa_\ell}{2}\right)}, & 1 \leq m,\ell \leq p, \ m \neq \ell \end{cases} . \quad (25)$$

Proposition 5 is proved in Appendix C.

B. End of proof

Consider $\boldsymbol{\mu}$, $\boldsymbol{\Delta}$ and $\boldsymbol{\Delta}_n$ as in Theorem 1. Assume moreover that n is large enough so that the Borel sets $(\Delta_{i,n}; 1 \leq i \leq p)$ are pairwise disjoint. As previously, denote $S_n(x,y;\boldsymbol{\lambda}) = S_n(x,y;\boldsymbol{\lambda},\boldsymbol{\Delta}_n)$; denote also $S_{i,n}(x,y;\lambda_i) = S_n(x,y;\lambda_i,\Delta_{i,n}) = \lambda_i \mathbf{1}_{\Delta_i}(x) K_n(x,y)$, for $1 \leq i \leq p$. Note that $S_n(x,y;\boldsymbol{\lambda}) = S_{i,n}(x,y;\lambda_i)$ if $x \in \Delta_{i,n}$.

For every $z \in \mathbb{C}$ and $\boldsymbol{\lambda} \in \mathbb{C}^p$, we use the following notations :

$$D_n(z,\boldsymbol{\lambda}) := \det(1 - zS_n(\boldsymbol{\lambda},\boldsymbol{\Delta}_n)) \quad \text{and} \quad D_{n,i}(z,\lambda_i) := \det(1 - zS_n(\lambda_i,\Delta_{i,n})) \tag{26}$$

The following controls will be of constant use in the sequel.

Proposition 6: 1) Let $\boldsymbol{\lambda} \in \mathbb{C}^p$ be fixed. The sequences of functions :

$$z \mapsto D_n(z,\boldsymbol{\lambda}) \qquad \text{and} \qquad z \mapsto D_{i,n}(z,\lambda_i), \quad 1 \leq i \leq p$$

are uniformly bounded on every compact subset of $\mathbb{C}$.

2) Let $z = 1$. The sequences of functions :

$$\boldsymbol{\lambda} \mapsto D_n(1,\boldsymbol{\lambda}) \qquad \text{and} \qquad \boldsymbol{\lambda} \mapsto D_{1,n}(1,\lambda_i), \quad 1 \leq i \leq p$$

are uniformly bounded on every compact subset of $\mathbb{C}^p$.

3) Let $\boldsymbol{\lambda} \in \mathbb{C}^p$ be fixed. For every $\delta > 0$, there exists $r > 0$ such that

$$\begin{aligned}\sup_n \sup_{z \in B(0,r)} |D_n(z,\boldsymbol{\lambda}) - 1| &< \delta \,, \\ \sup_n \sup_{z \in B(0,r)} |D_{i,n}(z,\lambda_i) - 1| &< \delta \,, \quad 1 \leq i \leq p \,.\end{aligned}$$

Proof of Proposition 6 is provided in Appendix D.

We introduce the following functions :

$$d_n : (z,\boldsymbol{\lambda}) \mapsto \det\left(1 - zS_n(\boldsymbol{\lambda},\boldsymbol{\Delta}_n)\right) - \prod_{i=1}^p \det\left(1 - zS_n(\lambda_i,\Delta_{i,n})\right) \,, \tag{27}$$

$$f_n : (z,\boldsymbol{\lambda}) \mapsto \frac{D_n'(z,\boldsymbol{\lambda})}{D_n(z,\boldsymbol{\lambda})} - \sum_{i=1}^p \frac{D_{i,n}'(z,\lambda_i)}{D_{i,n}(z,\lambda_i)} \,, \tag{28}$$

where $'$ denotes the derivative with respect to $z \in \mathbb{C}$. We first prove that f_n goes to zero as $z \to 0$.

1) Asymptotic study of f_n in a neighbourhood of $z = 0$: In this section, we mainly consider the dependence of f_n in $z \in \mathbb{C}$ while $\boldsymbol{\lambda} \in \mathbb{C}^p$ is kept fixed. We therefore drop the dependence in $\boldsymbol{\lambda}$ to lighten the notations. Equality (16) yields :

$$\frac{D_n'(z)}{D_n(z)} = -\sum_{k=0}^\infty T_n(k+1) z^k \quad \text{and} \quad \frac{D_{i,n}'(z)}{D_{i,n}(z)} = -\sum_{k=0}^\infty T_{i,n}(k+1) z^k \quad (1 \leq i \leq p) \tag{29}$$

where $'$ denotes the derivative with respect to $z \in \mathbb{C}$ and $T_n(k)$ and $T_{i,n}(k)$ are as in (17), respectively defined by :

$$T_n(k) := \int_{\mathbb{R}^k} S_n(x_1,x_2) S_n(x_2,x_3) \cdots S_n(x_k,x_1) dx_1 \cdots dx_k \,, \tag{30}$$

$$T_{i,n}(k) := \int_{\mathbb{R}^k} S_{i,n}(x_1,x_2) S_{i,n}(x_2,x_3) \cdots S_{i,n}(x_k,x_1) dx_1 \cdots dx_k \,. \tag{31}$$

Recall that D_n and $D_{i,n}$ are entire functions (of $z \in \mathbb{C}$). The functions $\dfrac{D_n'}{D_n}$ and $\dfrac{D_{i,n}'}{D_{i,n}}$ admit a power series expansion around zero given by (29). Therefore, the same holds true for $f_n(z)$, moreover :

Lemma 1: Define R as in Proposition 4. For n large enough, $f_n(z)$ defined by (28) is holomorphic on $B(0,R) := \{z \in \mathbb{C},\ |z| < R\}$, and converges uniformly to zero as $n \to \infty$ on each compact subset of $B(0,R)$.

Proof: Denote by $\xi_i^{(n)}(x) := \lambda_i \mathbf{1}_{\Delta_{i,n}}(x)$ and recall that $T_n(k)$ is defined by (30). Using the identity

$$\prod_{m=1}^k \sum_{i=1}^p a_{im} = \sum_{\sigma \in \{1,\cdots,p\}^k} \prod_{m=1}^k a_{\sigma(m)m}, \tag{32}$$

where a_{im} are complex numbers, $T_n(k)$ writes $(k \geq 2)$:

$$\begin{aligned} T_n(k) &= \int_{\mathbb{R}^k} \left(\prod_{m=1}^{k} \sum_{i=1}^{p} \xi_i^{(n)}(x_m) \right) K_n(x_1,x_2)\cdots K_n(x_k,x_1) dx_1 \cdots dx_k \,, \\ &= \sum_{\sigma \in \{1,\cdots,p\}^k} j_{n,k}(\sigma) \,, \end{aligned} \tag{33}$$

where we defined

$$j_{n,k}(\sigma) := \int_{\mathbb{R}^k} \left(\prod_{m=1}^{k} \xi_{\sigma(m)}^{(n)}(x_m) \right) K_n(x_1,x_2)\cdots K_n(x_k,x_1) dx_1 \cdots dx_k \,. \tag{34}$$

We split the sum in the right-hand side into two subsums. The first is obtained by gathering the terms with k-uples $\sigma = (i,i,\cdots,i)$ for $1 \leq i \leq p$ and writes :

$$\sum_{i=1}^{p} \int_{\mathbb{R}^k} \left(\prod_{m=1}^{k} \lambda_i \mathbf{1}_{\Delta_{i,n}}(x_m) \right) K_n(x_1,x_2)\cdots K_n(x_k,x_1) dx_1 \cdots dx_k = \sum_{i=1}^{p} T_{i,n}(k) \,,$$

where $T_{i,n}(k)$ is defined by (31). The remaining sum consists of those terms for which there exists at least one couple $(m,\ell) \in \{1,\cdots,k\}^2$ such that $\sigma(m) \neq \sigma(\ell)$. Let

$$\mathcal{S} = \left\{ \sigma \in \{1,\cdots,p\}^k \ : \ \exists (m,\ell) \in \{1,\cdots,k\}^2, \sigma(m) \neq \sigma(\ell) \right\} \,,$$

we obtain $T_n(k) = \sum_{i=1}^{p} T_{i,n}(k) + s_n(k)$ where

$$s_n(k) := \sum_{\sigma \in \mathcal{S}} j_{n,k}(\sigma) \,,$$

for each $k \geq 2$. For each $q \in \{1,\ldots,k-1\}$, denote by π_q the following permutation for any k-uplet $(a_1,\ldots,a_k)$:

$$\pi_q(a_1,\ldots,a_k) = (a_q, a_{q+1},\ldots,a_k,a_1,\ldots,a_{q-1}) \,.$$

In other words, π_q operates a circular shift of $q-1$ elements to the left. Clearly, any k-uple $\sigma \in \mathcal{S}$ can be written as $\sigma = \pi_q(m,\ell,\tilde{\sigma})$ for some $q \in \{1,\ldots,k-1\}$, $(m,\ell) \in \{1,\ldots,p\}$ such that $m \neq \ell$, and $\tilde{\sigma} \in \{1,\ldots,p\}^{k-2}$. Thus,

$$|s_n(k)| \leq \sum_{q=1}^{k-1} \sum_{\substack{(m,\ell)\in\{1\cdots p\}^2 \\ m \neq \ell}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} |j_{n,k}(\pi_q(m,\ell,\tilde{\sigma}))| \,.$$

From (34), function $j_{n,k}$ is invariant up to any circular shift π_q, so that $j_{n,k}(\sigma)$ coincides with $j_n(m,\ell,\tilde{\sigma})$ for any $\sigma = \pi_q(m,\ell,\tilde{\sigma})$ as above. Therefore,

$$\begin{aligned} |s_n(k)| &\leq \sum_{q=1}^{k-1} \sum_{\substack{(m,\ell)\in\{1\cdots p\}^2 \\ m \neq \ell}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} |j_{n,k}(m,\ell,\tilde{\sigma})| \\ &\leq k \sum_{\substack{(m,\ell)\in\{1\cdots p\}^2 \\ m \neq \ell}} \sum_{\tilde{\sigma}\in\{1\cdots p\}^{k-2}} \int_{\mathbb{R}^k} |\xi_m^{(n)}(x_1)\xi_\ell^{(n)}(x_2)\xi_{\tilde{\sigma}(1)}^{(n)}(x_3)\cdots\xi_{\tilde{\sigma}(k-2)}^{(n)}(x_k)| \\ &\quad \times |K_n(x_1,x_2)\cdots K_n(x_k,x_1)| dx_1 \cdots dx_k \end{aligned}$$

The latter writes

$$
\begin{aligned}
|s_n(k)| &= k \sum_{\substack{1\le m,\ell\le p\\ m\neq \ell}} \int_{\Delta_{m,n}\times\Delta_{\ell,n}} \left| K_n(x_1,x_2)\xi_m^{(n)}(x_1)\xi_\ell^{(n)}(x_2)\right| \\
&\quad \times \left(\int_{\mathbb{R}^{k-2}} \sum_{\tilde\sigma\in\{1\cdots p\}^{k-2}} \left|\xi^{(n)}_{\tilde\sigma(1)}(x_3)\cdots \xi^{(n)}_{\tilde\sigma(k-2)}(x_k)\right| |K_n(x_2,x_3)\cdots K_n(x_k,x_1)|\, dx_3\cdots dx_k \right) dx_1dx_2 \\
&= k \sum_{\substack{1\le m,\ell\le p\\ m\neq \ell}} \int_{\Delta_{m,n}\times\Delta_{\ell,n}} \left| K_n(x_1,x_2)\times \sum_{i=1}^p \xi_i^{(n)}(x_1)\right| \times \sum_{i=1}^p \left|\xi_i^{(n)}(x_2)\right| \\
&\quad \times \left(\int_{\mathbb{R}^{k-2}} \sum_{\tilde\sigma\in\{1\cdots p\}^{k-2}} \left|\xi^{(n)}_{\tilde\sigma(1)}(x_3)\cdots \xi^{(n)}_{\tilde\sigma(k-2)}(x_k)\right| |K_n(x_2,x_3)\cdots K_n(x_k,x_1)|\, dx_3\cdots dx_k \right) dx_1dx_2 .
\end{aligned}
$$

It remains to notice that

$$
\begin{aligned}
&\sum_{i=1}^p \left|\xi_i^{(n)}(x_2)\right| \int_{\mathbb{R}^{k-2}} \sum_{\tilde\sigma\in\{1\cdots p\}^{k-2}} \prod_{m=3}^k \left|\xi^{(n)}_{\tilde\sigma(m-2)}(x_m)\right| |K_n(x_2,x_3)\cdots K_n(x_k,x_1)|\, dx_3\cdots dx_k \\
&\overset{(a)}{=} \sum_{i=1}^p \left|\xi_i^{(n)}(x_2)\right| \int_{\mathbb{R}^{k-2}} \left(\prod_{m=3}^k \sum_{i=1}^p \left|\xi_i^{(n)}(x_m)\right|\right) |K_n(x_2,x_3)\cdots K_n(x_k,x_1)|\, dx_3\cdots dx_k \\
&= \int_{\mathbb{R}^{k-2}} |S_n(x_2,x_3)S_n(x_3,x_4)\cdots S_n(x_k,x_1)| dx_3\cdots dx_k \\
&\overset{(b)}{=} |S_n|^{(k-1)}(x_2,x_1) \ ,
\end{aligned}
$$

where (a) follows from (32), and (b) from the mere definition of the iterated kernel (24). Thus, for $k\geq 2$, the following inequality holds true :

$$
|s_n(k)| \le k \sum_{\substack{1\le m,\ell\le p\\ m\neq \ell}} \int_{\Delta_{m,n}\times\Delta_{\ell,n}} |S_n(x_1,x_2)| \times |S_n|^{(k-1)}(x_2,x_1) dx_1dx_2 \ . \tag{35}
$$

For $k=1$, let $s_n(1)=0$ so that equation $T_n(k)=\sum_i T_{i,n}(k)+s_n(k)$ holds for every $k\geq 1$.

According to (28), $f_n(z)$ writes :

$$
f_n(z) = -\sum_{k=1}^{\infty} s_n(k+1) z^k \ .
$$

Let us now prove that $f_n(z)$ is well-defined on the desired neighbourhood of zero and converges uniformly to zero as $n\to\infty$. Let $\beta > R^{-1}$, then Propositions 4 and 5 yield :

$$
\begin{aligned}
|s_n(k)| &\le k \sum_{\substack{1\le m,\ell\le p\\ m\neq \ell}} \int_{\Delta_{m,n}\times\Delta_{\ell,n}} |S_n(x,y)||S_n|^{(k-1)}(y,x)dxdy \ , \\
&\le k \sum_{\substack{1\le m,\ell\le p\\ m\neq \ell}} \mathcal{M}_{m\times\ell,n}\mathcal{M}^{(k-1)}_{\ell\times m,n}|\Delta_{m,n}||\Delta_{\ell,n}| \ , \\
&\le k\,\beta^{k-2} \sum_{\substack{1\le m,\ell\le p\\ m\neq \ell}} C^2 n^{\left(1-\frac{\kappa_m+\kappa_\ell}{2}\right)} n^{\left(1+\epsilon-\frac{\kappa_m+\kappa_\ell}{2}\right)} n^{-(\kappa_m+\kappa_\ell)} \times |\Delta_m\Delta_\ell| \ , \\
&\le k\,\beta^{k-2} \sum_{\substack{1\le m,\ell\le p\\ m\neq \ell}} \frac{C^2\ |\Delta_m\Delta_\ell|}{n^{2(\kappa_m+\kappa_\ell-1)-\epsilon}} \ , \\
&\overset{(a)}{\le} k\,\beta^{k-2} \times \left(\max_{1\le m\le p}|\Delta_m|\right)^2 \times \frac{p(p-1)C^2}{n^{\frac{2}{3}-\epsilon}} \ ,
\end{aligned}
$$

where (a) follows from the fact that $\kappa_m + \kappa_\ell - 1 \geq \frac{1}{3}$. Clearly, the power series $\sum_{k=1}^{\infty}(k+1)\beta^{k-1}z^k$ converges for $|z| < \beta^{-1}$. As β^{-1} is arbitrarily lower than R, this implies that $f_n(z)$ is holomorphic in $B(0,R)$. Moreover, for each compact subset K included in the open disk $B(0,\beta^{-1})$ and for each $z \in K$,

$$|f_n(z)| \leq \left(\sum_{k=1}^{\infty}(k+1)\beta^{k-1}(\sup_{z\in K}|z|)^k\right) \times \left(\max_{1\leq m\leq p}|\Delta_m|\right)^2 \times \frac{p(p-1)C^2}{n^{\frac{2}{3}-\epsilon}} \ .$$

The right-hand side of the above inequality converges to zero as $n \to \infty$. Thus, the uniform convergence of $f_n(z)$ to zero on K is proved ; in particular, as $\beta^{-1} < R$, $f_n(z)$ converges uniformly to zero on $B(0,R)$. Lemma 1 is proved.

■

2) Convergence of d_n to zero as $n \to \infty$: In this section, $\boldsymbol{\lambda} \in \mathbb{C}^p$ is fixed. We therefore drop the dependence in $\boldsymbol{\lambda}$ in the notations. Consider function F_n defined by :

$$F_n(z) := \log \frac{D_n(z)}{\prod_{i=1}^{p} D_{i,n}(z)} \ , \tag{36}$$

where $\log$ corresponds to the principal branch of the logarithm and D_n and $D_{i,n}$ are defined in (29). As $D_n(0) = D_{i,n}(0) = 1$, there exists a neighbourhood of zero where F_n is holomorphic. Moreover, using Proposition 6-3), one can prove that there exists a neighbourhood of zero, say $B(0,\rho)$, where $F_n(z)$ is a normal family. Assume that this neighbourhood is included in $B(0,R)$, where R is defined in Proposition 4 and notice that in this neighbourhood, $F_n'(z) = f_n(z)$ as defined in (28). Consider a compactly converging subsequence $F_{\phi(n)} \to F_\phi$ in $B(0,\rho)$ (by compactly, we mean that the convergence is uniform over any compact set $\boldsymbol{\Lambda} \subset B(0,\rho)$), then one has in particular $F_{\phi(n)}'(z) \to F_\phi'$ but $F_{\phi(n)}'(z) = f_{\phi(n)}(z) \to 0$. Therefore, F_ϕ is a constant over $B(0,\rho)$, in particular, $F_\phi(z) = F_\phi(0) = 0$. We have proved that every converging subsequence of F_n converges to zero in $B(0,\rho)$. This yields the convergence (uniform on every compact of $B(0,\rho)$) of F_n to zero in $B(0,\rho)$. This yields the existence of a neighbourhood of zero, say $B(0,\rho')$ where :

$$\frac{D_n(z)}{\prod_{i=1}^{p} D_{i,n}(z)} \xrightarrow[n\to\infty]{} 1 \tag{37}$$

uniformly on every compact of $B(0,\rho')$. Recall that $d_n(z) = D_n(z) - \prod_{i=1}^{p} D_{i,n}(z)$.

Combining (37) with Proposition 6-3) yields the convergence of $d_n(z)$ to zero in a small neighbourhood of zero. Now, Proposition 6-1) implies that $d_n(z)$ is a normal family in $\mathbb{C}$. In particular, every subsequence $d_{\phi(n)}$ compactly converges to a holomorphic function which coincides with 0 in a small neighbourhood of the origin, and thus is equal to 0 over $\mathbb{C}$. We have proved that

$$d_n(z) \xrightarrow[n\to\infty]{} 0, \qquad \forall z \in \mathbb{C} \ ,$$

with $\boldsymbol{\lambda} \in \mathbb{C}^p$ fixed.

3) Convergence of the partial derivatives of $\boldsymbol{\lambda} \mapsto d_n(1,\boldsymbol{\lambda})$ to zero: In order to establish Theorem 1, we shall rely on Proposition 3 where the probabilities of interest are expressed in terms of partial derivatives of Fredholm determinants. We therefore need to establish that the partial derivatives of $d_n(1,\boldsymbol{\lambda})$ with respect to $\boldsymbol{\lambda}$ converge to zero as well. This is the aim of this section.

In the previous section, we have proved that $\forall (z,\boldsymbol{\lambda}) \in \mathbb{C}^{p+1}$, $d_n(z,\boldsymbol{\lambda}) \to 0$ as $n \to \infty$. In particular,

$$d_n(1,\boldsymbol{\lambda}) \to 0, \quad \forall \boldsymbol{\lambda} \in \mathbb{C}^p \ .$$

We now prove the following facts (with a slight abuse of notation, denote $d_n(\boldsymbol{\lambda})$ instead of $d_n(1,\boldsymbol{\lambda})$) :

1) As a function of $\boldsymbol{\lambda} \in \mathbb{C}^p$, $d_n(\boldsymbol{\lambda})$ is holomorphic.
2) The sequence $(\boldsymbol{\lambda} \mapsto d_n(\boldsymbol{\lambda}))_{n\geq 1}$ is a normal family on $\mathbb{C}^p$.
3) The convergence $d_n(\boldsymbol{\lambda}) \to 0$ is uniform over every compact set $\boldsymbol{\Lambda} \subset \mathbb{C}^p$.

Proof of Fact 1) is straightforward and is thus omitted. Proof of Fact 2) follows from Proposition 6-2). Let us now turn to the proof of Fact 3). As (d_n) is a normal family, one can extract from every subsequence a compactly converging one in $\mathbb{C}^p$ (see for instance [4, Theorem 1.13]). But for every $\boldsymbol{\lambda} \in \mathbb{C}^p$, $d_n(\boldsymbol{\lambda}) \to 0$, therefore every compactly converging subsequence converges toward 0. In particular, d_n itself compactly converges toward zero, which proves Fact 3).

In order to conclude the proof, it remains to apply standard results related to the convergence of partial derivatives of compactly converging holomorphic functions of several complex variables, as for instance [4, Theorem 1.9]. As $d_n(\boldsymbol{\lambda})$ compactly converges to zero, the following convergence holds true : Let $(\ell_1, \cdots, \ell_p) \in \mathbb{N}^p$, then

$$\forall \boldsymbol{\lambda} \in \mathbb{C}^p, \qquad \left(\frac{\partial}{\partial \lambda_1}\right)^{\ell_1} \cdots \left(\frac{\partial}{\partial \lambda_p}\right)^{\ell_p} d_n(\boldsymbol{\lambda}) \xrightarrow[n\to\infty]{} 0 \ .$$

This, together with Proposition 3, completes the proof of Theorem 1.

Appendix

A. Proof of Proposition 3

Denote by $E_n(\boldsymbol{\ell}, \boldsymbol{\Delta})$ the probability that for every $i \in \{1, \cdots, p\}$, the set Δ_i contains exactly ℓ_i eigenvalues :

$$E_n(\boldsymbol{\ell}, \boldsymbol{\Delta}) = \mathbb{P}\left\{\mathcal{N}(\Delta_1) = \ell_1, \cdots, \mathcal{N}(\Delta_p) = \ell_p\right\} \ . \tag{38}$$

Let $\mathcal{P}_n(m)$ be the set of subsets of $\{1, \cdots, n\}$ with exactly m elements. If $A \in \mathcal{P}_n(m)$, then A^c is its complementary subset in $\{1, \cdots, n\}$. The mere definition of $E_n(\boldsymbol{\ell}, \boldsymbol{\Delta})$ yields :

$$E_n(\boldsymbol{\ell}, \boldsymbol{\Delta}) = \int_{\mathbb{R}^n} \sum_{\substack{(A_1,\cdots,A_p)\in \\ \mathcal{P}_n(\ell_1)\times\cdots\times\mathcal{P}_n(\ell_p)}} \prod_{k=1}^p \left\{ \prod_{i\in A_k} \mathbf{1}_{\Delta_k}(x_i) \prod_{j\in A_k^c} (1 - \mathbf{1}_{\Delta_k}(x_j)) \right\} p_n(x_1 \cdots x_n) dx_1 \cdots dx_n$$

Using the following formula :

$$\frac{1}{\ell!}\left(-\frac{d}{d\lambda}\right)^{\ell} \prod_{i=1}^n (1 - \lambda\alpha_i) = \sum_{A\in\mathcal{P}_n(\ell)} \prod_{i\in A} \alpha_i \prod_{j\in A^c} (1 - \lambda\alpha_j) \ ,$$

we obtain :

$$E_n(\boldsymbol{\ell}, \boldsymbol{\Delta}) \;=\; \frac{1}{\ell_1! \cdots \ell_p!}\left(-\frac{\partial}{\partial \lambda_1}\right)^{\ell_1} \cdots \left(-\frac{\partial}{\partial \lambda_p}\right)^{\ell_p} \Gamma(\boldsymbol{\lambda}, \boldsymbol{\Delta})\Big|_{\lambda_1=\cdots=\lambda_p=1}$$

where

$$\Gamma(\boldsymbol{\lambda}, \boldsymbol{\Delta}) = \int_{\mathbb{R}^n} \prod_{i=1}^n (1 - \lambda_1 \mathbf{1}_{\Delta_1}(x_i)) \cdots (1 - \lambda_p \mathbf{1}_{\Delta_p}(x_i))\, p_n(x_1 \cdots x_n)\, dx_1 \cdots dx_n \ .$$

Expanding the inner product and using the fact that the Δ_k's are pairwise disjoint yields :

$$(1 - \lambda_1 \mathbf{1}_{\Delta_1}(x)) \cdots (1 - \lambda_p \mathbf{1}_{\Delta_p}(x)) = \left(1 - \sum_{k=1}^p \lambda_k \mathbf{1}_{\Delta_k}(x)\right) \ .$$

Thus

$$\begin{aligned}
\Gamma(\boldsymbol{\lambda}, \boldsymbol{\Delta}) &= \int_{\mathbb{R}^n} \prod_{i=1}^n \left(1 - \sum_{k=1}^p \lambda_k \mathbf{1}_{\Delta_k}(x_i)\right) p_n(x_1 \cdots x_n)\, dx_1 \cdots dx_n \ , \\
&\overset{(a)}{=} 1 + \int_{\mathbb{R}^n} \sum_{m=1}^n (-1)^m \sum_{A\in\mathcal{P}_n(m)} \prod_{i\in A} \left(\sum_{k=1}^p \lambda_k \mathbf{1}_{\Delta_k}(x_i)\right)\ p_n(x_1 \cdots x_n)\, dx_1 \cdots dx_n \ , \\
&= 1 + \sum_{m=1}^n (-1)^m \sum_{A\in\mathcal{P}_n(m)} \int_{\mathbb{R}^n} \prod_{i\in A} \left(\sum_{k=1}^p \lambda_k \mathbf{1}_{\Delta_k}(x_i)\right) p_n(x_1 \cdots x_n)\, dx_1 \cdots dx_n \ , \\
&\overset{(b)}{=} 1 + \sum_{m=1}^n (-1)^m \binom{n}{m} \int_{\mathbb{R}^n} \prod_{i=1}^m \left(\sum_{k=1}^p \lambda_k \mathbf{1}_{\Delta_k}(x_i)\right) p_n(x_1 \cdots x_n)\, dx_1 \cdots dx_n \ , \\
&\overset{(c)}{=} 1 + \sum_{m=1}^n \frac{(-1)^m}{m!} \int_{\mathbb{R}^m} \prod_{i=1}^m \left(\sum_{k=1}^p \lambda_k \mathbf{1}_{\Delta_k}(x_i)\right) \det\left\{K_n(x_i, x_j)\right\}_{1\le i,j\le m} dx_1 \cdots dx_m \ ,
\end{aligned}$$

where (a) follows from the expansion of $\prod_i (1 - \sum_k \lambda_k \mathbf{1}_{\Delta_k}(x_i))$, (b) from the fact that the inner integral in the third line of the previous equation does not depend upon E due to the invariance of p_n with respect to any permutation of the x_i's, and (c) follows from the determinantal representation (14).

Therefore, $\Gamma(\boldsymbol{\lambda}, \boldsymbol{\Delta})$ writes :

$$\Gamma(\boldsymbol{\lambda}, \boldsymbol{\Delta}) = 1 + \sum_{m=1}^{n} \frac{(-1)^m}{m!} \int_{\mathbb{R}^m} \det \left\{ S_n(x_i, x_j; \boldsymbol{\lambda}, \boldsymbol{\Delta}) \right\}_{1 \le i,j \le m} dx_1 \cdots dx_m \tag{39}$$

where $S_n(x, y; \boldsymbol{\lambda}, \boldsymbol{\Delta})$ is the kernel defined in (18). As the operator $S_n(\boldsymbol{\lambda}, \boldsymbol{\Delta})$ has finite rank n, (39) coincides with the Fredholm determinant $\det(1 - S_n(\boldsymbol{\lambda}, \boldsymbol{\Delta}))$ (see [7] for details). Proof of Proposition 3 is completed.

B. Proof of Proposition 4

In the sequel, $C > 0$ will be a constant independent from n, but whose value may change from line to line.

First consider the case $i = j$. Denote by $S_{\mu_i}(x, y)$ the following limiting kernel :

$$S_{\mu_i}(x, y) := \begin{cases} \dfrac{\sin \pi \rho(\mu_i)(x - y)}{\pi(x - y)} & \text{if } -2 < \mu_i < 2 \\ \dfrac{Ai(x)Ai'(y) - Ai(y)Ai'(x)}{x - y} & \text{if } \mu_i = 2, \\ \dfrac{Ai(-x)Ai'(-y) - Ai(-y)Ai'(-x)}{-x + y} & \text{if } \mu_i = -2, \end{cases}$$

Proposition 1 implies that $n^{-\kappa_i} K_n(\mu_i + x/n^{\kappa_i}, \mu_i + y/n^{\kappa_i})$ converges uniformly to $S_{\mu_i}(x, y)$ on every compact subset of $\mathbb{R}^2$, where κ_i is defined by (20). Moreover, $S_{\mu_i}(x, y)$ being bounded on every compact subset of $\mathbb{R}^2$, there exists a constant C_i such that :

$$\begin{aligned} \mathcal{M}_{i \times i, n}(\boldsymbol{\Lambda}) &= \left(\sup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} |\lambda_i| \right) \sup_{(x,y) \in \Delta_{i,n}^2} |K_n(x, y)| = \left(\sup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} |\lambda_i| \right) \sup_{(x,y) \in \Delta_i^2} \left| K_n \left(\mu_i + \frac{x}{n^{\kappa_i}}, \mu_i + \frac{y}{n^{\kappa_i}} \right) \right| \\ &\le \left(\sup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} |\lambda_i| \right) n^{\kappa_i} \left(\sup_{(x,y) \in \Delta_i^2} \left| \frac{1}{n^{\kappa_i}} K_n \left(\mu_i + \frac{x}{n^{\kappa_i}}, \mu_i + \frac{y}{n^{\kappa_i}} \right) - S_{\mu_i}(x, y) \right| + \sup_{(x,y) \in \Delta_i^2} |S_{\mu_i}(x, y)| \right) \\ &\le n^{\kappa_i} C_i \,, \end{aligned} \tag{40}$$

It remains to take R as $R^{-1} = \max(C_1, \cdots, C_p)$ to get the pointwise or uniform estimate.

Consider now the case where $i \neq j$. Using notation κ_i, inequalities (11) and (12) can be conveniently merged as follows : There exists a constant C such that for $1 \le i \le p$,

$$\sup_{x \in \Delta_{i,n}} \left| \psi_{n-k}^{(n)}(x) \right| \le n^{\frac{1 - \kappa_i}{2}} C \,. \tag{41}$$

For n large enough, we obtain, using (8) :

$$\begin{aligned} \mathcal{M}_{i \times j, n}(\boldsymbol{\Lambda}) &\overset{(a)}{\le} \left(\sup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} |\lambda_i| \right) \sup_{(x,y) \in \Delta_{i,n} \times \Delta_{j,n}} \frac{|\psi_n^{(n)}(x)||\psi_{n-1}^{(n)}(y)| + |\psi_n^{(n)}(y)||\psi_{n-1}^{(n)}(x)|}{|x - y|} \,, \\ &\overset{(b)}{\le} \left(\sup_{\boldsymbol{\lambda} \in \boldsymbol{\Lambda}} |\lambda_i| \right) n^{\frac{1-\kappa_i}{2} + \frac{1-\kappa_j}{2}} \frac{2C^2}{\inf_{(x,y) \in \Delta_{i,n} \times \Delta_{j,n}} |x - y|} \,, \\ &\overset{(c)}{\le} C \, n^{1 - \frac{\kappa_i + \kappa_j}{2}} \,, \end{aligned}$$

where (a) follows from (8), (b) from (41) and (c) from the fact that

$$\liminf_{n \to \infty} \inf_{(x,y) \in \Delta_{i,n} \times \Delta_{j,n}} |x - y| = |\mu_i - \mu_j| > 0.$$

Proposition 4 is proved.

C. *Proof of Proposition 5*

Let $\mathbf{\Lambda} = \{\boldsymbol{\lambda}\}$ be fixed. We drop, in the rest of the proof, the dependence in $\boldsymbol{\lambda}$ in the notations. The mere definition of $|S_n|^{(k)}$ yields :

$$\begin{aligned} 0 \leq |S_n|^{(k)}(x,y) &\leq \int_{\mathbb{R}} |S_n(x,u)| \times |S_n|^{(k-1)}(u,y)du \\ &= \sum_{i=1}^{p} \int_{\Delta_{i,n}} |S_n(x,u)| \times |S_n|^{(k-1)}(u,y)du \end{aligned}$$

From the above inequality, the following is straightforward :

$$\forall (x,y) \in \Delta_{m,n} \times \Delta_{\ell,n}, \quad |S_n|^{(k)}(x,y) \leq \sum_{i=1}^{p} |\Delta_{i,n}| \mathcal{M}_{m\times i,n} \mathcal{M}^{(k-1)}_{i\times \ell,n} \ .$$

Using Proposition 4, we obtain :

$$\mathcal{M}^{(k)}_{m\times\ell,n} \leq R^{-1}\mathcal{M}^{(k-1)}_{m\times\ell,n} + \alpha \sum_{i\neq m} n^{(1-\frac{\kappa_m+3\kappa_i}{2})} \mathcal{M}^{(k-1)}_{i\times\ell,n} \ , \tag{42}$$

where $\alpha := \max(C|\Delta_1|, \cdots, C|\Delta_p|)$. Now take $\beta > R^{-1}$ and $\epsilon \in (0, \frac{1}{3})$. Property (25) holds for $k = 1$ since

$$\mathcal{M}_{m\times m,n} \leq R^{-1} n^{\kappa_m} \leq \beta n^{\kappa_m} \quad \text{and} \quad \mathcal{M}_{m\times \ell,n} \leq C n^{\left(1-\frac{\kappa_m+\kappa_\ell}{2}\right)} \leq C n^{\left(1+\epsilon-\frac{\kappa_m+\kappa_\ell}{2}\right)}$$

for every $m \neq \ell$ by Proposition 4. Assume that the same holds at step $k-1$.

Consider first the case where $m = \ell$. Eq. (42) becomes

$$\begin{aligned} \mathcal{M}^{(k)}_{m\times m,n} &\leq R^{-1}\beta^{k-1}n^{\kappa_m} + \alpha C \beta^{k-2} \sum_{i\neq m} n^{(1-\frac{\kappa_m}{2}-\frac{3\kappa_i}{2})} n^{(1+\epsilon-\frac{\kappa_i}{2}-\frac{\kappa_m}{2})} \\ &\leq \beta^k n^{\kappa_m} \left(\frac{R^{-1}}{\beta} + \sum_{i\neq m} \frac{\alpha C}{\beta^2} n^{(2+\epsilon-2\kappa_m-2\kappa_i)} \right) \\ &\leq \beta^k n^{\kappa_m} \qquad \text{for } n \text{ large enough} \ , \end{aligned}$$

where the last inequality follows from the fact that $2+\epsilon-2\kappa_m-2\kappa_i < 0$, which implies that $n^{2+\epsilon-2\kappa_m-2\kappa_i} \to 0$, which in turn implies that the term inside the parentheses is lower than one for n large enough.

Now if $m \neq \ell$, Eq. (42) becomes :

$$\begin{aligned} \mathcal{M}^{(k)}_{m\times\ell,n} &\leq R^{-1}C\beta^{k-2}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)} + \alpha\beta^{k-1}n^{\left(1-\frac{\kappa_\ell+\kappa_m}{2}\right)} + \sum_{i\neq m,\ell} C\alpha\beta^{k-2} n^{\left(1-\frac{\kappa_m+3\kappa_i}{2}\right)} n^{\left(1+\epsilon-\frac{\kappa_i+\kappa_\ell}{2}\right)} \\ &= C\beta^{k-1}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)} \left(\frac{R^{-1}}{\beta} + \frac{\alpha}{Cn^\epsilon} + \frac{\alpha}{\beta} \sum_{i\neq m,\ell} n^{1-2\kappa_i} \right) \\ &\leq C\beta^{k-1}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)} \left(\frac{R^{-1}}{\beta} + \frac{\alpha}{Cn^\epsilon} + \frac{\alpha p^2}{\beta n^{\frac{1}{3}}} \right) \\ &\leq C\beta^{k-1}n^{\left(1+\epsilon-\frac{\kappa_\ell+\kappa_m}{2}\right)} \ , \end{aligned}$$

where the last inequality follows from the fact that the term inside the parentheses is lower than one for n large enough. Therefore, (25) holds for each $k \geq 1$ and for n large enough.

D. Proof of Proposition 6

Define $U_n(k,\boldsymbol{\lambda}) := \int_{\mathbb{R}^k} \left|\det\{S_n(x_i,x_j;\boldsymbol{\lambda})\}_{i,j=1\cdots k}\right| dx_1\cdots dx_k$. Using Hadamard's inequality,

$$\begin{aligned} U_n(k,\boldsymbol{\lambda}) &\leq \int_{\mathbb{R}^k} \prod_{i=1}^k \sqrt{\sum_{j=1}^k |S_n(x_i,x_j;\boldsymbol{\lambda})|^2 dx_1\cdots dx_k} \\ &\leq \int_{\mathbb{R}^k} \prod_{i=1}^k \sqrt{\sum_{j=1}^k \left|\sum_{m=1}^p \lambda_m \mathbf{1}_{\Delta_{m,n}}(x_i)\right|^2 |K_n(x_i,x_j)|^2 dx_1\cdots dx_k} \end{aligned}$$

Therefore,

$$\begin{aligned} U_n(k,\boldsymbol{\lambda}) &\leq \int_{\mathbb{R}^k} \prod_{i=1}^k \left(\sum_{m=1}^p \lambda_m \mathbf{1}_{\Delta_{m,n}}(x_i)\right) \sqrt{\sum_{j=1}^k |K_n(x_i,x_j)|^2 dx_1\cdots dx_k} \\ &= \int_{\mathbb{R}^k} \sum_{\sigma\in\{1\cdots p\}^k} \prod_{i=1}^k \lambda_{\sigma(i)} \mathbf{1}_{\Delta_{\sigma(i),n}}(x_i) \sqrt{\sum_{j=1}^k |K_n(x_i,x_j)|^2 dx_1\cdots dx_k} \\ &= \sum_{\sigma\in\{1\cdots p\}^k} \int_{\mathbb{R}^k} \prod_{i=1}^k \sqrt{\sum_{j=1}^k |\lambda_{\sigma(i)} \mathbf{1}_{\Delta_{\sigma(i),n}}(x_i) K_n(x_i,x_j)|^2 dx_1\cdots dx_k}\,. \end{aligned}$$

In the above equation, integral $\int_{\mathbb{R}^k}$ clearly reduces to an integral on the set $\Delta_{\sigma(1),n}\times\cdots\times\Delta_{\sigma(p),n}$. Thus,

$$\begin{aligned} \sup_{\boldsymbol{\lambda}\in\Lambda} U_n(k,\boldsymbol{\lambda}) &\leq \sum_{\sigma\in\{1\cdots p\}^k} \int_{\Delta_{\sigma(1),n}\times\cdots\times\Delta_{\sigma(p),n}} \prod_{i=1}^k \sqrt{\sum_{j=1}^k \mathcal{M}^2_{\sigma(i)\times\sigma(j)}(\boldsymbol{\Lambda}) dx_1\cdots dx_k} \\ &= \sum_{\sigma\in\{1\cdots p\}^k} \prod_{i=1}^k \sqrt{\sum_{j=1}^k \left(|\Delta_{\sigma(i),n}|\mathcal{M}_{\sigma(i)\times\sigma(j)}(\boldsymbol{\Lambda})\right)^2} \end{aligned} \tag{43}$$

We now use Proposition 4 to bound the right-hand side. Clearly, when $\sigma(i)=\sigma(j)$, Proposition 4 implies that $|\Delta_{\sigma(i),n}|\mathcal{M}_{\sigma(i)\times\sigma(i),n}(\boldsymbol{\Lambda}) \leq R_{\boldsymbol{\Lambda}}^{-1}\Delta_{\max}$, where $\Delta_{\max}=\max_{1\leq i\leq p}|\Delta_i|$. This inequality still holds when $\sigma(i)\neq\sigma(j)$ as a simple application of Proposition 4. Therefore,

$$\sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} U_n(k,\boldsymbol{\lambda}) \leq \sum_{\sigma\in\{1,\cdots,p\}^k} k^{\frac{k}{2}} \Delta_{\max}^k R_{\boldsymbol{\Lambda}}^{-k} = \left(\frac{p\,\Delta_{\max}\sqrt{k}}{R_{\boldsymbol{\Lambda}}}\right)^k .$$

Using this inequality, it is straightforward to show that the serie $\sum_k \frac{\sup_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} U_n(k,\boldsymbol{\lambda})}{k!} z^k$ converges for every $z\in\mathbb{C}$ and every compact set $\boldsymbol{\Lambda}$. Parts 1) and 2) of the proposition are proved. Based on the definition of $D_n(z,\boldsymbol{\lambda})$ and $D_{i,n}(z,\lambda_i)$, we obtain :

$$\max\left(|D_n(z,\boldsymbol{\lambda})-1|, |D_{i,n}(z,\lambda_i)-1|,\ 1\leq i\leq p\right) \leq |z|\sum_{k=1}^{\infty} \frac{|z|^{k-1}}{k!} U_n(k,\boldsymbol{\lambda})\,,$$

which completes the proof of Proposition 6.

References

[1] Z. D. Bai and Y. Q. Yin. Necessary and sufficient conditions for almost sure convergence of the largest eigenvalue of a Wigner matrix. *Ann. Probab.*, 16(4) :1729–1741, 1988.

[2] M. L. Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.

[3] L. Pastur. Eigenvalue distribution of random matrices. In *Random Media 2000 (Proceedings of the Mandralin Summer School), June 2000, Poland*, pages 93–206, Warsaw, 2007. Interdisciplinary Centre of Mathematical and Computational Modelling.

[4] R. M. Range. *Holomorphic functions and integral representations in several complex variables*, volume 108 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.

[5] F. Smithies. *Integral equations*. Cambridge Tracts in Mathematics and Mathematical Physics, no. 49. Cambridge University Press, New York, 1958.

[6] C. A. Tracy and H. Widom. Level-spacing distributions and the Airy kernel. *Comm. Math. Phys.*, 159(1) :151–174, 1994.

[7] F. G. Tricomi. *Integral equations*. Dover Publications Inc., New York, 1985. Reprint of the 1957 original.

[8] A. W. van der Vaart. *Asymptotic statistics*, volume 3 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 1998.

[9] E. P. Wigner. On the distribution of the roots of certain symmetric matrices. *Ann. of Math. (2)*, 67 :325–327, 1958.