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A velocity and *Level-Sets* based approach of dynamic contact problems with applications

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Abstract By introducing unknown *Level-Sets* fields defined on contact interfaces, the *Signorini-Moreau* contact conditions and the Coulomb friction laws are written as equations. From this, a continuous weak-strong formulation of dynamic frictional contact problems is derived; the *Level-Sets* being the intrinsic contact unknown fields. The problem is then discretized by time, space and collocation schemes. Some numerical tests are carried out, showing the effectiveness of our global approach. The paper is ended by showing a promising application of the multiscale *Arlequin* method [1] to contact-impact problems.

Key words: *Level-Sets*, contact, impact, Lagrange velocity formulation, *Arlequin* method

INTRODUCTION

Impact problems are nonlinear, irregular and multiscale. Their numerical approximation needs special numerical tools. Since the late seventies, Hughes et al. [2] have designed *a posteriori* numerical schemes to treat shocks due to impact and increasing efforts have been since furnished by the computational contact community to address this numerical issue (see e.g. [3,4,5]). Very important advances have this way been done and some of the most pertinent schemes are now capitalized in commercial codes. But due to the aforementioned intrinsic difficulties of the impact problems (strong nonlinearities, both space and time irregularities and multiscale intrinsic characters), more effort is needed to address for instance dynamic contact problems such as those involved by vibrating structures under impact. This work is a contribution to this wide and complex theme.

By applying the "viability lemma" of J.J. Moreau [6], the unilateral contact laws are written in terms of positions, velocities and contact pressions. Moreover, by introducing two unknown *Level-Sets* [7] type fields standing for a location of the positions of contact surfaces with respect to each other and for the *Sign* of the normal velocity jump field on the contact interface, the obtained (*Signorini-Moreau*) contact model is converted to multi-valued equalities. The Coulomb's frictional laws are written similarly as equations by using a *Level-Set* field whose *iso-1* values characterize the sticking contact regions. From these local equations, a seemingly new continuous weak-strong Lagrange formulation of a dynamic 3-D frictional contact problem is carried out in a straightforward manner. The associated discrete nonlinear systems are derived by using three discretization methods, namely a θ-time discretization scheme (e.g. [5]), a *Galerkin* method and a collocation one. The numerical algorithm used to solve the nonlinear and irregular discrete problems is described. Some numerical dynamic (frictionless and frictional) contact tests exemplifying our methodology are given. It is shown in particular that our numerical solutions do not exhibit pathologies such as spurious oscillations of the numerical mechanical fields (velocities and contact loads). Finally a promising application of the multiscale *Arlequin* method [1] to the (multiscale) impact problems is shown.

A DYNAMIC FRICTIONAL CONTACT MODEL

We consider the problem of dynamic frictional contact between two deformable solids contained in IR^d (d=2 or 3). The *Signorini-Moreau* and Coulomb conditions are written as equations *via* the use of unknown *Level-Sets* like fields defined on known potential contact zones.

1. *Signorini-Moreau* contact model The classical Signorini contact laws read:

$$d_n \le 0,\ \lambda \le 0 \ \text{ and } \ \lambda\, d_n = 0, \ \text{ on } \Gamma_c \times \mathrm{I} \tag{1}$$

where Γ_c, I, d_n and λ denote the known potential "slave" contact surface, the time interval, the so-called *signed distance* and the normal contact pression, respectively.

The *viability lemma* of J.J. Moreau [6] asserts that if the Signorini laws are satisfied at a given time in I, then they are satisfied at all future times as far as the following conditions are fulfilled:

$$\lambda = 0 \qquad \text{if } \ d_n < 0 \tag{2.a}$$
$$[[v_n]] \le 0,\ \lambda \le 0,\ \lambda\,[[v_n]] = 0 \qquad \text{otherwise} \tag{2.b}$$

with $[[v_n]]$ standing for the normal velocity jump field on the contact surface (in the classical contact mechanics sense (see e.g. [8])).

By using two *Sign*-like functions (as introduced in [9] for a penalized unilateral contact model), what could be labelled the *Signorini-Moreau* contact laws (2) are transformed to the following equalities:

$$\lambda = S_u S_v \left(\lambda - h_n [[v_n]]\right), \ \text{on } \Gamma_c \times \mathrm{I} \tag{3.a}$$
$$S_u = 1_{\mathrm{IR}^-}\left(-d_n\right), \ \text{on } \Gamma_c \times \mathrm{I} \tag{3.b}$$
$$S_v = 1_{\mathrm{IR}^-}\left(\lambda - \rho_n [[v_n]]\right), \ \text{on } \Gamma_c \times \mathrm{I} \tag{3.c}$$

with $h_n \neq 0$, $\rho_n > 0$ and 1_K denoting the characteristic function of the set K. We notice that $S_c = S_u S_v$ is a *Level-Set* field whose *iso-1* values characterize the effective contact zone. This unknown field takes into account both of the relative positions of the contact surfaces and their relative velocities.

2. Coulomb friction laws Similarly, the classical Coulomb's laws can be equivalently written as the following equations [10]:

$$\mathbf{R}_\tau = \mu\lambda\boldsymbol{\Lambda}, \ \text{ on } \Gamma_c \times \mathrm{I} \tag{4.a}$$

with,

$$\boldsymbol{\Lambda} - P_{B(0,1)}\left(\boldsymbol{\Lambda} + h_\tau [[\mathbf{v}_\tau]]\right) = 0 \ \text{ on } \Gamma_c \times \mathrm{I} \qquad \text{if } \ S_u = 1 \tag{4.b}$$
$$\boldsymbol{\Lambda} = 0 \ \text{ on } \Gamma_c \times \mathrm{I} \qquad \text{otherwise} \tag{4.c}$$

where μ is the friction coefficient, h_τ is a non zero real parameter, $[[\mathbf{v}_\tau]]$ is the relative tangential velocity and $P_{B(0,1)}$ is the orthogonal projection on the unit ball, with respect to the Euclidian scalar product of IR^d.

Now, by introducing the following *Level-Set* field S_f whose *iso-1* values characterize the sticking zone:

$$S_f = 1_{B(0,1)}\left(\boldsymbol{\Lambda} + \rho_\tau [[\mathbf{v}_\tau]]\right), \ \text{ on } \Gamma_c \times \mathrm{I} \qquad (\rho_\tau > 0) \tag{5}$$

one can express the Coulomb's laws as follows:

$$\mathbf{R}_\tau = \mu\lambda\boldsymbol{\Lambda}, \ \text{ on } \Gamma_c \times \mathrm{I} \tag{6.a}$$

with:

$$S_u\left[\boldsymbol{\Lambda}-\left[S_f\mathbf{G}_{\boldsymbol{\tau}}+\left(1-S_f\right)\frac{\mathbf{G}_{\boldsymbol{\tau}}}{\left\|\mathbf{G}_{\boldsymbol{\tau}}\right\|}\right]\right]+\left(1-S_u\right)\boldsymbol{\Lambda}=0\,,\ \text{ on } \Gamma_c\times \mathrm{I} \tag{6.b}$$

where $\mathbf{G}_{\boldsymbol{\tau}}=\boldsymbol{\Lambda}+h_\tau[[\mathbf{v}_{\boldsymbol{\tau}}]]$

The new local settings of *Signorini-Moreau* contact conditions and Coulomb laws are interesting from a numerical point of view since they lead themselves to rather "natural" weak formulations, well-suited to numerical approximations.

WEAK-STRONG FORMULATION OF THE DYNAMIC FRICTIONAL CONTACT PROBLEM

By using the *Virtual Work Principle* (VWP) and the weak formulations of equations (3.a) and (6.b), keeping equations (3.b), (3.c) and (5) as local strong ones, the following (*seemingly new*) weak-strong mixed formulation of the dynamic frictional contact problem is derived:
assuming that the displacement and velocity fields $\mathbf{u}^i$ and $\mathbf{v}^i$ (i=1,2) are known at a given instant $t_0\in$I, then for all $t>t_0$, $t\in$I, the problem to be solved is: (without explicit reference to time for clarity of notations)

Find $\left(\mathbf{v}^i,\mathbf{u}^i,\lambda,\boldsymbol{\Lambda};S_u,S_v,S_f\right)\ ;\ \forall\left(\mathbf{w}^i,\lambda^*,\boldsymbol{\Lambda}^*\right)$

$$\sum_{i=1}^{2}\int_{\Omega_0^i}\rho_0^i\dot{\mathbf{v}}^i.\mathbf{w}^i+\sum_{i=1}^{2}\int_{\Omega_0^i}\boldsymbol{\Pi}^i(\mathbf{v}^i):\left(\nabla(\mathbf{w}^i)\right)^T-\int_{\Gamma_c}S_uS_v\lambda[[w_n]]-\int_{\Gamma_c}\mu\lambda S_u\left[S_f\boldsymbol{\Lambda}+\left(1-S_f\right)\frac{\boldsymbol{\Lambda}+h_\tau[[\mathbf{v}_{\boldsymbol{\tau}}]]}{\left\|\boldsymbol{\Lambda}+h_\tau[[\mathbf{v}_{\boldsymbol{\tau}}]]\right\|}\right].[[\mathbf{w}_{\boldsymbol{\tau}}]]=0 \tag{7.a}$$

$$-\frac{1}{h_n}\int_{\Gamma_c}\left[\lambda-S_uS_v\left(\lambda-h_n[[v_n]]\right)\right]\lambda^*=0 \tag{7.b}$$

$$\frac{1}{h_\tau}\int_{\Gamma_c}\mu\lambda S_u\left[\boldsymbol{\Lambda}-\left[S_f\left(\boldsymbol{\Lambda}+h_\tau[[\mathbf{v}_{\boldsymbol{\tau}}]]\right)+\left(1-S_f\right)\frac{\boldsymbol{\Lambda}+h_\tau[[\mathbf{v}_{\boldsymbol{\tau}}]]}{\left\|\boldsymbol{\Lambda}+h_\tau[[\mathbf{v}_{\boldsymbol{\tau}}]]\right\|}\right]\right]\boldsymbol{\Lambda}^*+\int_{\Gamma_c}\left(1-S_u\right)\boldsymbol{\Lambda}\boldsymbol{\Lambda}^*=0 \tag{7.c}$$

$$\mathbf{u}^i(t)=\mathbf{u}^i(t_0)+\int_{t0}^{t}\mathbf{v}^i(s)ds\quad \text{ in } \Omega_0^i \tag{7.d}$$

$$S_u-1_{\mathrm{IR}^-}\left(-d_n\right)=0\,,\ S_v-1_{\mathrm{IR}^-}\left(\lambda-\rho_n[[v_n]]\right)=0 \text{ and } S_f-1_{B(0,1)}\left(\boldsymbol{\Lambda}+\rho_\tau[[\mathbf{v}_{\boldsymbol{\tau}}]]\right)=0 \text{ on } \Gamma_c\times\mathrm{I} \tag{7.e}$$

where ρ_0^i denotes the mass density in the reference state, an over-dot refers to partial time derivative, $\boldsymbol{\Pi}^i$ is the first Piola-Kirchhoff stress tensor defined in Ω_0^i (given by a behaviour law) and where the initial conditions are known.

The above formulation may recall available Lagrange ones of frictional contact problems [11]. It is however an new one (to our best knowledge) since it is a *non augmented* and *non constrained* Lagrange formulation of dynamic contact problems, based on equalities.

NUMERICAL SOLUTION STRATEGY

To be solved, the problem (7) is discretized by means of three numerical tools:

i) A first-order finite time difference θ-scheme is used to approximate partial time derivatives.
ii) A Galerkin space discretisation method of the finite element type is used to approximate the continuous fields $\mathbf{v}^i$, (and $\mathbf{u}^i$), λ and $\boldsymbol{\Lambda}$ while respecting the compatibility condition between approximating spaces (e.g. [12], in the quasistatic case).
iii) The *Level-Sets* like fields S_u, S_v and S_f are approximated by means of a collocation method which consists in evaluating these fields only in a collection of selected points of the potential contact zone, corresponding to the contact numerical integration points.

The nonlinear systems are solved, at each time step, by fixed point strategies coupled with a Newton-Raphson method. The global algorithm is described by the following steps.

1. Loop over all time steps

$\mathbf{v}_k$, $\mathbf{u}_k$, λ_k and $\mathbf{\Lambda}_k$ being known, increment the time and calculate $\mathbf{v}_{k+1}$, $\mathbf{u}_{k+1}$, λ_{k+1} and $\mathbf{\Lambda}_{k+1}$ by:

2. A fixed point geometry loop

$(\overline{\mathbf{p}},\mathbf{n},\boldsymbol{\tau})= (\overline{\mathbf{p}},\mathbf{n},\boldsymbol{\tau})^g$

↺ if converged GOTO 1.

$\overline{\mathbf{p}}$: paired point to a potential contacting point $\mathbf{p}$
$(\mathbf{n},\boldsymbol{\tau})$: local frame at $\overline{\mathbf{p}}$

3. A fixed point friction threshold loop

$\lambda = \lambda^g$ (in $\mathbf{R}_{\boldsymbol{\tau}}$)

↺ if converged GOTO 2.

4. A fixed point *Level-Sets* values loop

$(S_u, S_v, S_f)= (S_u, S_v, S_f)^g$

↺ if converged GOTO 3.

5. The Newton-Raphson method

↺ if converged GOTO 4.

Fig. 1 Global algorithm

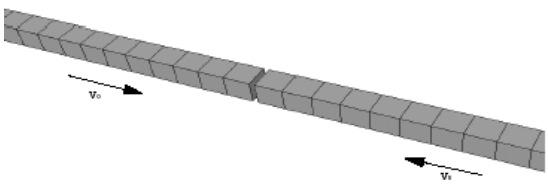
In this algorithm, k is an index referring to the time step and g indicates that the unknown fields are frozen (actually they are updated iteratively). The algorithm is stopped whenever all the loops reach convergence.

Remark: The time discretization scheme used here is of implicit type in order to approximate friction and elastoplastic phenomena with high accuracy. The reader is referred to [13,14], for explicit-like approaches.

NUMERICAL RESULTS

To show the effectiveness of our global approach, we consider two frictionless and one frictional contact-impact examples.

1. Impact of two identical rods We consider the classical test of impact of two elastic prismatic rods moving with equal speed in opposite directions (Fig. 2). The rods are initially undeformed and the problem is symmetric about the points at which the two rods experience impact. We mesh the two rods similarly with Hexa 3D-elements. The contact loads are approximated by a bilinear finite element space, defined at the contact interface. Two 3D-finite element solutions are plotted in figure 3, one with a dissipative Newmark scheme used for the time discretization of a classical displacement-based formulation and the other with the use of our continuous velocity-based formulation.



$E = 2\ 10^{11}$ Pa ,$\nu = 0.3$,ρ =7800kg.m^{-3} ,
V_0=10m.s^{-1}, $\Delta t = 10^{-5}$ s, θ=1,
l=1m , S=10^{-4}m^2, Initial gap= 1mm.

Fig. 2 Impact of two 3D elastic and identical rods

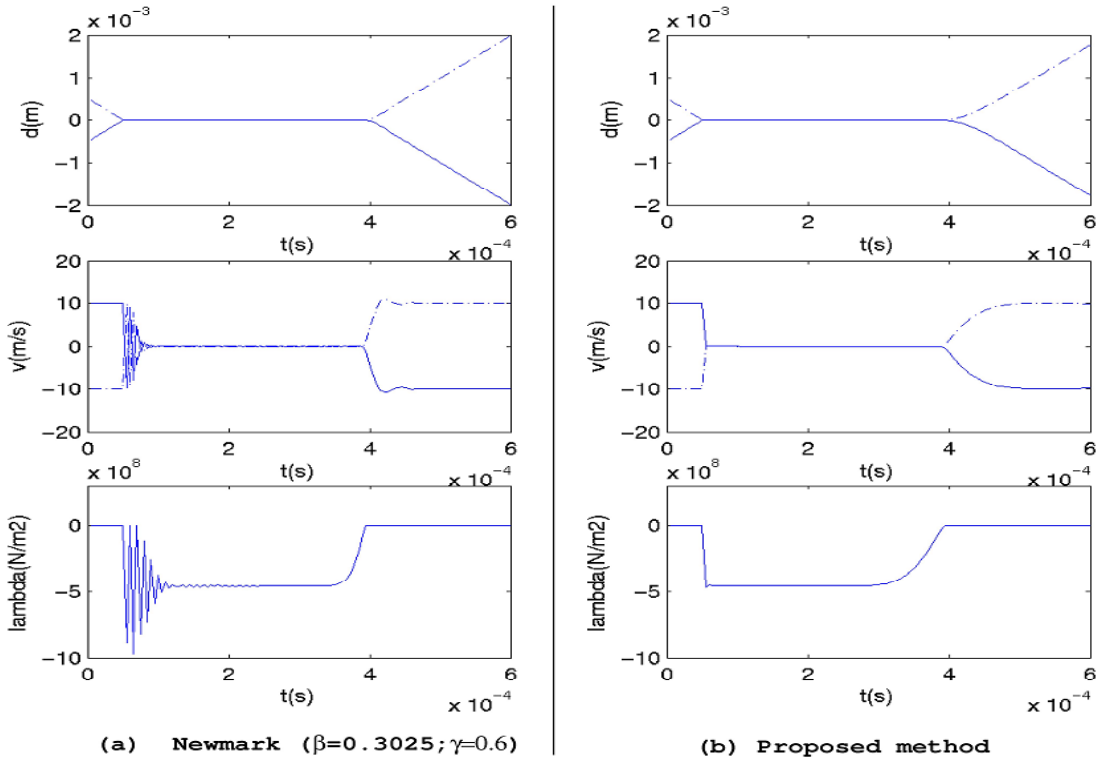


Fig. 3 Impact of two identical rods; histories of tip displacement, velocity and contact multiplier

2. Impact of a cylinder on a wall The second example concerns the impact of an elastic cylinder on a quasi-rigid wall under plane strain conditions. The geometric and the material properties are given in figure 4. The velocity and contact fields are approximated by bilinear finite 2D and 1D elements and the nodes of the potential slave surface (belonging to the cylinder) have been taken as the collocation points for the *Level-Sets* fields. In figure 5, we show the time history of the displacement, velocity and contact pression of the bottom contacting point of the cylinder obtained by both of a dissipative Newmark scheme (β=0.4, γ=0.7) (for a displacement-based formulation) and the proposed method.

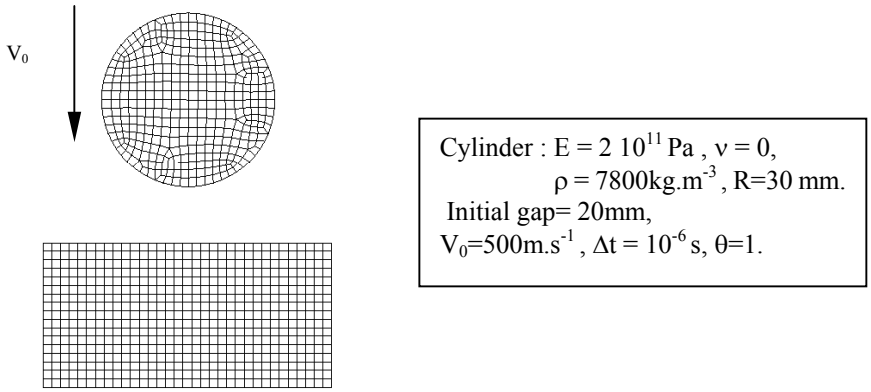


Fig. 4 Impact of an elastic cylinder on a wall

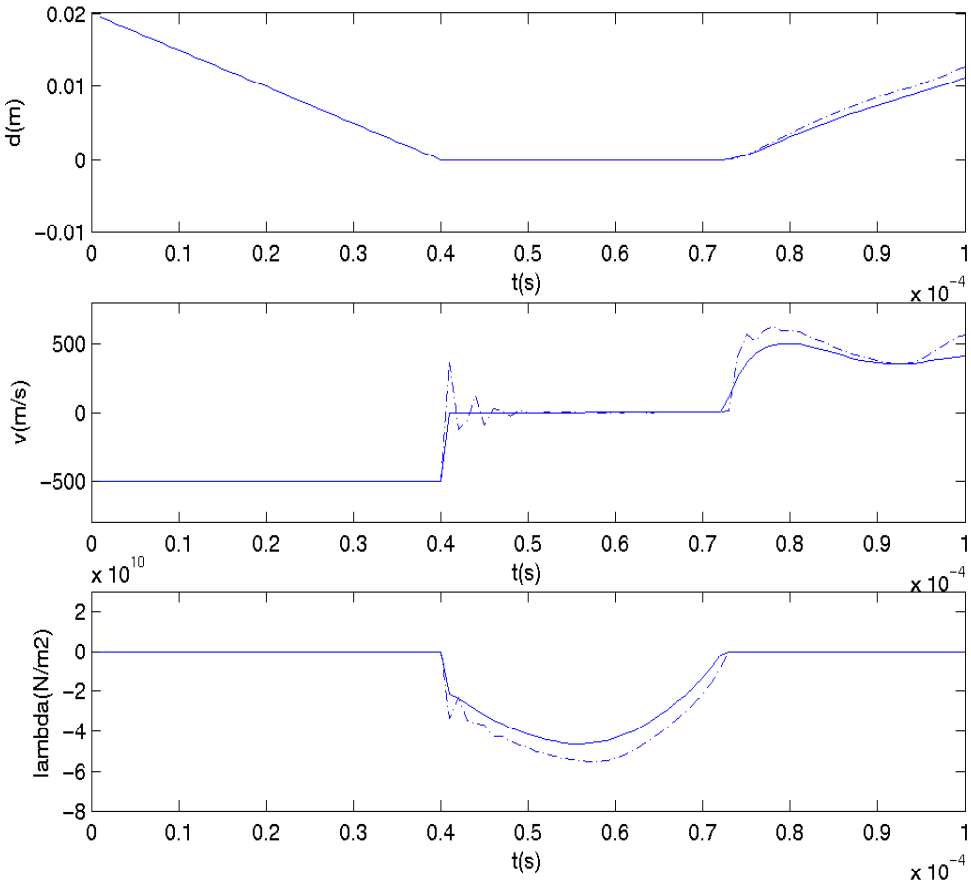


Fig. 5 Displacement, velocity and contact pression histories of the bottom contacting point of the cylinder

3. Elastoplastic impact of a plane rod This example involves the frictional elastoplastic impact of a plane rod ($E = 1.17\ 10^5$ MPa ,$\nu = 0.35$, $\rho = 8930 kg.m^{-3}$, $\sigma_Y = 400$MPa , $E_t = 100$MPa) against a quasi-rigid wall. The rod has an initial velocity of $227 m.s^{-1}$. The friction coefficient μ is equal to 0.25. Bilinear 2D and 1D finite elements are used to approximate unknown fields. Plane strain hypotheses are used. This test is treated by two approaches. The first one is based on the proposed method within the classical monomodel framework and the second is carried out within the multi-model Arlequin framework.

3.1 the monomodel framework We present in figure 6 both of computed deformed geometries and the equivalent plastic strain spread in the rod obtained by the application of the velocity based formulation in the classical modelling framework. The time step is equal to $2\ 10^{-7}$s. The histories of the displacement, velocity, contact pression and Lagrange semi-multiplier of friction at selected points of the impacting face of the rod are depicted in Figure 7. Notice that there are no numerical oscillations on the unknown fields such as velocities and contact pression. The values of the Lagrange semi-multiplier of friction indicate the states of the contacting points: $||\mathbf{\Lambda}|| \in]0,1[$ whenever a contacting point is sticking and $||\mathbf{\Lambda}||=1$ when it slides.

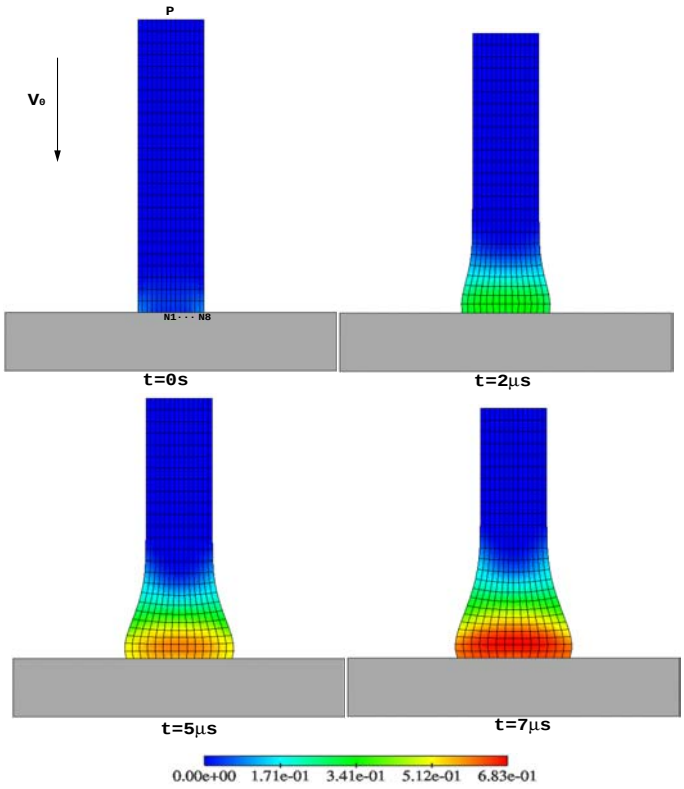


Fig. 6 Frictional elastoplastic impact of a bar: deformed geometries and equivalent plastic strain

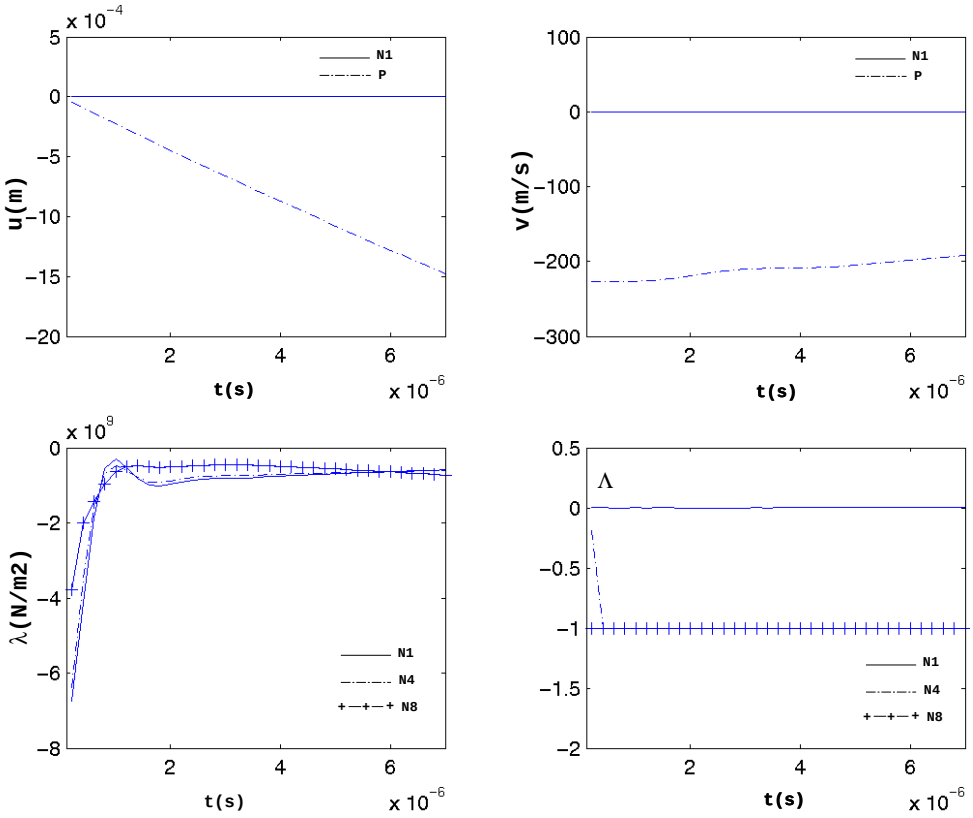


Fig. 7 Time history of displacement, velocity, contact pression and Lagrange semi-multiplier of friction at selected points

3.2 the Arlequin framework The previous mechanical test show that impact involves sharp localized variation of mechanical fields both in time and space. With coarse meshes and times steps, the numerical results can suffer precision. Adaptive methods have been designed to achieve the computations with a prescribed tolerance for the errors [15]. For a flexibility purpose we use here the multiscale approach named Arlequin [1]. This method of global-local type consists in superimposing a local refined model to the global coarse one. The coexistence of the two different models allows to use.

i) different formulations,
ii) different time integration schemes,
iii) different refinements in space and time.

In this framework, the impact problem can be written as given in appendix 1. It is applied here to indicate the way to solve the previous problem (under frictionless hypothesis). As shown by figure 8.b, a local refined model of the rod is superposed to a global coarse one. The first is elastoplastic and encompasses the contact friction phenomena. The second is elastic. With the same data as for the previous test, we give the deformed geometries and the equivalent plastic strain evolutions in the rod obtained within the refined monomodel (Fig. 8.a) and the *Arlequin* (Fig. 8.b) frameworks, the latter being only locally refined. One can notice the similarity of the obtained results. Notice that in both models, the classical rising of the contact edge is obtained (see figure 8.c)

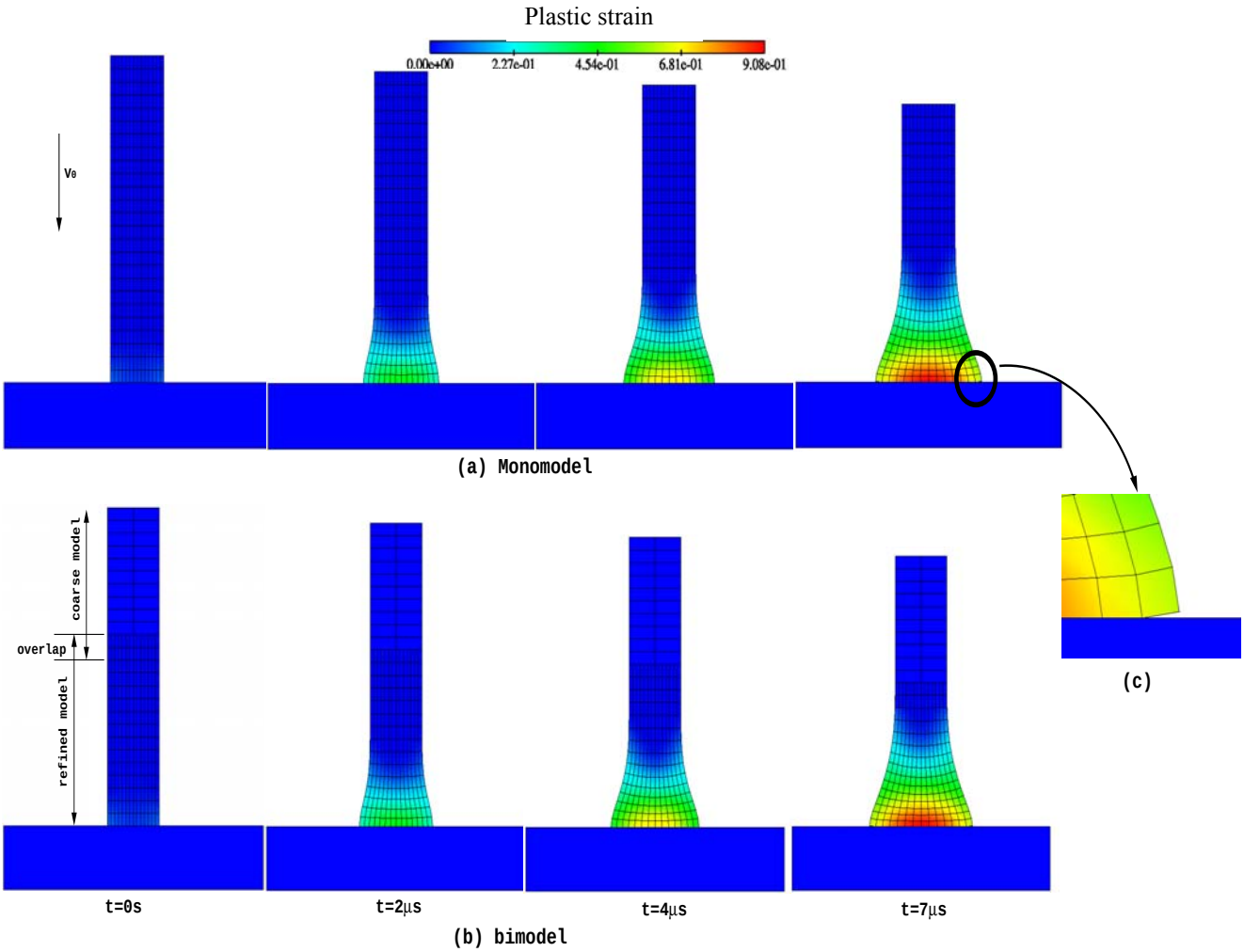


Fig. 8 Impact of a rod by the velocity and Level-Sets based formulation in (a) the monomodel framework (b) the Arlequin framework

CONCLUSION

A velocity and *Level-Sets* based continuous Lagrange weak-strong formulation of dynamic frictional has been developed in this paper. The continuous formulation is derived from an equivalent setting of the *Signorini-Moreau* conditions and the Coulomb friction laws by using unknown *Level-Sets* fields. The problem is discretized by means of time, space and collocation method and solved by fixed-point strategies mixed with the Newton-Raphson method. Numerical examples show the effectiveness of our approach particularly for the treatment of spurious numerical oscillations. First promising results using the multiscale *Arlequin* framework are given. A mixing of time integration schemes (explicit/implicit and/or dissipative/conservative) in this framework is now in progress.

APPENDIX 1

In this appendix, we show how the Arlequin method can be applied to dynamic contact problems. For this, we consider the following domain decomposition of the elastoplastic frictionless impact of a plane rod on a quasi-rigid wall (paragraph 3.2). The domain Ω_0^1 occupied by the rod is assumed to be partitioned into two overlapping domains Ω_{nc}^1 and Ω_c^1. The overlap is denoted by Ω_s^1. The Arlequin mixed dynamic contact problem reads (for each time t).

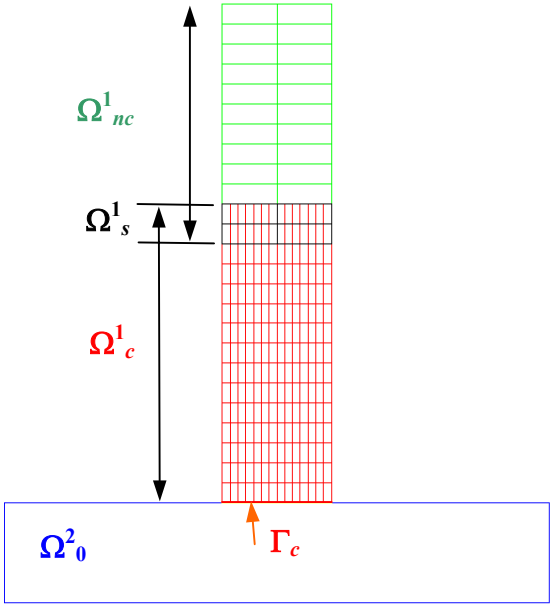


Fig. 9: Domain decomposition in the Arlequin framework

Find $(\mathbf{v}^1,\mathbf{u}^1,\mathbf{u}^2,\lambda,\mathcal{F}\,;\,S_u,S_v)$ such for all $(\mathbf{v}^{*1},\mathbf{w}^1,\mathbf{w}^2,\lambda^*,\mathcal{F}^*)$:

$$\left(m_{nc}^1(\ddot{\mathbf{u}}^1,\mathbf{w}^1)+m_c^1(\dot{\mathbf{v}}^1,\mathbf{v}^{*1})+m^2(\ddot{\mathbf{u}}^2,\mathbf{w}^2)\right)+\left(k_{nc}^1(\mathbf{u}^1,\mathbf{w}^1)+k_c^1(\mathbf{v}^1,\mathbf{v}^{*1})+k^2(\mathbf{u}^2,\mathbf{w}^2)\right) - \int_{\Gamma_c} S_u S_v \lambda[[v_n^*]]d\Gamma-\left(\mathcal{F},\mathbf{v}^{*1}-\mathbf{w}^1\right)=0 \tag{8.a}$$

$$-\frac{1}{h_n}\int_{\Gamma_c}\left(\lambda-S_uS_v\left(\lambda-h_n[[v_n]]\right)\right)\lambda^* d\Gamma=0 \tag{8.b}$$

$$\left(\mathbf{v}^1-\dot{\mathbf{u}}^1,\mathcal{F}^*\right)=0 \tag{8.c}$$

$$\mathbf{u}^1(t)=\mathbf{u}^1(t_0)+\int_{t0}^{t}\mathbf{v}^1(s)ds \quad \text{in}\,\Omega_c^1 \tag{8.d}$$

$$S_u-1_{\mathrm{IR}^-}\left(-d_n\right)=0 \text{ and } S_v-1_{\mathrm{IR}^-}\left(\lambda-\rho_n[[v_n]]\right)=0 \text{ on } \Gamma_c\times\mathrm{I} \tag{8.e}$$

where **(.,.)** is the duality bracket between the space of displacements defined in Ω_s^1 and its dual space.

$$m_{nc}^{1}(\ddot{\mathbf{u}}^1,\mathbf{w}^1)=\int_{\Omega_{nc}^1}\alpha_{nc}^1\rho_0^1\ddot{\mathbf{u}}^1.\mathbf{w}^1\ ;\ m_c^{1}(\dot{\mathbf{v}}^1,\mathbf{v}^{*1})=\int_{\Omega_c^1}\alpha_c^1\rho_0^1\dot{\mathbf{v}}^1.\mathbf{v}^{*1}\ \ ;\ m^{2}(\ddot{\mathbf{u}}^2,\mathbf{w}^2)=\int_{\Omega_0^2}\rho_0^2\ddot{\mathbf{u}}^2\mathbf{w}^2$$

$$k_{nc}^{1}(\mathbf{u}^1,\mathbf{w}^1)=\int_{\Omega_{nc}^1}\delta_{nc}^1\boldsymbol{\Pi}^1(\mathbf{u}^1):\varepsilon(\mathbf{w}^1)\ ;\ k_c^{1}(\mathbf{v}^1,\mathbf{v}^{*1})=\int_{\Omega_c^1}\delta_c^1\overline{\boldsymbol{\Pi}}^1(\mathbf{v}^1):\boldsymbol{\varepsilon}(\mathbf{v}^{*1})\ ;\ k^{2}(\mathbf{u}^2,\mathbf{w}^2)=\int_{\Omega_0^2}\boldsymbol{\Pi}^2(\mathbf{u}^2):\boldsymbol{\varepsilon}(\mathbf{w}^2)$$

$\alpha_{nc}^1,\alpha_c^1,\delta_{nc}^1$ and δ_c^1 are 4 positive parameters permitting the distribution of the kinetic and the strain energies between the two superposed models. They are defined as follows:

$$\alpha_{nc}^1+\alpha_c^1=1\ \text{ on }\Omega_{\mathrm{s}}^1\ ;\ \alpha_c^1=0\text{ on }\Omega_{\mathrm{nc}}^1\setminus\Omega_{\mathrm{s}}^1\ \text{ and }\ \alpha_{nc}^1=0\text{ on }\Omega_{\mathrm{c}}^1\setminus\Omega_{\mathrm{s}}^1$$

$$\delta_{nc}^1+\delta_c^1=1\ \text{ on }\Omega_{\mathrm{s}}^1\ ;\ \delta_c^1=0\text{ on }\Omega_{\mathrm{nc}}^1\setminus\Omega_{\mathrm{s}}^1\ \text{ and }\ \delta_{nc}^1=0\text{ on }\Omega_{\mathrm{c}}^1\setminus\Omega_{\mathrm{s}}^1$$

The gluing forces field is denoted by $\mathcal{F}$.

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REFERENCES

[1] Ben Dhia H., Multiscale mechanical problems: The Arlequin method. *C.R. Acad. Sc., Paris,* 326*, Ser. IIb*, (1998), 899-904.

[2] Hughes T.J.R. et al., A finite element method for a class of contact-impact problems. *Comp. Methods App. Mech. Eng.* 8, (1976), 249-276.

[3] Armero F., Petocz E., Formulation and analysis of conserving algorithms for frictionless dynamic contact/impact problems, *Comp. Methods Appl. Mech. Eng.* 185, (1998), 269-300.

[4] Laursen T.A., Love G.R., Improved implicit integrators for transient impact problems, *Int. J. Num. Methods Eng.* 53, (2002), 245-274.

[5] Vola D., Pratt E., Jean M., Raous M., Consistent time discretization for a dynamic frictional contact problem and complementary techniques, *Finite Element European Review* 7, (1998), 149-162.

[6] Moreau J.-J., In Nonsmooth Mechanics and application. *CISM courses and lectures*, 1-82. Springer-Verlag. Wien/New York 1988.

[7] Sethian J.A., *Level-Set* Methods evolving interfaces in geometry, fluid mechanics, computer vision, and materials science, *Cambridge University Press* (1996).

[8] Klarbring A., Large displacement frictional contact: a continuum framework for finite element discretization, *Euro. J. Mech. A/Solids* 14, n° 2, (1995), 237-253.

[9] Ben Dhia H., Modelling and solution by penalty duality method of unilateral contact problems. *Calcul des Structures et Intelligence Artificielle* 2, (1988), 1-18.

[10] Ben Dhia H., Vautier I., Zarroug M., Problèmes de contact frottant en grandes transformations: du continu au discret, *Finite Element European Review* 9, (2000), 243-261.

[11] Wrigger P., Computational Contact Mechanics, John Wiley & Sons (2002).

[12] Ben Dhia H., Zarroug M., Hybrid frictional contact particles-in elements, *Finite Element European Review* 9, (2002), 417-430.

[13] Hallquist J.O., Goudreau G.L., Benson D.J., Sliding interfaces with contact-impact in large-scale Lagrangian computations, *Comp. Methods Appl. Mech. Eng.* 51, (1985), 107-137.

[14] Belytschko T., Neal N.O., Contact-impact by the pinball algorithm with penalty and lagrangian methods, *Int. J. Num. Methods Eng.* 31, (1991), 547-572.

[15] Lee C.Y., Oden J.T., *A posteriori* error estimation of *h-p* finite element approximations of frictional contact problems, *Comp. Methods Appl. Mech. Eng.* 113, (1994), 11-45.