Chapter 1
Reliability estimation by advanced Monte Carlo simulation

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Abstract Monte Carlo Simulation (MCS) offers a powerful means for evaluating the reliability of a system, due to the modeling flexibility that it offers indifferently of the type and dimension of the problem. The method is based on the repeated sampling of realizations of system configurations, which however are seldom of failure so that a large number of realizations must be simulated in order to achieve an acceptable accuracy in the estimated failure probability, with costly large computing times. For this reason, techniques of efficient sampling of system failure realizations are of interest, in order to reduce the computational effort.

In this paper, the recently developed Subset Simulation (SS) and Line Sampling (LS) techniques are considered for improving the MCS efficiency in the estimation of system failure probability. The SS method is founded on the idea that a small failure probability can be expressed as a product of larger conditional probabilities of some intermediate events: with a proper choice of the intermediate events, the conditional probabilities can be made sufficiently large to allow accurate estimation with a small number of samples. The LS method employs lines instead of random points in order to probe the failure domain of interest. An "important direction" is determined, which points towards the failure domain of interest; the high-dimensional reliability problem is then reduced to a number of conditional one-dimensional problems which are solved along the "important direction".

The two methods are applied on two structural reliability models of literature, i.e. the cracked plate model and the Paris-Erdogan model for thermal fatigue crack growth. The efficiency of the proposed techniques is evaluated in comparison to other stochastic simulation methods of literature, i.e., standard MCS, Importance Sampling (IS), Dimensionality Reduction (DR) and Orthogonal Axis (OA).

1.1 Introduction

In the performance-based design and operation of modern engineered systems, the accurate assessment of reliability is of paramount importance, particularly for civil, nuclear, aerospace and chemical systems and plants which are safety-critical and must be designed and operated within a risk-informed approach (Thunnissen et al. 2007; Patalano et al. 2008).

The reliability assessment requires the realistic modelling of the structural/mechanical components of the system and the characterization of their material constitutive behaviour, loading conditions and mechanisms of deterioration and failure that are anticipated to occur during the working life of the system (Schueller and Pradlwarter 2007).

In practice, not all the characteristics of the system under analysis can be fully captured in the model. This is due to: i) the intrinsically random nature of several of the phenomena occurring during the system life; ii) the incomplete knowledge about some of these phenomena. Thus, uncertainty is always present in the hypotheses underpinning the model (model uncertainty) and in the values of its parameters (parameter uncertainty); this leads to uncertainty in the model output, which must be quantified for a realistic assessment of the system (Nutt and Wallis, 2004).

In mathematical terms, the probability of system failure can be expressed as a multi-dimensional integral of the form

$$P(F) = P(\boldsymbol{x} \in F) = \int I_F(\boldsymbol{x}) q(\boldsymbol{x}) d\boldsymbol{x} \tag{1.1}$$

where $\boldsymbol{x} = \{x_1, x_2, ..., x_j, ..., x_n\} \in \Re^n$ is the vector of the uncertain input parameters/variables of the model, with multidimensional probability density function (PDF) $q : \Re^n \to [0, \infty)$, $F \subset \Re^n$ is the failure region and $I_F : \Re^n \to \{0,1\}$ is an indicator function such that $I_F(\boldsymbol{x}) = 1$, if $\boldsymbol{x} \in F$ and $I_F(\boldsymbol{x}) = 0$, otherwise. The failure domain *F* is commonly defined by a so-called Performance Function (PF) or Limit State Function (LSF) $g_x(\boldsymbol{x})$ which is lower than or equal to zero if $\boldsymbol{x} \in F$ and greater than zero, otherwise.

In practical cases, the multi-dimensional integral (1.1) can not be easily evaluated by analytical methods nor by numerical schemes. On the other hand, Monte Carlo Simulation (MCS) offers an effective means for estimating the integral, because the method does not suffer from the complexity and dimension of the domain of integration, albeit it implies the nontrivial task of sampling from the multidimensional PDF. The MCS solution to (1.1) entails that a large number of samples of the values of the uncertain parameters $\boldsymbol{x}$ be drawn from $q(\boldsymbol{x})$ and that these be used to compute an unbiased and consistent estimate of the system failure probability as the fraction of the number of samples that lead to failure. However,

a large number of samples (inversely proportional to the failure probability) is necessary to achieve an acceptable estimation accuracy: in terms of the integral in (1.1) this can be seen as due to the high dimensionality n of the problem and the large dimension of the relative sample space compared to the failure region of interest (Schueller 2007). This calls for new simulation techniques for performing robust estimations with a limited number of input samples (and associated low computational time).

In this respect, effective approaches are offered by Subset Simulation (SS) (Au and Beck 2001; Au and Beck 2003b) and Line Sampling (LS) (Koutsourelakis et al. 2004; Pradlwarter et al. 2005).

In the SS method, the failure probability is expressed as a product of conditional failure probabilities of some chosen intermediate events, whose evaluation is obtained by simulation of more frequent events. The evaluation of small failure probabilities in the original probability space is thus tackled by a sequence of simulations of more frequent events in the conditional probability spaces. The necessary conditional samples are generated through successive Markov Chain Monte Carlo (MCMC) simulations (Metropolis et al. 1953; Hastings 1970; Fishman 1996), gradually populating the intermediate conditional regions until the final target failure region is reached.

In the LS method, *lines*, instead of random *points*, are used to probe the failure domain of the high-dimensional problem under analysis (Pradlwarter et al. 2005). An "important direction" is optimally determined to point towards the failure domain of interest and a number of conditional, one-dimensional problems are solved along such direction, in place of the high-dimensional problem (Pradlwarter et al. 2005). The approach has been shown to perform always better than standard MCS; furthermore, if the boundaries of the failure domain of interest are not too rough (i.e., almost linear) and the "important direction" is almost perpendicular to them, the variance of the failure probability estimator could be ideally reduced to zero (Koutsourelakis et al. 2004).

In this Chapter, SS and LS schemes are developed for application to two structural reliability models of literature, i.e., the cracked plate model (Ardillon and Venturini 1995) and the Paris-Erdogan thermal fatigue crack growth model (Paris 1961). The problem is rather challenging as it entails estimating failure probabilities of the order of 10^{-7}. The effectiveness of SS and LS is compared to that of other simulation methods, e.g. the Importance Sampling (IS), Dimensionality Reduction (DR) and Orthogonal Axis (OA) methods (Gille 1998 and 1999). In the IS method, the PDF $q(\boldsymbol{x})$ in (1.1) is replaced with an Importance Sampling Distribution (ISD) arbitrarily chosen so as to generate samples that lead to failure more frequently (Au and Beck 2003a); in the DR method, the failure event is re-expressed in such a way as to highlight one important variable (say, x_j) and the failure probability is then computed as the expected value of the Cumulative Distribution Function (CDF) of x_j conditional on the remaining $(n - 1)$ variables; fi-

nally, in the OA method, a sort of importance sampling is performed around the most likely point in the failure domain (Gille 1998 and 1999).

The remainder of the Chapter is organized as follows. In Section 1.2, a general presentation of the SS and LS schemes implemented for this study is given. In Section 1.3, the IS, DR and OA methods taken as terms of comparison are briefly summarized. The results of the application of SS and LS to the cracked plate and thermal fatigue crack growth models are reported in Sections 1.4 and 1.5, respectively. Based on the results obtained, a critical discussion of the simulation techniques adopted and compared in this work is offered in the last Section. For completeness of the contents of the paper, detailed descriptions of the Markov Chain Monte Carlo (MCMC) simulation method used for the development of the SS and LS algorithms are provided in Appendices 1 and 2, respectively.

1.2 Simulation methods implemented in this study

1.2.1 The Subset Simulation method

Subset Simulation (SS) is an adaptive stochastic simulation method originally developed for efficiently computing small failure probabilities in structural reliability analysis (Au and Beck 2001). The underlying idea is to express the (small) failure probability as a product of (larger) probabilities conditional on some intermediate events. This allows converting a rare event simulation into a sequence of simulations of more frequent events. During simulation, the conditional samples are generated by means of a Markov chain designed so that the limiting stationary distribution is the target conditional distribution of some adaptively chosen event; by so doing, the conditional samples gradually populate the successive intermediate regions up to the final target (rare) failure region (Au and Beck 2003b).

1.2.1.1 The basic principles

For a given target failure event F of interest, let $F_1 \supset F_2 \supset \ldots \supset F_m = F$ be a sequence of intermediate events, so that $F_k = \cap_{i=1}^{k} F_i$, $k = 1, 2, \ldots, m$. By sequentially conditioning on the event F_i, the failure probability $P(F)$ can be written as

$$P(F) = P(F_m) = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1} \mid F_i) \tag{1.2}$$

Notice that even if $P(F)$ is small, the conditional probabilities involved in (1.2) can be made sufficiently large by appropriately choosing m and the intermediate events $\{F_i, i = 1, 2, \ldots, m-1\}$.

The original idea of SS is to estimate the failure probability $P(F)$ by estimating $P(F_1)$ and $\{P(F_{i+1} \mid F_i) : i = 1, 2, \ldots, m-1\}$. Considering for example $P(F) \approx 10^{-5}$ and choosing $m = 4$ intermediate events such that $P(F_1)$ and $\{P(F_{i+1} \mid F_i) : i = 1, 2, 3, 4\} \approx 0.1$, the conditional probabilities can be evaluated efficiently by simulation of the relatively frequent intermediate events (Au and Beck 2001).

Standard MCS can be used to estimate $P(F_1)$. On the contrary, computing the conditional probabilities in (1.2) by MCS entails the non-trivial task of sampling from the conditional distributions of $\boldsymbol{x}$ given that it lies in F_i, $i = 1, 2, \ldots, m-1$, i.e. from $q(\boldsymbol{x} \mid F_i) = q(\boldsymbol{x}) I_{F_i}(\boldsymbol{x}) / P(F)$. In this regard, Markov Chain Monte Carlo (MCMC) simulation provides a powerful method for generating samples condi-

tional on the intermediate regions F_i, i = 1, 2, ..., m – 1 (Au and Beck 2001; Au and Beck 2003b). For completeness of the paper, the related algorithm is presented in Appendix 1.

1.2.1.2 The algorithm

In the actual SS implementation, with no loss of generality it is assumed that the failure event of interest can be defined in terms of the value of a critical system response variable Y being lower than a specified threshold level y, i.e., $F = \{Y < y\}$. The sequence of intermediate events $\{F_i : i = 1, 2, ..., m\}$ can then be correspondingly defined as $F_i = \{Y < y_i\}$, $i = 1, 2, ..., m$ where $y_1 > y_2 > ... > y_i > ... > y_m = y > 0$ is a decreasing sequence of intermediate threshold values (Au and Beck 2001; Au and Beck 2003b).

The choice of the sequence $\{y_i : i = 1, 2, ..., m\}$ affects the values of the conditional probabilities $\{P(F_{i+1} \mid F_i) : i = 1, 2, ..., m-1\}$ in (1.2) and hence the efficiency of the SS procedure. In particular, choosing the sequence $\{y_i : i = 1, 2, ..., m\}$ a priori makes it difficult to control the values of the conditional probabilities $\{P(F_{i+1} \mid F_i) : i = 1, 2, ..., m-1\}$. For this reason, in this work, the intermediate threshold values are chosen adaptively in such a way that the estimated conditional probabilities are equal to a fixed value p_0 (Au and Beck 2001; Au and Beck 2003b).

Schematically, the SS algorithm proceeds as follows (Figure 1.1):

1. Sample N vectors $\{\boldsymbol{x}_0^k : k = 1, 2, ..., N\}$ by standard MCS, i.e., from the original probability density function $q(\cdot)$. The subscript '0' denotes the fact that these samples correspond to 'Conditional Level 0';
2. Set i = 0;
3. Compute the values of the response variable $\{Y(\boldsymbol{x}_i^k) : k = 1, 2, ..., N\}$;
4. Choose the intermediate threshold value y_{i+1} as the $(1 - p_0)N^{\text{th}}$ value in the decreasing list of values $\{Y(\boldsymbol{x}_i^k) : k = 1, 2, ..., N\}$ (computed at step 3. above) to define $F_{i+1} = \{Y < y_{i+1}\}$. By so doing, the sample estimate of $P(F_{i+1}|F_i) = P(Y < y_{i+1}|Y < y_i)$ is equal to p_0 (note that it has been implicitly assumed that p_0N is an integer value);
5. If y_{i+1} y_m, proceed to 10. below;
6. Viceversa, i.e. if $y_{i+1} > y_m$, with the choice of y_{i+1} performed at step 4. above, identify the p_0N samples $\{\boldsymbol{x}_i^u : u = 1, 2, ..., p_0N\}$ among $\{\boldsymbol{x}_i^k : k = 1, 2, ..., N\}$ whose response Y lies in $F_{i+1} = \{Y < y_{i+1}\}$: these samples are at 'Conditional level i + 1' and distributed as $q(\cdot \mid F_{i+1})$ and function as seeds of the MCMC simulation (step 7. below);

7. Starting from each one of the samples $\{x_i^u : u = 1,2,...,p_0 N\}$ (identified at step 6. above), use MCMC simulation to generate $(1 - p_0)N$ additional conditional samples distributed as $q(\cdot \mid F_{i+1})$, so that there are a total of N conditional samples $\{x_{i+1}^k : k = 1,2,...,N\} \in F_{i+1}$, at 'Conditional level $i + 1$';
8. Set $i \leftarrow i + 1$;
9. Return to step 3. above;
10. Stop the algorithm.

For clarity sake, a step-by-step illustration of the procedure for Conditional levels 0 and 1 is provided in Figure 1.2 by way of example.

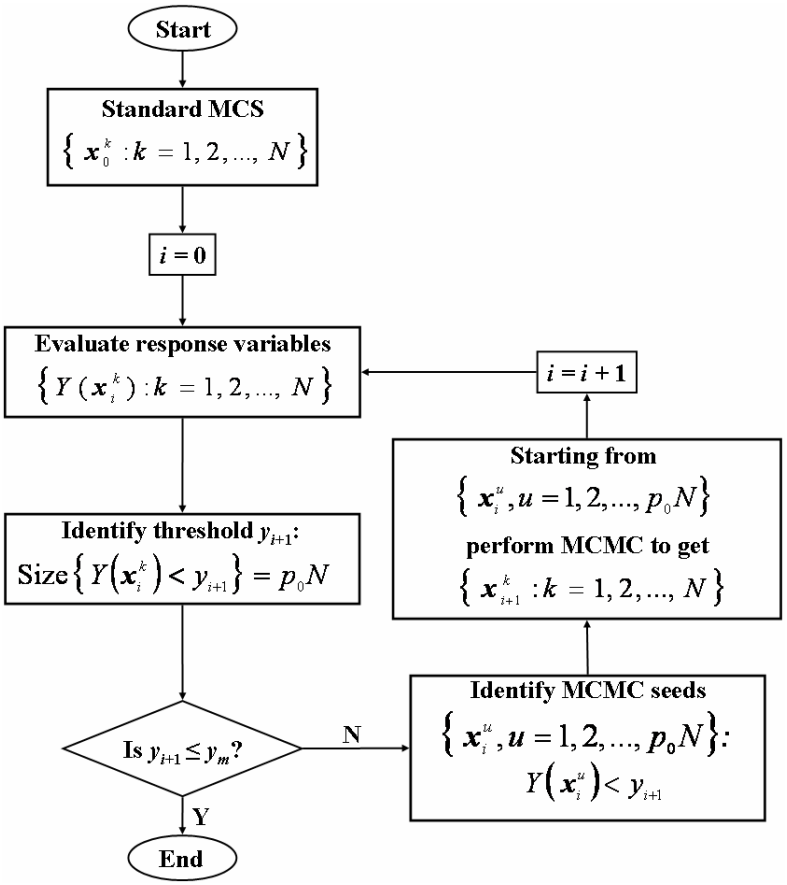


Fig. 1.1. Sketch of the SS algorithm

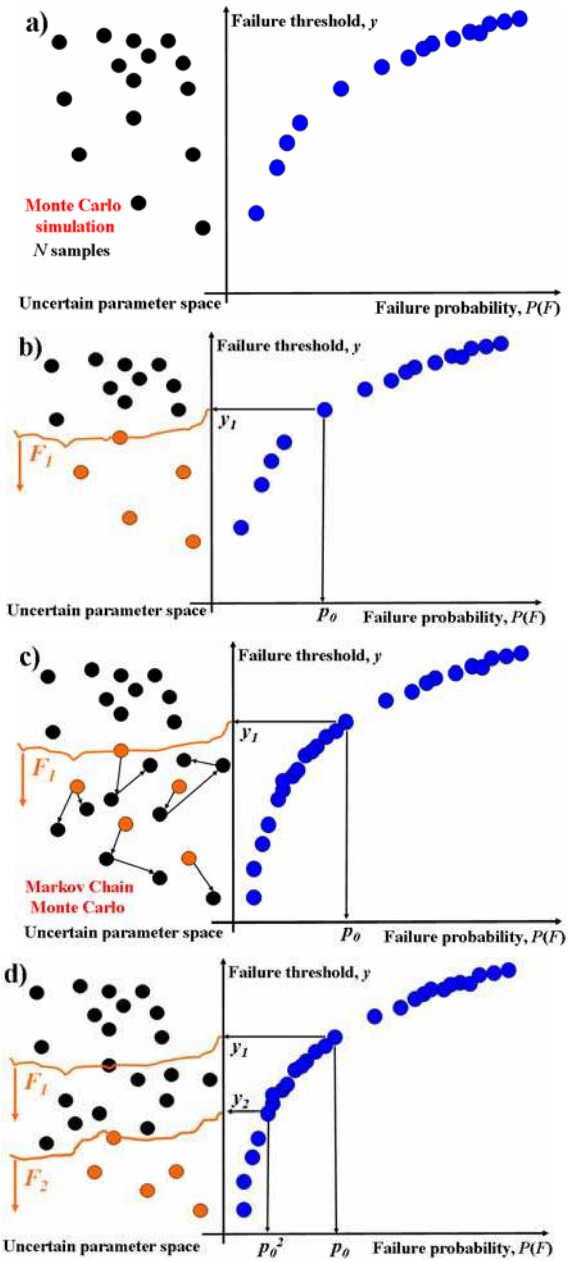


Fig. 1.2. Illustration of the SS procedure: a) Conditional level 0: Standard Monte Carlo simulation; b) Conditional level 0: adaptive selection of y_1; c) Conditional level 1: Markov Chain Monte Carlo simulation; d) Conditional level 1: adaptive selection of y_2 (Au 2005)

Notice that the procedure is such that the response values $\{y_i : i = 1, 2, ..., m\}$ at the specified probability levels $P(F_1) = p_0$, $P(F_2) = p(F_2 \mid F_1)P(F_1) = p_0^2$, …, $P(F_m) = p_0^m$ are estimated, rather than the event probabilities $P(F_1)$, $P(F_2 \mid F_1)$, …, $P(F_m \mid F_{m-1})$, which are a priori fixed at p_0. In this view, SS is a method for generating samples whose response values correspond to specified probability levels, rather than for estimating probabilities of specified failure events. As a result, it produces information about $P(Y < y)$ versus y at all the simulated values of Y rather than at a single value of y. This feature is important because the whole trend of $P(Y < y)$ versus y provides much more information than a point estimate (Au 2005).

1.2.2 The Line Sampling method

Line Sampling (LS) was also originally developed for the reliability analysis of complex structural systems with small failure probabilities (Koutsourelakis et al. 2004). The underlying idea is to employ lines instead of random points in order to probe the failure domain of the high-dimensional system under analysis (Pradlwarter et al. 2005).

In extreme synthesis, the problem of computing the multidimensional failure probability integral (1.1) in the original "physical" space is transformed into the so-called "standard normal space", where each random variable is represented by an independent central unit Gaussian distribution. In this space, a *unit* vector $\boldsymbol{\alpha}$ (hereafter also called "important unit vector" or "important direction") is determined, pointing towards the failure domain F of interest (for illustration purposes, two plausible important unit vectors, $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2$, pointing towards two different failure domains, F^1 and F^2, are visually represented in Figure 1.3, left and right, respectively, in a two-dimensional uncertain parameter space). The problem of computing the high-dimensional failure probability integral (1.1) is then reduced to a number of conditional one-dimensional problems, which are solved along the "important direction" $\boldsymbol{\alpha}$ in the standard normal space. The conditional one-dimensional failure probabilities (associated to the conditional one-dimensional problems) are readily computed by using the standard normal cumulative distribution function (Pradlwarter et al. 2005).

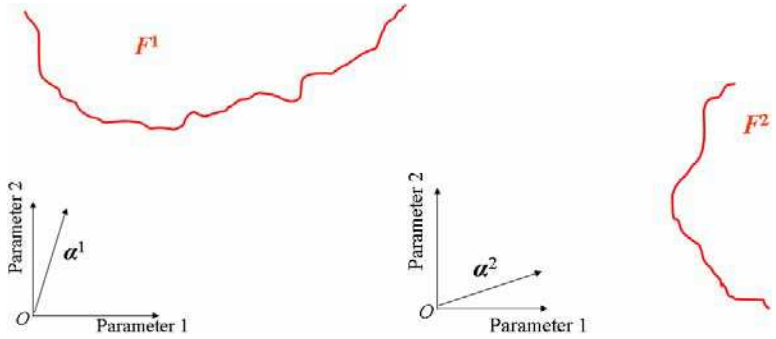


Fig. 1.3. Examples of possible important unit vectors $\boldsymbol{\alpha}^1$ (left) and $\boldsymbol{\alpha}^2$ (right) pointing towards the corresponding failure domains F^1 (left) and F^2 (right) in a two-dimensional uncertain parameter space

1.2.2.1 Transformation of the physical space into the standard normal space

Let $\boldsymbol{x} = \{x_1, x_2, ..., x_j, ..., x_n\} \in \Re^n$ be the vector of uncertain parameters defined in the original physical space $\boldsymbol{x} \in \Re^n$. For problems where the dimension n is not so small, the parameter vector $\boldsymbol{x}$ can be transformed into the vector $\boldsymbol{\theta} \in \Re^n$, where each element of the vector θ_j, $j = 1, 2, \ldots, n$, is associated with a central unit Gaussian standard distribution (Schueller et al. 2004). The joint probability density function of the random parameters $\{\theta_j : j = 1, 2, ..., n\}$ is, then:

$$\varphi(\boldsymbol{\theta}) = \prod_{j=1}^{n} \phi_j(\theta_j) \tag{1.3}$$

where $\phi_j(\theta_j) = \left(1/\sqrt{2\pi}\right) e^{-\theta_j^2/2}$, $j = 1, 2, ..., n$.

The mapping from the original, physical vector of random variables $\boldsymbol{x} \in \Re^n$ to the standard normal vector $\boldsymbol{\theta} \in \Re^n$ is denoted by $T_{x\theta}(\cdot)$ and its inverse by $T_{\theta x}(\cdot)$, i.e.:

$$\boldsymbol{\theta} = T_{x\theta}(\boldsymbol{x}) \tag{1.4}$$

$$\boldsymbol{x} = T_{\theta x}(\boldsymbol{\theta}) \tag{1.5}$$

Transformations (1.4) and (1.5) are in general nonlinear and are obtained by applying Rosenblatt's or Nataf's transformations, respectively (Rosenblatt 1952;

Nataf 1962; Huang and Du 2006). They are linear only if the random vector $\boldsymbol{x}$ is jointly Gaussian distributed. By transformation (1.4), also the Performance Function (PF) or Limit State Function (LSF) $g_x(\cdot)$ defined in the physical space (Section 1) can be transformed into $g_\theta(\cdot)$ in the standard normal space:

$$g_\theta(\boldsymbol{\theta}) = g_x(\boldsymbol{x}) = g_x(T_{\theta x}(\boldsymbol{\theta})) \tag{1.6}$$

Since in most cases of practical interest the function $g_\theta(\boldsymbol{\theta})$ is not known analytically, it can be evaluated only point-wise. According to (1.6), the evaluation of the system performance function $g_\theta(\cdot)$ at a given point $\boldsymbol{\theta}^k$, k = 1, 2, ..., N_T, in the standard normal space requires i) a transformation into the original space, ii) a complete simulation of the system response and iii) the computation of the system response from the model. The computational cost of evaluating the failure probability is governed by the number of system performance analyses that have to be carried out (Schueller et al. 2004).

1.2.2.2 The important direction α for Line Sampling

Three methods have been proposed to estimate the important direction $\boldsymbol{\alpha}$ for Line Sampling. In (Koutsourelakis et al. 2004), the important unit vector $\boldsymbol{\alpha}$ is taken as pointing in the direction of the "design point" in the standard normal space. According to a geometrical interpretation, the "design point" is defined as the vector point $\boldsymbol{\theta}^*$ on the limit state surface $g_\theta(\boldsymbol{\theta}) = 0$ which is closest to the origin in the standard normal space (Schueller et al. 2004). It can be demonstrated that $\boldsymbol{\theta}^*$ is also the point of maximum likelihood (Freudenthal 1956; Schueller and Stix 1987). Then, the unit important vector $\boldsymbol{\alpha}$ can be easily obtained by normalizing $\boldsymbol{\theta}^*$, i.e., $\boldsymbol{\alpha} = \boldsymbol{\theta}^* / \|\boldsymbol{\theta}^*\|_2$, where $\|\cdot\|_2$ denotes the usual Euclidean measure of a vector.

However, the design points, and their neighborhood, do not always represent the most important regions of the failure domain, especially in high-dimensional spaces (Schueller et al. 2004). Moreover, the computational cost associated with the calculation of the design point can be quite high, in particular if long-running numerical codes are required to simulate the response of the system to its uncertain input parameters (Schueller et al. 2004), as it is frequently the case in structural reliability.

In (Pradlwarter et al. 2005), the direction of $\boldsymbol{\alpha}$ is taken as the normalized gradient of the performance function in the standard normal space. Since the unit vector $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, ..., \alpha_j, ..., \alpha_n\}$ points towards the failure domain F, it can be used to draw information about the relative importance of the random parameters $\{\theta_j : j = 1, 2, ..., n\}$ with respect to the failure probability $P(F)$: the more relevant a random variable in determining the failure of the system, the larger the corre-

sponding component of the unit vector $\boldsymbol{\alpha}$ will be (Pradlwarter et al. 2005). Such quantitative information is obtained from the gradient of the performance function $g_\theta(\boldsymbol{\theta})$ in the standard normal space, $\nabla g_\theta(\boldsymbol{\theta})$:

$$\nabla g_\theta(\boldsymbol{\theta}) = \left[\frac{\partial g_\theta(\boldsymbol{\theta})}{\partial \theta_1} \frac{\partial g_\theta(\boldsymbol{\theta})}{\partial \theta_2} \cdots \frac{\partial g_\theta(\boldsymbol{\theta})}{\partial \theta_j} \cdots \frac{\partial g_\theta(\boldsymbol{\theta})}{\partial \theta_n} \right]^T \tag{1.7}$$

The gradient (1.7) measures in a unique way the relative importance of a particular random variable with respect to the failure probability $P(F)$: the larger the (absolute) value of a component of (1.7), the greater the "impact" of the corresponding random variable on the performance function $g_\theta(\boldsymbol{\theta})$ in the standard normal space. In other words, given a specified finite variation $\Delta\boldsymbol{\theta}$ in the parameter vector $\boldsymbol{\theta}$, the performance function $g_\theta(\boldsymbol{\theta})$ will change most if this variation is taken in the direction of (1.7). Thus, it is reasonable to identify the LS important direction with the direction of the gradient (1.7) and compute the corresponding unit vector $\boldsymbol{\alpha}$ as the *normalized* gradient of the performance function $g_\theta(\cdot)$ in the standard normal space, i.e. $\boldsymbol{\alpha} = \nabla g_\theta(\boldsymbol{\theta}) / \left\| \nabla g_\theta(\boldsymbol{\theta}) \right\|_2$ (Pradlwarter et al. 2005).

On the other hand, when the performance function is defined on a high-dimensional space, i.e. when many parameters of the system under analysis are random, the computation of the gradient $\nabla g_\theta(\boldsymbol{\theta})$ in (1.7) becomes a numerically challenging task. Actually, as the function $g_\theta(\boldsymbol{\theta})$ is known only implicitly through the response of a numerical code, for a given vector $\boldsymbol{\theta} = \{\theta_1, \theta_2, ..., \theta_j, ..., \theta_n\}$ at least n system performance analyses are required to determine accurately the gradient at a given point of the performance function $g_\theta(\cdot)$ by straightforward numerical differentiation, e.g. the secant method (Ahammed and Melchers 2006; Fu 2006).

Finally, the important unit vector $\boldsymbol{\alpha}$ can also be computed as the normalized "center of mass" of the failure domain F of interest (Koutsourelakis et al. 2004). A point $\boldsymbol{\theta}^0$ is taken in the failure domain F. This can be done by traditional Monte Carlo sampling or by engineering judgment when possible. Subsequently, $\boldsymbol{\theta}^0$ is used as the initial point of a Markov chain which lies entirely in the failure domain F. For that purpose a MCMC Metropolis-Hastings algorithm is employed to generate a sequence of N_s points $\{\boldsymbol{\theta}^u : u = 1, 2, ..., N_s\}$ lying in the failure domain F (Metropolis et al. 1956). The unit vectors $\boldsymbol{\theta}^u / \left\| \boldsymbol{\theta}^u \right\|_2$, u = 1, 2, …, N_s, are then averaged in order to obtain the LS important unit vector as $\boldsymbol{\alpha} = \frac{1}{N_s} \cdot \sum_{u=1}^{N_s} \boldsymbol{\theta}^u / \left\| \boldsymbol{\theta}^u \right\|_2$ (Figure 1.4). This direction is not optimal, but it is clear that it provides a good approximation of the important regions of the failure domain (at least as the sample

size N_s is large). On the other hand, it should be noticed that the procedure implies N_s additional system analyses by the deterministic model simulating the system, which substantially increase the computational cost associated to the simulation method.

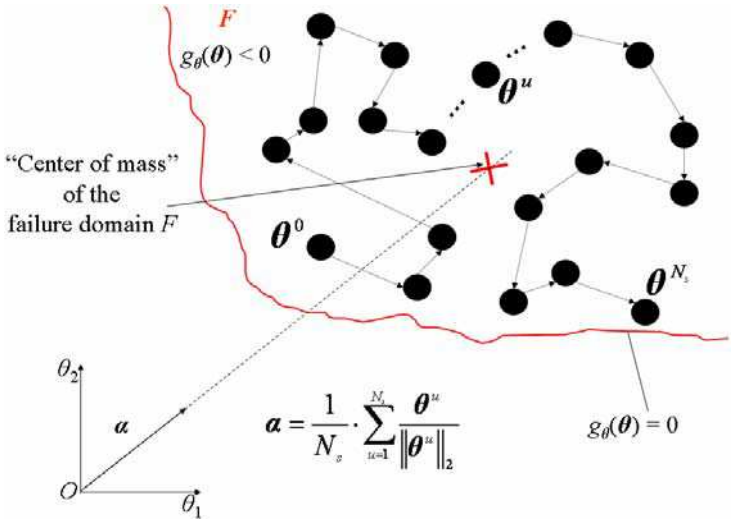


Fig. 1.4. Line Sampling important unit vector $\boldsymbol{\alpha}$ taken as the normalized "center of mass" of the failure domain F in the standard normal space. The "center of mass" of F is computed as an average of N_s failure points generated by means of a Markov chain starting from an initial failure point $\boldsymbol{\theta}^0$ (Koutsourelakis et al. 2004)

In the implementation of LS for this work, the method based on the normalized "center of mass" of the failure domain F has been employed, because it relies on a "map" approximating the failure domain F under analysis (given by the failure samples generated through a Markov chain) and thus it provides in principle the most realistic and reliable estimate for the LS important direction $\boldsymbol{\alpha}$.

For completeness, a thorough description of the Line Sampling algorithm and its practical implementation issues is given in Appendix 2 at the end of the paper.

1.3 Simulation methods considered for comparison

The performances of Subset Simulation (Section 1.2.1) and Line Sampling (Section 1.2.2) will be compared to those of the Importance Sampling (IS) (Section 1.3.1), Dimensionality Reduction (DR) (Section 1.3.2) and Orthogonal Axis (OA) (Section 1.3.3) methods; the comparison will be made with respect to the results reported in (Gille 1998 and 1999) for the two literature case studies considered, of the cracked plate and thermal fatigue crack growth models.

1.3.1 The Importance Sampling method

The concept underlying the Importance Sampling (IS) method is to replace the original PDF $q(\boldsymbol{x})$ with an Importance Sampling Distribution (ISD) $\tilde{q}(\boldsymbol{x})$ arbitrarily chosen by the analyst so as to generate a large number of samples in the "important region" of the sample space, i.e. the failure region F (Au and Beck 2003a; Schueller et al. 2004).

The IS algorithm proceeds as follows (Schueller et al. 2004):

1. Identify a proper Importance Sampling Density (ISD), $\tilde{q}(\cdot)$, in order to increase the probability of occurrence of the failure samples.
2. Express the failure probability $P(F)$ in (1.1) as a function of the ISD $\tilde{q}(\cdot)$:

$$\begin{aligned} P(F) &= \int I_F(\boldsymbol{x}) q(\boldsymbol{x}) d\boldsymbol{x} \\ &= \int \left[\frac{I_F(\boldsymbol{x}) q(\boldsymbol{x})}{\tilde{q}(\boldsymbol{x})} \right] \tilde{q}(\boldsymbol{x}) d\boldsymbol{x} \\ &= E_{\tilde{q}} \left[\frac{I_F(\boldsymbol{x}) q(\boldsymbol{x})}{\tilde{q}(\boldsymbol{x})} \right] \end{aligned} \tag{1.8}$$

3. Draw N_T independent and identically distributed (i.i.d.) samples $\{\boldsymbol{x}^k : k = 1, 2, ..., N_T\}$ from the ISD $\tilde{q}(\cdot)$; if a good choice for the ISD $\tilde{q}(\cdot)$ has been made, the samples $\{\boldsymbol{x}^k : k = 1, 2, ..., N_T\}$ should be concentrated in the failure region F of interest.
4. Compute an estimate $\hat{P}(F)$ for the failure probability $P(F)$ in (1.1) by resorting to the last expression in (1.8):

$$\hat{P}(F) = \frac{1}{N_T} \sum_{k=1}^{N_T} \frac{I_F(\boldsymbol{x}^k) q(\boldsymbol{x}^k)}{\tilde{q}(\boldsymbol{x}^k)} \tag{1.9}$$

5. The variance $V\left[\hat{P}(F)\right]$ of the estimator $\hat{P}(F)$ in (1.9) is given by

$$\begin{aligned} V\left[\hat{P}(F)\right] &= \frac{1}{N_T} V_{\tilde{q}}\left[\frac{I_F(\boldsymbol{x}) q(\boldsymbol{x})}{\tilde{q}(\boldsymbol{x})}\right] \\ &= \frac{1}{N_T}\left(\int \frac{I_F(\boldsymbol{x}) q^2(\boldsymbol{x})}{\tilde{q}^2(\boldsymbol{x})} \tilde{q}(\boldsymbol{x}) d\boldsymbol{x} - P(F)^2\right) \end{aligned} \tag{1.10}$$

It is straightforward to verify that the quantity (1.10) becomes zero when

$$\tilde{q}(\boldsymbol{x}) = \tilde{q}_{opt}(\boldsymbol{x}) = \frac{I_F(\boldsymbol{x}) q(\boldsymbol{x})}{P(F)} \tag{1.11}$$

This represents the optimal choice for the importance sampling density which is practically unfeasible since it requires the a priori knowledge of $P(F)$. Several techniques have been developed in order to approximate the optimal sampling density (1.11) or to at least find one giving small variance of the estimator (1.9). Recent examples include the use of engineering judgment (Pagani et al. 2005), design points (Schueller et al. 2004) and kernel density estimators (Au and Beck 2003a).

1.3.2 The Dimensionality Reduction method

Objective of the Dimensionality Reduction (DR) method is to reduce the variance associated to the failure probability estimates by exploiting the property of conditional expectation (Gille 1998 and 1999). In extreme synthesis, the failure event $g_x(\boldsymbol{x}) \leq 0$ is re-expressed in such a way as to highlight one of the n uncertain input variables of $\boldsymbol{x}$ (say, x_j); then, the failure probability estimate is computed as the expected value of the CDF of x_j conditional on the remaining $(n - 1)$ input variables. By so doing, the zero values contained in the standard MCS estimator (i.e., $I_F(\boldsymbol{x}) = 0$, if $\boldsymbol{x} \in F$) are removed: this allows to i) reach any level of probability (even very small) and ii) reduce the variance of the failure probability estimator (Gille 1998 and 1999).

The DR algorithm proceeds as follows (Gille 1998 and 1999):

1. Write the failure event $g_x(\boldsymbol{x}) = g_x(x_1, x_2, ..., x_j, ..., x_n) \leq 0$ in such a way as to highlight one of the *n* uncertain input variables (e.g., x_j):

$$x_j \leq h_x(\boldsymbol{x}_{-j}),\ j = 1, 2, \ldots, n \tag{1.12}$$

   where $h_x(\cdot)$ is a function defined on $\Re^{n-1}$ which takes values on the set of all (measurable) subsets of $\Re$ and $\boldsymbol{x}_{-j}$ is a vector containing all the uncertain input variables except x_j, i.e., $\boldsymbol{x}_{-j} = (x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_n)$;
2. Write the failure probability $P(F)$ as follows:

$$\begin{aligned} P(F) &= P[g_x(\boldsymbol{x}) \leq 0] \\ &= P[x_j \leq h_x(\boldsymbol{x}_{-j})] \\ &= E_{\boldsymbol{x}_{-j}}\left\{F_{x_j|\boldsymbol{x}_{-j}}[h_x(\boldsymbol{x}_{-j})]\right\} \end{aligned} \tag{1.13}$$

   where $F_{x_j|\boldsymbol{x}_{-j}}(\cdot)$ is the Cumulative Distribution Function of x_j conditional on $\boldsymbol{x}_{-j}$, i.e., $\boldsymbol{x}_{-j} = (x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_n)$;
3. Draw N_T samples $\{\boldsymbol{x}_{-j}^k : k = 1, 2, ..., N_T\}$, where $\boldsymbol{x}_{-j}^k = (x_1^k, x_2^k, ..., x_{j-1}^k, x_{j+1}^k, ..., x_n^k)$, from the $(n-1)$-dimensional marginal probability density function $q_m(\boldsymbol{x}_{-j})$, i.e., $q_m(\boldsymbol{x}_{-j}) = q_m(x_1, x_2, ..., x_{j-1}, x_{j+1}, ..., x_n) = \int_{x_j} q(x_1, x_2, ..., x_j, ..., x_n)dx_j$;
4. Using the last expression in (1.13), compute an unbiased and consistent estimate $\hat{P}(F)$ for the failure probability $P(F)$ as follows:

$$\hat{P}(F) = \frac{1}{N_T}\sum_{k=1}^{N_T} F_{x_j|\boldsymbol{x}_{-j}}[h_x(\boldsymbol{x}_{-j}^k)] \tag{1.14}$$

It is worth noting that in (1.14) the failure probability estimate is computed as the expected value of the cumulative distribution function $F_{x_j|\boldsymbol{x}_{-j}}(\cdot)$ of x_j conditional on the remaining $(n-1)$ input variables. Since this quantity takes values *between* 0 and 1, the zero values contained in the standard MCS estimator (i.e., $I_F(\boldsymbol{x}) = 0$, if $\boldsymbol{x} \in F$) are removed: this allows to i) reach any level of failure probability (even very small) and ii) reduce the variance of the failure probability estimator. However, such method can not always be applied: first, the performance function $g_x(\cdot)$ must be known analytically; second, it must have the property that one of the uncertain input variables can be separated from the others to allow re-writing the failure condition $g_x(\boldsymbol{x}) \leq 0$ in the form of (1.12) (Gille 1998 and 1999).

Finally, notice that DR can be considered a very special case of LS (Section 1.2.2) where the performance function $g_x(\cdot)$ is analytically known and the important direction $\boldsymbol{\alpha}$ coincides with the "direction" of the variable x_j, i.e., $\boldsymbol{\alpha} = (0, 0, ..., x_j, ..., 0, 0)$.

1.3.3 The Orthogonal Axis method

The Orthogonal Axis (OA) method combines the Fist Order Reliability Method (FORM) approximation (Der Kiureghian 2000) and Monte Carlo Simulation (MCS) in a sort of importance sampling around the "design point" of the problem (see Section 1.2.2.2).

The OA algorithm proceeds as follows (Gille 1998 and 1999):

1. Transform $\boldsymbol{x} = \{x_1, x_2, ..., x_j, ..., x_n\} \in \Re^n$, i.e., the vector of uncertain parameters defined in the original physical space $\boldsymbol{x} \in \Re^n$, into the vector $\boldsymbol{\theta} \in \Re^n$, where each element of the vector θ_j, j = 1, 2, …, n, is associated with a central unit Gaussian standard distribution (Schueller et al. 2004) (see Section 1.2.2.1). Thus, the joint probability density function of $\boldsymbol{\theta}$ can simply be written as

$$\varphi_n(\boldsymbol{\theta}) = \prod_{j=1}^{n} \phi(\theta_j) \tag{1.15}$$

   where $\phi(\theta_j) = \left(1/\sqrt{2\pi}\right) e^{-\left(\theta_j^2/2\right)}$, j = 1, 2, ..., n;
2. Find the "design point" $\boldsymbol{\theta}^*$ of the problem (see Section 1.2.2.2);
3. Rotate the coordinate system (i.e., by means of a proper rotation matrix $\boldsymbol{R}$) so that the new coordinate θ_n is in the direction of the axis defined by the design point $\boldsymbol{\theta}^*$;
4. Define a new failure function $g_{axis}(\boldsymbol{\theta})$ as

$$g_{axis}(\boldsymbol{\theta}) = g(\boldsymbol{R}\boldsymbol{\theta}) \tag{1.16}$$

5. Writing $\boldsymbol{\theta}$ as $(\tilde{\boldsymbol{\theta}}, \theta_n)$, where $\tilde{\boldsymbol{\theta}} = (\theta_1, \theta_2, ..., \theta_{n-1})$, express the failure probability $P(F)$ as follows:

$$
\begin{aligned}
P(F) &= P\left[g_{axis}\left(\tilde{\boldsymbol{\theta}}, \theta_n\right) \le 0\right] \\
&= \int P\left[g_{axis}\left(\tilde{\boldsymbol{\theta}}, \theta_n\right) \le 0 \mid \tilde{\boldsymbol{\theta}}\right] \varphi_{n-1}\left(\tilde{\boldsymbol{\theta}}\right) d\tilde{\boldsymbol{\theta}} \\
&= E_{\tilde{\theta}}\left\{P\left[g_{axis}\left(\tilde{\boldsymbol{\theta}}, \theta_n\right) \le 0\right]\right\}
\end{aligned}
\tag{1.17}
$$

6. Generate N_T i.i.d. (n – 1)-dimensional samples $\left\{\tilde{\boldsymbol{\theta}}^k : k = 1, 2, ..., N_T\right\}$, where $\tilde{\boldsymbol{\theta}}^k = \left(\theta_1^k, \theta_2^k, ..., \theta_{n-1}^k\right)$;
7. Compute an estimate $\hat{P}(F)$ for the failure probability $P(F)$ as follows:

$$
\hat{P}(F) = \frac{1}{N_T} \sum_{k=1}^{N_T} P\left[g_{axis}\left(\tilde{\boldsymbol{\theta}}^k, \theta_n\right) \le 0\right] \tag{1.18}
$$

The terms $P\left[g_{axis}\left(\tilde{\boldsymbol{\theta}}^k, \theta_n\right) \le 0\right]$, k = 1, 2, …, N_T, are evaluated with an iterative algorithm which searches for the roots of the equation $g_{axis}\left(\tilde{\boldsymbol{\theta}}^k, \theta_n\right) = 0$ (Gille 1998 and 1999).

It is worth noting that the idea underlying the OA method is essentially the same as that of LS (Section 1.2.2). However, in OA the “important direction” is forced to coincide with that of the design point of the problem; moreover, OA employs a rotation of the coordinate system which can be difficult to define in very high-dimensional problems.

1.4 Application 1: the cracked plate model

The cracked plate model is a classical example in Fracture Mechanics and its relative simplicity allows a detailed and complete study of different simulation techniques. A thorough description of this model can be found in (Ardillon and Venturini 1995).

1.4.1 The mechanical model

A metal plate of infinite length with a defect of initial length equal to a [m] is considered. The plate is supposed to be subject to a uniform normal loading (i.e., stress) s [MPa]. The intensity factor K [MPa $\sqrt{\text{m}}$], determined by the uniform loading in the neighborhood of the defect is defined as follows:

$$K = Fs_{\infty}\sqrt{\pi a} \tag{1.19}$$

where F is the shape factor of the defect. The plate is supposed to break (i.e., fail) when the intensity factor K in (1.19) becomes greater than or equal to a critical value K_c, i.e.:

$$K = Fs_{\infty}\sqrt{\pi a} \geq K_c \tag{1.20}$$

The variables of the mechanical model are summarized in Table 1.1.

Table 1.1. Names, descriptions and units of measure of the variables of the cracked plate model

Variables of the cracked plate model		
Name	**Description**	**Unit of measure**
K_c	Critical stress intensity factor	MPa m
a	Initial length of the defect	m
F	Shape factor of the defect	/
s	Uniform normal loading (stress) to which the plate is subject	MPa

1.4.2 The structural reliability model

From the point of view of a structural reliability analysis, the cracked plate mechanical model of Section 1.4.1 is analyzed within a probabilistic framework in which the variables K_c, a, F and s are uncertain (for simplicity of illustration with

respect to the notation of the previous Sections, the four variables are hereafter named x_1, x_2, x_3 and x_4, respectively).

Referring to (1.20), the performance function $g_x(\boldsymbol{x})$ of the system is

$$g_x(\boldsymbol{x}) = g_x(x_1, x_2, x_3, x_4) = x_1 - x_3 x_4 \sqrt{\pi x_2} \tag{1.21}$$

The failure region F is then expressed as

$$F = \{\boldsymbol{x} : g_x(\boldsymbol{x}) \le 0\} = \left\{(x_1, x_2, x_3, x_4) : x_1 \le x_3 x_4 \sqrt{\pi x_2}\right\} \tag{1.22}$$

Finally, the probability of system failure $P(F)$ is written as follows:

$$P(F) = P(\boldsymbol{x} \in F) = P[g_x(\boldsymbol{x}) \le 0] = P\left(x_1 \le x_3 x_4 \sqrt{\pi x_2}\right). \tag{1.23}$$

1.4.3 Case studies

Four case studies, namely Case 0 (Reference case), 1, 2 and 3, are considered with respect to the structural reliability model of the previous Section 1.4.2. Each case study is characterized by different PDFs for the uncertain variables x_1, x_2, x_3 and x_4 and by different failure probabilities $P(F)$: these features are summarized in Table 1.2. Notice that in Cases 0, 1 and 2 the random variables are independent and normally distributed, whereas in Case 3 they are independent and lognormally distributed. Moreover, it is worth noting that the exact (i.e., analytically computed) failure probabilities $P(F)$ approximately range from 10^{-3} to 10^{-7}, allowing a deep exploration of the capabilities of the simulation algorithms considered and a meaningful comparison between them (Gille 1998 and 1999).

Table 1.2. Probability distributions and parameters (i.e., means and standard deviations) of the uncertain variables x_1, x_2, x_3, and x_4 of the cracked plate model of Section 1.4.2 for the four case studies considered (i.e., Cases 0, 1, 2 and 3); the last row reports the values of the corresponding exact (i.e., analytically computed) failure probabilities, $P(F)$ (Gille 1998 and 1999). N = Normal distribution; LG = Lognormal distribution

	Case 0	**Case 1**	**Case 2**	**Case 3**
x_1 **(*K*)**	*N*(149.3, 22.2)	*N*(149.3, 22.2)	*N*(160, 18)	*LG*(149.3, 22.2)
x_2 **(*a*)**	*N*($5\cdot10^{-3}$, 10^{-3})	*N*($5\cdot10^{-3}$, 10^{-3})	*N*($5\cdot10^{-3}$, 10^{-3})	*LG*($5\cdot10^{-3}$, 10^{-3})
x_3 **(*F*)**	*N*(0.99, 0.01)	*N*(0.99, 0.01)	*N*(0.99, 0.01)	*LG*(0.99, 0.01)
x_4 **(*s*)**	*N*(600, 60)	*N*(300, 30)	*N*(500, 45)	*LG*(600, 60)
***P*(*F*)**	$1.165\cdot10^{-3}$	$4.500\cdot10^{-7}$	$4.400\cdot10^{-7}$	$3.067\cdot10^{-4}$

1.4.4 Results

In this Section, the results of the application of SS and LS for the reliability analysis of the cracked plate model of Section 1.4.1 are illustrated with reference to Case studies 0, 1, 2 and 3 described in the previous Section 1.4.3.

For fair comparison, all methods have been run with a total of N_T = 50000 samples in all four cases. The efficiency of the simulation methods under analysis is evaluated in terms of four quantities: the failure probability estimate $\hat{P}(F)$, the sample standard deviation $\hat{\sigma}$ of the failure probability estimate $\hat{P}(F)$, the coefficient of variation (c.o.v.) δ of $\hat{P}(F)$ (defined as the ratio of the sample standard deviation $\hat{\sigma}$ to the estimate $\hat{P}(F)$) and the Figure Of Merit (FOM) of the method (defined as $1/(\hat{\sigma}^2 t_{comp})$, where t_{comp} is the computational time required by the simulation method). The closer is the estimate $\hat{P}(F)$ to the exact (i.e., analytically computed) failure probability $P(F)$, the more accurate is the simulation method. The sample standard deviation $\hat{\sigma}$ and the c.o.v. δ of $\hat{P}(F)$ are used to quantify the variability of the failure probability estimator; in particular, the lower are the values of $\hat{\sigma}$ and δ, the lower is the variability of the corresponding failure probability estimator and thus the higher is the efficiency of the simulation method adopted. Finally, the FOM is introduced to take into account the computational time required by the method. The value of the FOM increases as the sample variance $\hat{\sigma}^2$ of the failure probability estimate $\hat{P}(F)$ and the computational time t_{comp} required by the method decrease; thus, in this case the higher is the value of the index, the higher is the efficiency of the method (Gille 1998 and 1999).

The different simulation methods are also compared with respect to two direct performance indicators relative to standard MCS. First, the ratio of the sample standard deviation $\hat{\sigma}_{MC}$ obtained by Standard MCS to that obtained by the simulation method under analysis $\hat{\sigma}_{meth}$ is computed. This ratio *only* quantifies the improvement in the *precision* of the estimate achieved by using a given simulation method instead of standard MCS. Then, the ratio of the FOM of the simulation method in object, namely FOM_{meth}, to that of standard MCS, namely FOM_{MC}, is considered to quantify the *overall* improvement in *efficiency* achieved by a given simulation method with respect to standard MCS, since it takes into account also the computational time required. Obviously, the higher are the values of these two indices for a given method, the higher is the efficiency of that method (Gille 1998 and 1999).

Table 1.3 reports the values of $\hat{P}(F)$, $\hat{\sigma}$, δ, FOM, $\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$ and FOM_{meth}/FOM_{MC} obtained by Standard MCS, SS and LS in Cases 0, 1, 2 and 3 (Section

1.4.3); the *actual* number N_{sys} of system response analyses (i.e., model evaluations) is also reported. Notice that for both SS and LS the actual number N_{sys} of system analyses does not coincide with the total number N_T of random samples drawn (i.e., $N_T = 50000$). In particular, in the SS method, the presence of *repeated* conditional samples in each Markov chain (used to gradually populate the intermediate event regions) allows a *reduction* in the number of model evaluations required: actually, one evaluation is enough for all identical samples (see Appendix 1). In the LS method, instead, the actual number N_{sys} of system analyses is given by $N_{sys} = N_s + 2 \cdot N_T$: in particular, $N_s = 2000$ analyses are performed to generate the Markov chain used to compute the important unit vector $\boldsymbol{\alpha}$ as the normalized "center of mass" of the failure domain F (Section 1.2.2.2); the $2 \cdot N_T$ analyses are carried out to compute the N_T conditional one-dimensional probability estimates $\left\{\hat{P}^k(F) : k = 1, 2, ..., N_T\right\}$ by linear interpolation (equation (1.5') in Appendix 2).

Table 1.3. Results of the application of standard MCS, SS and LS to the reliability analysis of Cases 0 (Reference), 1, 2 and 3 of the cracked plate model of Section 1.4.2; the values of the performance indicators used to compare the effectiveness of the methods (i.e., $\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$ and FOM$_{meth}$/FOM$_{MC}$) are highlighted in bold

Case 0 (Reference)							
	$\hat{P}(F)$	$\hat{\sigma}$	***c.o.v., δ***	N_{sys}	***FOM***	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	***FOM$_{meth}$/FOM$_{MC}$***
Standard MCS	$1.120 \cdot 10^{-3}$	$1.496 \cdot 10^{-4}$	$1.336 \cdot 10^{-1}$	50000	893.65	1	1
SS	$1.274 \cdot 10^{-3}$	$7.136 \cdot 10^{-5}$	$5.597 \cdot 10^{-2}$	49929	3936.67	2.10	4.41
LS	$1.169 \cdot 10^{-3}$	$5.142 \cdot 10^{-7}$	$4.399 \cdot 10^{-4}$	102000	$3.782 \cdot 10^{7}$	290.92	42318

Case 1							
	$\hat{P}(F)$	$\hat{\sigma}$	***c.o.v., δ***	N_{sys}	***FOM***	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	***FOM$_{meth}$/FOM$_{MC}$***
Standard MCS	$4.500 \cdot 10^{-7}$	$3.000 \cdot 10^{-6}$	6.667	50000	$2.222 \cdot 10^{6}$	1	1
SS	$4.624 \cdot 10^{-7}$	$7.295 \cdot 10^{-8}$	$1.578 \cdot 10^{-1}$	49937	$3.762 \cdot 10^{9}$	41.12	$1.7 \cdot 10^{3}$
LS	$4.493 \cdot 10^{-7}$	$1.791 \cdot 10^{-10}$	$3.986 \cdot 10^{-4}$	102000	$3.117 \cdot 10^{14}$	16750	$1.4 \cdot 10^{8}$

Case 2							
	$\hat{P}(F)$	$\hat{\sigma}$	***c.o.v., δ***	N_{sys}	***FOM***	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	***FOM$_{meth}$/FOM$_{MC}$***
Standard MCS	$4.400 \cdot 10^{-7}$	$3.000 \cdot 10^{-6}$	6.667	50000	$2.222 \cdot 10^{6}$	1	1
SS	$4.679 \cdot 10^{-7}$	$6.890 \cdot 10^{-8}$	$1.473 \cdot 10^{-1}$	49888	$4.222 \cdot 10^{9}$	43.54	$1.9 \cdot 10^{3}$
LS	$4.381 \cdot 10^{-7}$	$4.447 \cdot 10^{-10}$	$1.015 \cdot 10^{-3}$	102000	$4.959 \cdot 10^{13}$	6746.7	$2.2 \cdot 10^{7}$

Case 3							
	$\hat{P}(F)$	$\hat{\sigma}$	***c.o.v., δ***	N_{sys}	***FOM***	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	***FOM$_{meth}$/FOM$_{MC}$***
Standard MCS	$3.000 \cdot 10^{-4}$	$7.745 \cdot 10^{-5}$	$2.582 \cdot 10^{-1}$	50000	$3.334 \cdot 10^{3}$	1	1
SS	$3.183 \cdot 10^{-4}$	$2.450 \cdot 10^{-5}$	$7.697 \cdot 10^{-2}$	49907	$3.339 \cdot 10^{4}$	3.16	10.01
LS	$3.068 \cdot 10^{-4}$	$1.817 \cdot 10^{-7}$	$5.923 \cdot 10^{-4}$	102000	$3.028 \cdot 10^{8}$	426.16	$9.1 \cdot 10^{4}$

It can be seen that SS performs consistently better than standard MCS and its performance significantly grows as the failure probability to be estimated decreases: for instance, in Case 0 (Reference), where $P(F) \sim 10^{-3}$, the FOM of SS, namely FOM_{SS}, is only four times larger than that of Standard MCS, namely FOM_{MC}; whereas in Case 1, where $P(F) \sim 10^{-7}$, the ratio FOM_{SS}/FOM_{MC} is about 557. On the other hand, LS outperforms SS with respect to both $\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$ and FOM_{meth}/FOM_{MC} in all the Cases considered. For instance, in Case 2, where the failure probability $P(F)$ to be estimated is very small, i.e., $P(F) = 4.4 \cdot 10^{-7}$, the ratio $\hat{\sigma}_{MC}/\hat{\sigma}_{LS}$ is 155 times larger than the ratio $\hat{\sigma}_{MC}/\hat{\sigma}_{SS}$, whereas the ratio FOM_{LS}/FOM_{MC} is 11750 times larger than the ratio FOM_{SS}/FOM_{MC}. Notice that for the LS method even though the determination of the sampling important direction $\boldsymbol{\alpha}$ (Section 1.2.2.2) and the calculations of the conditional one-dimensional failure probability estimates $\left\{\hat{P}^k(F) : k = 1, 2, ..., N_T\right\}$ (equation (1.5') in Appendix 2) require much more than N_T system analyses by the model, this is significantly overweighed by the accelerated convergence rate that can be attained by the LS method with respect to SS.

1.4.4.1 Comparison with other stochastic simulation methods

The results obtained by SS and LS are compared to those obtained by the Importance Sampling (IS), Dimensionality Reduction (DR), Orthogonal Axis (OA) methods and by a combination of IS and DR (Section 1.3) (Gille 1998 and 1999). For DR, the variable x_1 is explicited.

The values of the performance indicators $\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$ and FOM_{meth}/FOM_{MC} obtained by the four methods in Cases 0, 1, 2 and 3 are summarized in Table 1.4.

Table 1.4. Values of the performance indicators $\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$ and FOM_{meth}/FOM_{MC} obtained by IS, DR (with variable x_1 specified), OA and IS + DR when applied for the reliability analysis of Cases 0 (Reference), 1, 2 and 3 of the cracked plate model of Section 1.4.2 (Gille 1998 and 1999)

Case 0 (Reference)		
	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	FOM_{meth}/FOM_{MC}
IS	17	100
DR (Variable x_1)	14	14
OA	340	$7.7\cdot10^{3}$
IS + DR	194	$2.1\cdot10^{4}$

Case 1		
	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	FOM_{meth}/FOM_{MC}
IS	630	376
DR (Variable x_1)	856	$7.3\cdot10^{5}$
OA	17255	$2.0\cdot10^{7}$
IS + DR	8300	$1.3\cdot10^{8}$

Case 2		
	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	FOM_{meth}/FOM_{MC}
IS	643	$1.5\cdot10^{5}$
DR (Variable x_1)	242	242
OA	10852	$7.9\cdot10^{6}$
IS + DR	8077	$3.6\cdot10^{7}$

Case 3		
	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	FOM_{meth}/FOM_{MC}
IS	29	289
DR (Variable x_1)	7	7
OA	4852	$4.9\cdot10^{5}$
IS + DR	150	$1.2\cdot10^{4}$

Comparing Table 1.3 and Table 1.4, it can be seen that LS performs significantly better than IS and DR in all the case studies considered: in particular, in Cases 1 and 2 the values of the performance indicators $\hat{\sigma}_{MC}/\hat{\sigma}_{LS}$ (16750 and 6746.7) and FOM_{LS}/FOM_{MC} ($1.4\cdot10^{8}$ and $2.2\cdot10^{7}$) are more than one order of magnitude larger than those reported in (Gille 1998 and 1999) for IS (630, 376 and 643, $1.5\cdot10^{5}$ for Cases 1 and 2, respectively) and DR (856, $7.3\cdot10^{5}$ and 242, 242 for Cases 1 and 2, respectively). Moreover, it is worth noting that in the reference studies by (Gille 1998 and 1999) a significant number of simulations has been run to properly tune the parameters of the ISDs for the IS method (in particular, 8, 6, 6

and 8 simulations have been performed for Cases 0, 1, 2 and 3, respectively), with a significant increase in the associated computational effort.

LS is found to perform slightly worse than OA in all the case studies considered: actually, the values of both $\hat{\sigma}_{MC}/\hat{\sigma}_{LS}$ and FOM_{LS}/FOM_{MC} are slightly lower than those reported in (Gille 1998 and 1999) for OA. However, it should be considered that in these studies the OA method has been applied to a simplified version of the problem described in Sections 1.4.1 and 1.4.2; actually, only three uncertain variables (i.e., x_1, x_2 and x_4) have been considered by keeping variable x_3 (i.e., F) fixed to its mean value (i.e., 0.99): this certainly reduces the variability of the model output and contributes to the reduction of the variability of the associated failure probability estimator.

Further, LS performs consistently better than the combination of IS and DR in the task of estimating failure probabilities around $10^{-3} \div 10^{-4}$ (for instance, in Case 0 $\hat{\sigma}_{MC}/\hat{\sigma}_{IS+DR}$ = 194 and $\hat{\sigma}_{MC}/\hat{\sigma}_{LS}$ = 290, whereas in Case 4 $\hat{\sigma}_{MC}/\hat{\sigma}_{IS+DR}$ = 150 and $\hat{\sigma}_{MC}/\hat{\sigma}_{LS}$ = 426). In addition, LS performs comparably to the combination of IS and DR in the estimation of failure probabilities around 10^{-7}: actually, in Case 1 $\hat{\sigma}_{MC}/\hat{\sigma}_{IS+DR}$ = 8300 and $\hat{\sigma}_{MC}/\hat{\sigma}_{LS}$ = 16750, whereas in Case 2 $\hat{\sigma}_{MC}/\hat{\sigma}_{IS+DR}$ = 8077 and $\hat{\sigma}_{MC}/\hat{\sigma}_{LS}$ = 6746. However, it has to be noticed again that in the reference studies by (Gille 1998 and 1999) a significant number of simulations has been run to properly tune the parameters of the ISDs for the IS method (in particular, 4, 8, 8 and 10 simulations have been performed in Cases 0, 1, 2 and 3, respectively).

Finally, it is worth noting that in these cases SS performs worse than the other methods proposed.

1.5 Application 2: thermal fatigue crack growth model

The thermal fatigue crack growth model considered in this study is based on the deterministic Paris-Erdogan model which describes the propagation of a manufacturing defect due to thermal fatigue (Paris 1961).

1.5.1 The mechanical model

The evolution of the size a of a defect satisfies the following equation:

$$\frac{da}{dN_c} = C \cdot \left(f(R) \cdot \Delta K\right)^m \tag{1.24}$$

where N_c is the number of fatigue cycles, C and m are parameters depending on the properties of the material, $f(R)$ is a correction factor which is a function of the material resistance R and K is the variation of the intensity factor, defined as

$$\Delta K = \Delta s \cdot Y(a) \cdot \sqrt{\pi a} \tag{1.25}$$

In (1.25), s is the variation of the uniform loading (stress) applied to the system and $Y(a)$ is the shape factor of the defect. Let $S_i = s_i$ be the variation of the uniform normal stress at cycle $i = 1, 2, \ldots, N_c$. The integration of equation (1.24) gives

$$\int_{a_0}^{a_{N_c}} \frac{da}{\left(Y(a)\sqrt{\pi a}\right)^m} = C \cdot \sum_{i=1}^{N_c} \left(f(R) \cdot S_i\right)^m \tag{1.26}$$

where a_0 and a_{N_c} are the initial and final size of the defect, respectively. In (1.26) the following approximation can be adopted

$$\sum_{i=1}^{N_c} \left(f(R) \cdot S_i\right)^m \approx \left(T - T_0\right) \cdot N_c \cdot \left(f(R) \cdot S\right)^m \tag{1.27}$$

where T and T_0 are the initial and final times of the thermal fatigue treatment (of N_c cycles).

The system is considered failed when the size a_{Nc} of the defect at the end of the N_c cycles exceeds a critical dimension a_c, i.e.:

$$a_c - a_{N_c} \leq 0 \tag{1.28}$$

which in the integral form (1.26) reads

$$\psi(a_c) - \psi(a_{N_c}) \leq 0 \tag{1.29}$$

where

$$\psi(a) = \int_{a_0}^{a} \frac{da'}{\left(Y(a') \cdot \sqrt{\pi a'}\right)^m} \tag{1.30}$$

Using (1.27), a safety margin $M(T)$ can then be defined as follows:

$$M(T) = \int_{a_0}^{a_c} \frac{da}{\left(Y(a) \cdot \sqrt{\pi a}\right)^m} - C \cdot (T - T_0) \cdot N_c \cdot \left(f(R) \cdot S\right)^m \tag{1.31}$$

The failure criterion can then be expressed in terms of the safety margin (1.31):

$$M(T) \leq 0 \tag{1.32}$$

The variables of the model are summarized in Table 1.5.

Table 1.5. Names, descriptions and units of measure of the variables of the thermal fatigue crack growth model

Variables of the thermal fatigue crack growth model		
Name	**Description**	**Unit of measure**
a_0	Initial size of the defect	[m]
a_c	Critical size of the defect	[m]
T_0	Initial time	[years]
T	Final time	[years]
C	Parameter of the material	/
m	Parameter of the material	/
$f(R)$	Correction factor	/
N_c	Number of cycles per year	/
S	Stress per cycle	[MPa]

1.5.2 The structural reliability model

For the purpose of a structural reliability analysis, the thermal fatigue crack growth model is framed within a probabilistic representation of the uncertainties

affecting the nine variables a_0, a_c, T_0, T, C, m, $f(R)$, N_c and S (hereafter named x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 and x_9, respectively).

From (1.32), the probability of system failure $P(F)$ is written as

$$P(F) = P[M(T) \le 0] = P\left[\int_{a_0}^{a_c} \frac{da}{\left(Y(a)\cdot\sqrt{\pi a}\right)^m} - C\cdot(T - T_0)\cdot N_c\cdot\left(f(R)\cdot S\right)^m \le 0\right] \quad (1.33)$$

or

$$P(F) = P[M(T) \le 0] = P\left[\int_{x_1}^{x_2} \frac{da}{\left(Y(a)\cdot\sqrt{\pi a}\right)^{x_6}} - x_5\cdot(x_4 - x_3)\cdot x_8\cdot\left(x_7\cdot x_9\right)^{x_6} \le 0\right] \quad (1.34)$$

It is worth noting the highly nonlinear nature of expressions (1.33) and (1.34) which increases the complexity of the problem.

1.5.3 Case studies

Two different case studies, namely Case 1 and Case 2, are built with reference to the structural reliability model of the previous Section 1.5.2. The characteristics of the PDFs of the uncertain variables of Table 1.5 are summarized in Table 1.6; the values of the exact (i.e., analytically computed) failure probabilities, $P(F)$, for both Cases 1 and 2 are also reported in the last row of Table 1.6.

Table 1.6. Probability distributions and parameters (i.e., means and standard deviations) of the uncertain variables x_1, x_2, …, x_9 of the thermal fatigue crack growth model of Section 1.5.2 for the two case studies considered (i.e., Cases 1 and 2); the last row reports the values of the corresponding exact (i.e., analytically computed) failure probabilities, $P(F)$ (Gille 1998 and 1999). Exp=exponential distribution; LG=Lognormal distribution; N=Normal distribution

	Case 1	**Case 2**
x_1 (a_0)	Exp($0.61\cdot10^{-3}$)	Exp($0.81\cdot10^{-3}$)
x_2 (a_c)	N($21.4\cdot10^{-3}$, $0.214\cdot10^{-3}$)	N($21.4\cdot10^{-3}$, $0.214\cdot10^{-3}$)
x_3 (T_0)	0	0
x_4 (T)	40	40
x_5 (C)	LG($6.5\cdot10^{-13}$, $5.75\cdot10^{-13}$)	LG($1.00\cdot10^{-12}$, $5.75\cdot10^{-13}$)
x_6 (m)	3.4	3.4
x_7 ($f(R)$)	2	2
x_8 (N_c)	N(20, 2)	N(20, 2)
x_9 (S)	LG(300, 30)	LG(200, 20)
$P(F)$	$3.3380\cdot10^{-4}$	$1.780\cdot10^{-5}$

1.5.4 Results

In this Section, the results of the application of SS and LS for the reliability analysis of the thermal fatigue crack growth model of Sections 1.5.1 and 1.5.2 are illustrated with reference to Cases 1 and 2 (Table 1.6 of Section 1.5.3).

Again for fair comparison all simulation methods have been run with the same total number of samples (N_T = 40000) in both Cases 1 and 2. The efficiency of the methods has been evaluated in terms of the same indices and performance indicators defined in Section 1.4.4.

Table 1.7 reports the values of $\hat{P}(F)$, $\hat{\sigma}$, δ, FOM, $\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$ and FOM$_{meth}$/FOM$_{MC}$ obtained by Standard MCS, SS and LS in the Cases 1 and 2 of Section 1.5.3; the *actual* number N_{sys} of system response analyses (i.e., model evaluations) is also reported.

Table 1.7. Results of the application of standard MCS, SS and LS to the reliability analysis of Cases 1 and 2 of the thermal fatigue crack growth model of Section 1.5.2; the values of the performance indicators used to compare the effectiveness of the methods (i.e., $\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$ and FOM$_{meth}$/FOM$_{MC}$) are highlighted in bold

Case 1							
	$\hat{P}(F)$	$\hat{\sigma}$	***c.o.v.*, δ**	N_{sys}	***FOM***	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	**FOM$_{meth}$/FOM$_{MC}$**
Standard MCS	$2.500\cdot10^{-4}$	$7.905\cdot10^{-5}$	$3.162\cdot10^{-1}$	40000	$4.001\cdot10^{3}$	1	1
SS	$3.006\cdot10^{-4}$	$3.214\cdot10^{-5}$	$1.069\cdot10^{-1}$	40019	$2.419\cdot10^{4}$	2.46	6.05
LS	$3.768\cdot10^{-4}$	$4.610\cdot10^{-7}$	$1.223\cdot10^{-3}$	82000	$5.737\cdot10^{7}$	171.46	$1.434\cdot10^{4}$

Case 2							
	$\hat{P}(F)$	$\hat{\sigma}$	***c.o.v.*, δ**	N_{sys}	***FOM***	$\hat{\sigma}_{MC}/\hat{\sigma}_{meth}$	**FOM$_{meth}$/FOM$_{MC}$**
Standard MCS	$1.780\cdot10^{-5}$	$2.269\cdot10^{-5}$	1.102	40000	$4.860\cdot10^{4}$	1	1
SS	$1.130\cdot10^{-5}$	$1.653\cdot10^{-6}$	$1.462\cdot10^{-1}$	39183	$9.341\cdot10^{6}$	13.73	192.36
LS	$1.810\cdot10^{-5}$	$2.945\cdot10^{-8}$	$1.627\cdot10^{-3}$	81999	$1.188\cdot10^{13}$	770.02	$2.892\cdot10^{5}$

Also in this application, the LS methodology is found to outperform SS in both Cases 1 and 2: for example, in Case 2, where the failure probability $P(F)$ to be estimated is around 10^{-5}, the ratio FOM$_{LS}$/FOM$_{MC}$ is about 1500 times larger than the ratio FOM$_{SS}$/FOM$_{MC}$.

1.5.4.1 Comparison with other stochastic simulation methods

As done for the previous application of Section 1.4, the results obtained by SS and LS have been compared to those obtained by other literature methods, in particular the Importance Sampling (IS) and a combination of Importance Sampling and

Dimensionality Reduction (Section 1.3) which have turned out to give the best results in the case studies considered (Gille 1998 and 1999). Notice that the Orthogonal Axis (OA) method has not been implemented for this application in the reference study (Gille 1998 and 1999): this is due to the high dimensionality of the problem which makes the definition of a proper rotation matrix very difficult (step 3. in Section 1.3.3).

The values of the performance indicators $\hat{\sigma}_{MC} / \hat{\sigma}_{meth}$ and FOM_{meth}/FOM_{MC} obtained by IS and IS and DR for Cases 1 and 2 of the thermal fatigue crack growth model of Sections 1.5.1 and 1.5.2 are summarized in Table 1.8.

Table 1.8. Values of the performance indicators $\hat{\sigma}_{MC} / \hat{\sigma}_{meth}$ and FOM_{meth}/FOM_{MC} obtained by IS and IS + DR when applied for the reliability analysis of Cases 1 and 2 of the thermal fatigue crack growth model of Section 1.5.2 (Gille 1998 and 1999)

Case 1		
	$\hat{\sigma}_{MC} / \hat{\sigma}_{meth}$	FOM_{meth}/FOM_{MC}
IS	16.9	424.36
IS + DR	65.4	864.36

Case 2		
	$\hat{\sigma}_{MC} / \hat{\sigma}_{meth}$	FOM_{meth}/FOM_{MC}
IS	41.1	$4.396 \cdot 10^3$
IS + DR	172.4	$8.317 \cdot 10^3$

In this application, LS is found to outperform both IS and the combination of IS and DR: for example, in Case 2, the ratio FOM_{LS}/FOM_{MC} is 65 and 35 times larger than FOM_{IS}/FOM_{MC} and FOM_{IS+DR}/FOM_{MC}, respectively. This confirms the capability of the LS method to efficiently probe complex high-dimensional domains of integration.

1.6 Summary and critical discussion of the techniques

One of the major obstacles in applying simulation methods for the reliability analysis of engineered systems and structures is the challenge posed by the estimation of small failure probabilities: the simulation of the rare events of failure occurrence implies a significant computational burden (Schueller 2007).

In order to overcome the rare-event problem, the Importance Sampling (IS) method has been introduced (Au and Beck 2003a; Schueller et al. 2004). This technique amounts to replacing the original PDF of the uncertain random variables with an Importance Sampling Distribution (ISD) chosen so as to generate samples that lead to failure more frequently (Au and Beck 2003). IS has the capability to considerably reduce the variance compared with standard MCS, provided that the ISD is chosen similar to the theoretical optimal one (equation (1.11) of Section 1.3.1). However, generally substantial insights on the system stochastic behaviour and extensive modelling work is needed to identify a "good" ISD, e.g. by identifying "design points" (Schueller et al. 2004), setting up complex kernel density estimators (Au and Beck 2003a) or simply by tuning the parameters of the ISD based on expert judgment and trial-and-error (Gille 1998 and 1999; Pagani et al. 2005). Overall, this greatly increases the effort associated to the simulation for accurate failure probability estimation. Furthermore, there is always the risk that an inappropriate choice of the ISD may lead to worse estimates compared to Standard MCS (Schueller et al. 2004).

Subset Simulation (SS) offers a clever way out of this problem by breaking the small failure probability evaluation task into a sequence of estimations of larger conditional probabilities. During the simulation, more frequent samples conditional to intermediate regions are generated from properly designed Markov chains. The method has been proven much more effective than standard MCS in the very high-dimensional spaces characteristic of structural reliability problems in which the failure regions are just tiny bits (Au and Beck 2001).

The strength of Subset Simulation lies in the generality of its formulation and the straightforward algorithmic scheme. In contrast to some of the alternative methods (e.g., Line Sampling and Orthogonal Axis), it is not restricted to standard normal spaces and can provide equally good results irrespectively of the joint distribution of the uncertain variables as long as one can draw samples from it. Furthermore, a single run of the SS algorithm leads to the calculation of the probabilities associated with all the conditional events considered: if for example, the probability of exceeding a critical level by a system response statistic of a stochastic system (e.g., the mean or a percentile of the displacement, stress, temperature, etc) is sought, then by appropriate parametrization of the intermediate conditional events, a single run can provide the probabilities of exceedance associated with a wide range of values of the response statistic of interest irrespectively of their magnitude (Au 2005).

On the other hand, a word of caution is in order with respect to the fact that the conditional samples generated during the Markov Chain Monte Carlo (MCMC)

simulation are correlated by construction. Since it is demonstrated that a high correlation among conditional samples increases the variance of the SS estimates, a good choice/tuning of the SS parameters (i.e., the conditional probability p_0 and the proposal PDFs for MCMC simulation) is required to avoid it (Au and Beck 2003b). Finally, another drawback of the SS method is the need to express the failure event F in terms of a real valued parameter crossing a given threshold (i.e., $F = \{Y < y\}$). This parametrization is natural for the cases of practical interest in structural reliability and otherwise specific for other system reliability problems (Zio and Pedroni 2008).

An alternative way to perform robust estimations of small failure probabilities without the extensive modelling effort required by IS is offered by Line Sampling (LS). The LS method employs *lines* instead of random *points* in order to probe the high-dimensional failure domain of interest. An "important direction" is optimally determined to point towards the failure domain of interest and a number of conditional, one-dimensional problems are solved along such direction, in place of the original high-dimensional problem (Pradlwarter et al. 2005). In case the boundaries of the failure domain of interest are not too rough (i.e., approximately linear) and the "important direction" is almost perpendicular to them, only few simulations suffice to arrive at a failure probability with acceptable confidence. The determination of the important direction requires additional evaluations of the system performance which increases the computational cost (Section 1.2.2.2). Further, for each random sample (i.e., system configuration) drawn, two or three evaluations of the system performance are necessary to estimate the conditional one-dimensional failure probability estimates by linear or quadratic interpolation (equation (1.5') in Appendix 2). In case the "important direction" is not the optimal one, the variance of the estimator will increase. Of particular advantage of Line Sampling is its robustness: in the worst possible case where the "important direction" is selected orthogonal to the (ideal) optimal direction, line sampling performs at least as well as standard Monte Carlo simulation (Schueller et al. 2004).

Finally, the Dimensionality Reduction (DR) method and the Orthogonal Axis (OA) method employ simulation concepts similar to those of LS, but with important limitations (Gille 1998 and 1999). In the DR method, the failure event of interest is re-expressed in such a way as to highlight *one* (say, x_j) of the input random variables, recognized as more important; then, the failure probability estimate is computed as the expected value of the CDF of x_j conditional on the remaining $(n - 1)$ input variables. By so doing, the zero values contained in the standard MCS estimator (i.e., $I_F(\boldsymbol{x}) = 0$, if $\boldsymbol{x} \in F$) are removed: this allows to i) reach any level of probability (even very small) and ii) reduce the variance of the failure probability estimator (Gille 1998 and 1999). Notice that DR can be considered a very special case of LS where the important direction $\boldsymbol{\alpha}$ coincides with the "direction" of the variable x_j, i.e., $\boldsymbol{\alpha} = \left(0, 0, ..., x_j, ..., 0, 0\right)$. However, such method can not always be applied: first, the performance function of the system must be analytically known (which is never the case for realistic systems simulated by detailed

computer codes); second, the performance function must have the characteristic that one of the variables can be separated from the others (Gille 1998 and 1999).

Finally, the Orthogonal Axis (OA) method performs a sort of importance sampling around the design point of the problem in the standard normal space. Thus, if the design point is actually representative of the most important regions of the failure domain, the OA leads to an impressive reduction in the variance of the failure probability estimator. However, it is worth noting that the design points and their neighbors do not always represent the most important regions of the failure domain, especially in high-dimensional problems. Moreover, the computational cost associated with the identification of the design points may be quite relevant which adversely affect the efficiency of the method (Schueller et al. 2004). Finally, the implementation of the OA method requires the definition of a rotation matrix in order to modify the coordinate system, which can be very difficult for high-dimensional problems.

A synthetic comparison of the stochastic simulation methods considered in this work is given in Table 1.9 (the second column, namely “Decisions”, refers to parameters, distributions and other characteristics of the methods that have to be chosen or determined by the analyst in order to perform the simulation).

Table 1.9. Synthetic comparison of the stochastic simulation methods considered in this work (Part I)

Method	Simulation concepts	Decisions	Advantages	Drawbacks
Standard MCS	- repeat random sampling of possible system configurations	/	- samples the full range of each input variable - consistent performance in spite of complexity and dimension of the problem - accuracy easily assessed - no need for simplifying assumptions nor surrogate models - no complex elaborations of the original model - identification of nonlinearities, thresholds and discontinuities - simplicity	- high computational cost (in presence of long-running models for determining system response and small failure probabilities)
SS	- express a small probability as a product of larger conditional probabilities - generate conditional samples by Markov Chain Monte Carlo (MCMC) simulation	- conditional failure probability p_0 at each simulation level - proposal PDFs for MCMC Simulation	- general formulation - straightforward algorithmic scheme - no restriction to standard normal space - consistent performance in spite of complex joint PDFs - consistent performance in spite of irregularities in topology and boundary of the failure domain - one single run computes probabilities for more than one event - reduced computational effort with respect to other methods	- parametrization of the failure event in terms of intermediate conditional events - correlation among conditional samples: bias in the estimates and possibly increased variance
LS	- turn a high-dimensional problem in the physical space into one-dimensional problems in the standard normal space - project the problem onto a line $\boldsymbol{\alpha}$ pointing at the important regions of the failure domain - use line $\boldsymbol{\alpha}$ almost perpendicular to the failure domain to reduce the variance of the estimates	- one failure point to start the Markov chain for the determination of $\boldsymbol{\alpha}$	- no assumptions about regularity of the limit state function (robustness) - if limit state function is almost linear, few simulations suffice to achieve acceptable estimation accuracies - no necessity to estimate important direction $\boldsymbol{\alpha}$ with excessive accuracy - even in the worst possible case ($\boldsymbol{\alpha}$ orthogonal to optimal direction) the performance is at least comparable to standard MCS	- determination of important direction $\boldsymbol{\alpha}$ requires additional evaluation of system performance (with increase in the computational cost) - for each sample drawn, two or three evaluations of system performance are necessary to estimate failure probability (with increase in the computational cost) - essential restriction to standard normal space (Rosenblatt's or Nataf's transformations are required) (Rosenblatt 1952; Nataf 1962)

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Table 1.9. Synthetic comparison of the stochastic simulation methods considered in this work (Continued)

Method	Simulation concepts	Decisions	Advantages	Drawbacks
IS	- repeated random sampling of possible system configurations - sample from Importance Sampling Density (ISD) to generate more samples in the region of interest (e.g., low probability of occurrence)	- construction/choice of the ISD	- if the ISD is similar to optimal one: significant increase in estimation accuracy (or, conversely, reduction in sample size for given accuracy)	- many system behavior insights and and much modeling work needed for identification of good ISD - inappropriate ISD leads to worse estimates compared to Standard MCS
DR	- express failure event in such a way as to highlight one random variable - estimate failure probability as expected value of the CDF of the chosen variable conditional on the remaining $(n-1)$ variables	- random variable to be separated from others	- remove zero values included in the Standard MCS estimator (reduced variance) - any probability level can be reached (also the very small ones of rare events)	- analytical expression for the system performance function is required - performance function must have the characteristics that one of the variables can be separated out from the others
OA	- identification of the design point - rotation of system coordinates - solve one-dimensional problems along direction of design point	/	- if the design point is representative of the most important regions of the failure domain, then the variance is significantly reduced	- design point frequently not representative of the most important regions of the failure domain (high-dimensional problems) - high computational cost associated to design point (nonlinear constrained optimization problem) - rotation matrix difficult to introduce in high-dimensional spaces

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Appendix 1. Markov Chain Monte Carlo (MCMC) Simulation

Markov Chain Monte Carlo (MCMC) simulation comprises a number of powerful simulation techniques for generating samples according to any given probability distribution (Metropolis et al. 1953).

In the context of the reliability assessment of interest in the present work, MCMC simulation provides an efficient way for generating samples from the multidimensional conditional PDF $q(\boldsymbol{x} \mid F)$. The distribution of the samples thereby generated tends to the multidimensional conditional PDF $q(\boldsymbol{x} \mid F)$ as the length of the Markov chain increases. In the particular case of the initial sample $\boldsymbol{x}^1$ being distributed exactly as the multidimensional conditional PDF $q(\boldsymbol{x} \mid F)$, then so are the subsequent samples and the Markov chain is always stationary (Au and Beck 2001).

In the following it is assumed without loss of generality that the components of $\boldsymbol{x}$ are independent, that is, $q(\boldsymbol{x}) = \prod_{j=1}^{n} q_j(x_j)$, where $q_j(x_j)$ denotes the one-dimensional PDF of x_j (Au and Beck 2001).

To illustrate the MCMC simulation algorithm with reference to a generic failure region F_i, let $\boldsymbol{x}^u = \{x_1^u, x_2^u, ..., x_j^u, ..., x_n^u\}$ be the u^{th} Markov chain sample drawn and let $p_j^*(\xi_j \mid x_j^u)$, j = 1, 2, …, n, be a one-dimensional 'proposal PDF' for ξ_j, centered at the value x_j^u and satisfying the symmetry property $p_j^*(\xi_j \mid x_j^u) = p_j^*(x_j^u \mid \xi_j)$. Such distribution, arbitrarily chosen for each element x_j of $\boldsymbol{x}$, allows generating a 'precandidate value' ξ_j based on the current sample value x_j^u. The following algorithm is then applied to generate the next Markov chain sample $\boldsymbol{x}^{u+1} = \{x_1^{u+1}, x_2^{u+1}, ..., x_j^{u+1}, ..., x_n^{u+1}\}$, u = 1, 2, …, N_s – 1 (Au and Beck 2001):

1. Generation of a candidate sample $\tilde{\boldsymbol{x}}^{u+1} = \{\tilde{x}_1^{u+1}, \tilde{x}_2^{u+1}, ..., \tilde{x}_j^{u+1}, ..., \tilde{x}_n^{u+1}\}$: for each parameter x_j, j = 1, 2, …, n:

   Sample a precandidate value ξ_j^{u+1} from $p_j^*(\cdot \mid x_j^u)$;

   a. Compute the acceptance ratio:

$$r_j^{u+1} = \frac{q_j(\xi_j^{u+1})}{q_j(x_j^u)} \tag{1.1'}$$

b. Set the new value $\tilde{x}_j^{u+1}$ of the j^{th} element of $\tilde{\boldsymbol{x}}^{u+1}$ as follows:

$$\tilde{x}_j^{u+1} = \begin{cases} \xi_j^{u+1} & \text{with probability } \min(1, r_j^{u+1}) \\ x_j^u & \text{with probability } 1 - \min(1, r_j^{u+1}) \end{cases} \tag{1.2'}$$

2. Acceptance/rejection of the candidate sample vector $\tilde{\boldsymbol{x}}^{u+1}$:

If $\tilde{\boldsymbol{x}}^{u+1} = \boldsymbol{x}^u$ (i.e., no precandidate values have been accepted), set $\boldsymbol{x}^{u+1} = \boldsymbol{x}^u$. Otherwise, check whether $\tilde{\boldsymbol{x}}^{u+1}$ is a system failure configuration, i.e. $\tilde{\boldsymbol{x}}^{u+1} \in F_i$: if it is, then accept the candidate $\tilde{\boldsymbol{x}}^{u+1}$ as the next state, i.e., set $\boldsymbol{x}^{u+1} = \tilde{\boldsymbol{x}}^{u+1}$; otherwise, reject the candidate $\tilde{\boldsymbol{x}}^{u+1}$ and take the current sample as the next one, i.e., set $\boldsymbol{x}^{u+1} = \boldsymbol{x}^u$.

In synthesis, a candidate sample $\tilde{\boldsymbol{x}}^{u+1}$ is generated from the current sample $\boldsymbol{x}^u$ and then either the candidate sample $\tilde{\boldsymbol{x}}^{u+1}$ or the current sample $\boldsymbol{x}^u$ is taken as the next sample $\boldsymbol{x}^{u+1}$, depending on whether the candidate $\tilde{\boldsymbol{x}}^{u+1}$ lies in the failure region F_i or not.

Finally, notice that in this work, the one-dimensional proposal PDF p_j^*, j = 1, 2, …, n, is chosen as a symmetrical uniform distribution centered at the current sample x_j, j = 1, 2, …, n, with width $2l_j$, where l_j is the maximum step length, i.e. the maximum allowable distance that the next sample can depart from the current one. The choice of l_j is such that the standard deviation of p_j^* is equal to that of q_j, j = 1, 2, …, n.

Appendix 2. The Line Sampling algorithm

The LS algorithm proceeds as follows (Pradlwarter et al. 2005):

1. Determine the *unit* important direction $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, ..., \alpha_j, ..., \alpha_n\}$. Any of the methods summarized in Section 1.2.2.2 can be employed to this purpose.
   Notice that the computation of $\boldsymbol{\alpha}$ implies additional system analyses, which substantially increase the computational cost associated to the simulation method (Section 1.2.2.2).
2. From the original multidimensional joint probability density function $q(\cdot): \Re^n \to [0, \infty)$, sample N_T vectors $\{\boldsymbol{x}^k : k = 1, 2, ..., N_T\}$, with $\boldsymbol{x}^k = \{x_1^k, x_2^k, ..., x_j^k, ..., x_n^k\}$ by standard MCS.
3. Transform the N_T sample vectors $\{\boldsymbol{x}^k : k = 1, 2, ..., N_T\}$ defined in the original (i.e., physical) space of possibly dependent, non-normal random variables (step 2. above) into N_T samples $\{\boldsymbol{\theta}^k : k = 1, 2, ..., N_T\}$ defined in the standard normal space where each component of the vector $\boldsymbol{\theta}^k = \{\theta_1^k, \theta_2^k, ..., \theta_j^k, ..., \theta_n^k\}$, k = 1, 2, ..., N_T, is associated with an independent central unit Gaussian standard distribution (Section 1.2.2.1).
4. Estimate N_T conditional "one-dimensional" failure probabilities $\{\hat{P}^k(F) : k = 1, 2, ..., N_T\}$, corresponding to each one of the standard normal samples $\{\boldsymbol{\theta}^k : k = 1, 2, ..., N_T\}$ obtained in step 3. above. In particular, for each random sample $\boldsymbol{\theta}^k$, k = 1, 2, …, N_T, perform the following steps (Figure 1.1') (Schueller et al. 2004; Pradlwarter et al. 2005; Pradlwarter et al. 2007):

   a. Define the sample vector $\widetilde{\boldsymbol{\theta}}^k$, k = 1, 2, ..., N_T, as the sum of a deterministic multiple of $\boldsymbol{\alpha}$ and a vector $\boldsymbol{\theta}^{k,\perp}$, k = 1, 2, ..., N_T, perpendicular to the direction $\boldsymbol{\alpha}$, i.e.,

$$\widetilde{\boldsymbol{\theta}}^k = c^k \boldsymbol{\alpha} + \boldsymbol{\theta}^{k,\perp}, \; k = 1, 2, ..., N_T \tag{1.3'}$$

   where c^k is a real number in [- , +] and

$$\boldsymbol{\theta}^{k,\perp} = \boldsymbol{\theta}^k - \langle \boldsymbol{\alpha}, \boldsymbol{\theta}^k \rangle \boldsymbol{\alpha}, \; k = 1, 2, ..., N_T \tag{1.4'}$$

   In (1.4'), $\boldsymbol{\theta}^k$, k = 1, 2, ..., N_T, denotes a random realization of the input variables in the standard normal space of dimension n and

$\left\langle \boldsymbol{\alpha}, \boldsymbol{\theta}^k \right\rangle$ is the scalar product between $\boldsymbol{\alpha}$ and $\boldsymbol{\theta}^k$, k = 1, 2, ..., N_T. Finally, it is worth noting that since the standard Gaussian space is isotropic, both the scalar c^k and the vector $\boldsymbol{\theta}^{k,\perp}$ are also standard normally distributed (Pradlwarter et al. 2007).

b. Compute the value $\bar{c}^k$ as the intersection between the limit state function $g_\theta\left(\tilde{\boldsymbol{\theta}}^k\right) = g_\theta\left(c^k \boldsymbol{\alpha} + \boldsymbol{\theta}^{k,\perp}\right) = 0$ and the line $l^k\left(c^k, \boldsymbol{\alpha}\right)$ passing through $\boldsymbol{\theta}^k$ and parallel to $\boldsymbol{\alpha}$ (Figure 1.1'). The value of $\bar{c}^k$ can be approximated by evaluating the performance function $g_\theta(\cdot)$ at two or three different values of c^k (e.g., c_1^k, c_2^k and c_3^k in Figure 1.1'), fitting a first or second order polynomial and determining its root (Figure 1.1'). Hence, for each standard normal random sample $\boldsymbol{\theta}^k$, k = 1, 2, …, N_T, two or three system performance evaluations by the model are required.

c. Solve the conditional one-dimensional reliability problem associated to each random sample $\boldsymbol{\theta}^k$, k = 1, 2, …, N_T, in which the only (standard normal) random variable is c^k. The associated conditional failure probability $\hat{P}^k(F)$, k = 1, 2, …, N_T, is given by

$$\begin{aligned} \hat{P}^k(F) &= P\left[N(0,1) > \bar{c}^k\right] = \\ &1 - P\left[N(0,1) \le \bar{c}^k\right] = \\ &1 - \Phi\left(\bar{c}^k\right) = \Phi\left(-\bar{c}^k\right) \end{aligned} \tag{1.5'}$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function.

5. Using the independent conditional "one-dimensional" failure probability estimates $\left\{\hat{P}^k(F) : k = 1, 2, ..., N_T\right\}$ in (1.5') (step 4.c. above), compute the unbiased estimator $\hat{P}(F)$ for the failure probability $P(F)$ as

$$\hat{P}(F) = \frac{1}{N_T} \sum_{k=1}^{N_T} \hat{P}^k(F). \tag{1.6'}$$

The variance of the estimator (1.6') is

$$\sigma^2\left(\hat{P}(F)\right) = \frac{1}{N_T\left(N_T - 1\right)} \sum_{k=1}^{N_T} \left(\hat{P}^k(F) - \hat{P}(F)\right)^2 \tag{1.7'}$$

With the described approach the variance of the estimator $\hat{P}(F)$ of the failure probability $P(F)$ is considerably reduced. In general, a relatively low number N_T of simulations has to be carried out to obtain a sufficiently accurate estimate. A single evaluation would suffice for the ideal case in which the limit state function is linear and a Line Sampling direction $\boldsymbol{\alpha}$ perpendicular to it has been identified (Koutsourelakis et al. 2004).

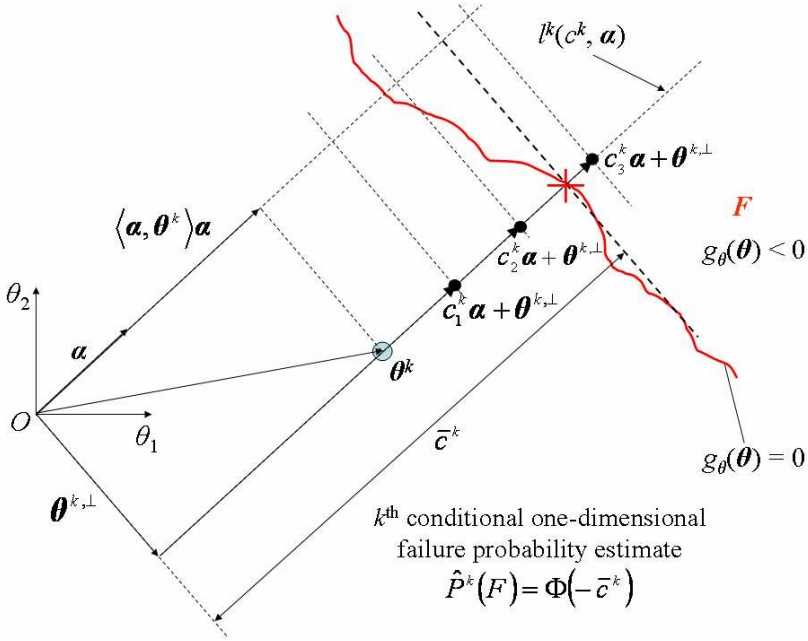


Fig. 1.1'. The Line Sampling procedure (Pradlwarter et al. 2005)