Robust tube-based constrained predictive control via zonotopic set-membership estimation

V.T.H. Le, C. Stoica, D. Dumur, T. Alamo and E.F. Camacho

Abstract— **This paper proposes an approach to deal with the problem of robust output feedback model predictive control for linear discrete-time systems subject to state and input constraints, in the presence of unknown but bounded disturbances and measurement noises. The estimation of the states is built using a zonotopic set-membership estimation. This set is time-decreasing and is computed off-line as the solution of a Linear Matrix Inequality optimization problem. The control law is designed by using tube-based model predictive control such that the closed-loop stability is guaranteed and the state and input constraints are fulfilled. The proposed methodology is illustrated through numerical simulations.**

I. Introduction

Model Predictive Control (MPC) can be considered today as a mature domain, both from the research and the industrial applications points of view. This is mainly due to its ability to handle hard constraints, which always appear when dealing with real plants [1]. MPC is based on the receding horizon strategy: an optimal control problem is solved at each time instant to find a sequence of control input and then only the first part of the control sequence is applied. The robustness of MPC in the presence of uncertainties and disturbances is studied by many authors [2], [3], [4], [5], [6]. The simplest ways to enhance robustness are based on the deterministic model predictive control by ignoring the disturbance over the prediction horizon [2], [3]. These methods require full knowledge of the state which usually cannot be reached due to measurement noises acting on real systems. In this case, the Luenberger observer and the Kalman filter can be used to estimate the state of the system [4], [6], [7]. The Kalman filter [8] is based on probabilistic assumptions about the perturbations and the noises which sometimes are difficult to validate. In [4] the authors built a Luenberger observer that contains a correction term computed by a gain and the difference between the actual measurement and the actual state estimation. Then a tube-based model predictive control is designed, replacing in the optimization problem the true states by the nominal estimated states. The state estimation error lies in a time varying compact set which converges to the minimal robust positive invariant set. The design of this gain plays an important role in the observer (and thus in the control performance) because it determines the convergence speed and also the size of the minimal robust positive invariant set.

When dealing with bounded disturbances and measurement noises or uncertain system, a substitute for Luenberger observer and Kalman filter is to use a set-membership estimation. The set-membership estimation has been developed in the last 40 years [9], [10]. This method relies on the description of uncertainties belonging to bounded compact sets. The state estimation set is a compact set containing all possible states of the system that are consistent with the uncertain model and the measurement noise. This set is built using some simple geometrical forms such as polytopes [11], parallelotopes [12] [5], ellipsoids [9], [10] or zonotopes [13], [14]. The set-membership estimation is proved to be robust with respect to disturbances, measurement noises and even in the presence of model uncertainties, which Luenberger observer and Kalman filter cannot cover. On the contrary the computation burden is more important for set-membership estimation than for the Luenberger observer or the Kalman filter. The set-membership estimation has been proposed also for output feedback MPC but the research remains limited due to its complexity [5], [15].

This paper considers the problem of robust output feedback MPC for constrained linear discrete-time systems subject to state disturbance and measurement noise. The main contribution of this paper is a combination between the set-membership estimation and the robust tube-based model predictive control. The set-membership estimation is used here in the perspective of future application of tube model predictive control for systems with interval uncertainties. When the set-membership estimation is used, a trade-off between the precision and the computation burden has to be achieved. In a linear formulation, polytopes can be used for an exact representation of the domains of the system state. However efficient results may be obtained only for a reasonable number of vertices of the polytopes [11]. To overcome this drawback, the representation by ellipsoids has been used in the literature, sometimes with a significant loss of performance [9], [10]. In this paper, the zonotopic form is chosen as a compromise for the set-membership estimation due to its better precision in comparison with the ellipsoidal form and less complexity compared to polytopic form. The zonotopic sets are computed off-line and the size of the zonotope is time decreasing. If the center of the zonotope can be stabilizable to the origin then the system has the propriety of input-to-state stability. Based on this idea, a tube predictive control is proposed to steer the nominal system to the origin, so that the real system state converges to a

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neighborhood of the origin in a similar way as in [3], [4].

The paper is organized as follows. Section II presents useful mathematical notations and basic definitions. Section III focuses on the problem formulation. In section IV, the zonotopic set-membership estimation of a linear discrete-time system is defined. The next section presents the robust tube-based model predictive control and its properties. An example is proposed in the Section VI in order to show the advantages of the developed methodology. Finally, some concluding remarks and future work are presented.

II. Mathematical notations and basic definitions

An *interval* $[a;b]$ is defined as the set $\{x : a \leq x \leq b\}$. The *unitary interval* is **B** = [-1;1].

A *box* $([a_1;b_1],...,[a_n;b_n])^T$ is an interval vector. A *unitary box* in $\mathbb{R}^m$, denoted $\mathbf{B}^m$, is a box composed by m unitary intervals.

The *Minkowski sum* and *Pontryagin difference* of two sets X and Y are defined by $X \oplus Y = \{x+y : x \in X, y \in Y\}$ and $X \ominus Y = \{z | z \oplus Y \subseteq X\}$ respectively.

A *polytope* Ω is the convex hull of its vertices $\Omega = Co\{v_1; v_2; ...; v_n\}$. This means that if $v \in \Omega$ then $v = \sum_{i=1}^{n} \alpha_i v_i$ with $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0$ for $i = 1, ..., n$.

Zonotopes are a special class of convex polytopes. A m-zonotope in $\mathbb{R}^n$ can be defined as the linear image of a m-dimensional hypercube in $\mathbb{R}^n$. Given a vector $p \in \mathbb{R}^n$ and a matrix $H \in \mathbb{R}^{n\times m}$, a m-zonotope is the set: $p \oplus H\mathbf{B}^m = \{p + Hz, z \in \mathbf{B}^m\}$. This is the Minkowski sum of the m-segments defined by m columns of matrix H in $\mathbb{R}^n$.

The *P-radius* of a zonotope $X = p \oplus H\mathbf{B}^m$ is defined as $d(x) = \max(\|x - p\|_P^2)$, $x \in X$. This notion is related to the ellipsoid $(x-p)^T P (x-p) \leq 1$.

A *strip* X is defined as the set $\{x \in \mathbb{R}^n : |c^T x - d| \leq \sigma\}$ with $c \in \mathbb{R}^n$, d, $\sigma \in \mathbb{R}$.

A matrix $M = M^T \in \mathbb{R}^{n\times n}$ is called a *positive-definite matrix* (respectively *negative-definite matrix*), denoted $M \succeq 0$ ($M \preceq 0$), if $z^T M z \geq 0$ ($z^T M z \leq 0$) for all non-zero vectors z with real entries ($z \in \mathbb{R}^n$).

A set $X \subset \mathbb{R}^n$ is called a *C-set* if X is compact, convex and contains the origin. This is a proper set if its interior is not empty.

Property 1: [13] Given two centered zonotopes $Z_1 = H_1\mathbf{B}^{m_1} \in \mathbb{R}^n$ and $Z_2 = H_2\mathbf{B}^{m_2} \in \mathbb{R}^n$. The Minkowski sum of two zonotopes is also a zonotope defined by $Z = Z_1 \oplus Z_2 = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \mathbf{B}^{m_1+m_2}$.

Property 2: [13] The image of a centered zonotope $Z_1 = H_1\mathbf{B}^{m_1} \in \mathbb{R}^n$ by a linear application K can be computed by a standard matrix product $K \cdot Z_1 = (K \cdot H_1)\mathbf{B}^{m_1}$.

Property 3: (Zonotope reduction) [13], [14] Given the zonotope $Z = p \oplus H\mathbf{B}^m \in \mathbb{R}^n$ and the integer s with $n < s < m$, denote $\hat{H}$ the matrix resulting from the reordering of the columns of the matrix H in decreasing order of Euclidean norm ($\hat{H} = \begin{bmatrix} \hat{h}_1 ... \hat{h}_i ... \hat{h}_m \end{bmatrix}$ with $\|\hat{h}_i\|_2 \geq \|\hat{h}_{i+1}\|_2$). Then $Z \subseteq p \oplus \begin{bmatrix} \hat{H}_T & Q \end{bmatrix} \mathbf{B}^s$ where $\hat{H}_T$ is obtained from the first $s-n$ columns of matrix $\hat{H}$ and $Q \in \mathbb{R}^{n\times n}$ is a diagonal matrix that satisfies $Q_{ii} = \sum_{j=s-n+1}^{m} |\hat{H}_{ij}|$, with $i \in \mathbb{N}_{[1,n]}$ and $\mathbb{N}_{[1,n]} = \{1, 2, ..., n\}$.

III. Problem formulation

Consider the following linear discrete-time invariant system of the form:

$$\begin{cases} x_{k+1} = Ax_k + Bu_k + F\omega_k \\ y_k = c^T x_k + \sigma v_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state of the system, $y_k \in \mathbb{R}$ is the measured output at sample time k. The vector $\omega_k \in \mathbb{R}^{n_\omega}$ represents the state perturbation vector and $v_k \in \mathbb{R}$ is the measurement perturbation (noise, offset, etc.). It is assumed that the uncertainties and the initial state are bounded by zonotopes: $\omega_k \in W, v_k \in V$ and $x_0 \in X_0$ that contain the origin. W and V are assumed to be unitary boxes.

Consider the system (1) subject to constraints on state and input: $x_k \in X$, $u_k \in U$, where X and U are two compact and convex sets containing the origin as an interior point. It is assumed that system (1) is controllable and observable.

With all these assumptions, it is clear that the state vector is not exactly known at each sampling time. A set-membership estimation is further used in order to find the zonotope containing the true state at a given moment. The size of the zonotope is decreasing at each sampling time, leading to a more accurate estimation. Using the estimated zonotope of the true state, a tube-based model predictive control is then developed in order to stabilize the considered system (subject to bounded disturbances, bounded noises and constraints). As the state is known to belong to a zonotope with a given center and radius, it is convenient to control the center of the zonotope in order to satisfy the stability of the considered system.

IV. Guaranteed state estimation

Definition 1: Given the system (1) without any control and with a measured output y_k, the *consistent state set* at time k is defined as $X_{y_k} = \{x_k \in \mathbb{R}^n : |c^T x_k - y_k| \leq \sigma\}$.

Definition 2: Consider the system (1). The *exact uncertain state set* X_k is equal to the set of states that are consistent with the measured output and the initial state set X_0: $X_k = (AX_{k-1} \oplus FW) \cap X_{y_k}$, for $k \geq 1$.

Remark: The exact computation of this set is difficult. In order to reduce the complexity of the computation, these sets are bounded by mean of conservative outer bounds. Let us consider that an outer bound of the exact uncertain state set denoted $\hat{X}_{k-1}$ is available at time instant $k-1$. Suppose also that a measured output y_k is obtained at time instant k. Under these assumptions, an outer bound of the exact uncertain state set can be estimated using the following algorithm.

Algorithm 1

Step 1: (*Prediction step*) Given the system (1), compute a zonotope $\bar{X}_k$ that offers a bound for the uncertain trajectory of the system ($\bar{X}_k = A\hat{X}_{k-1} \oplus FW$).

Step 2: (*Measurement*) Compute the consistent state set X_{y_k} by using the measurement. According to the assumption on v_k this set can be represented by a strip as

$\{x \in \mathbb{R}^n : |c^T x - y_k| \leq \sigma\}$.

Step 3: (*Correction step*) In order to find the state estimation set, compute an outer approximation $\hat{X}_k$ of the intersection between X_{y_k} and $\bar{X}_k$.

The proposed algorithm is similar to the Kalman filter: the first step is a prediction step, while the second and third steps constitute a correction step. To obtain a zonotope bounding the uncertain trajectory of the system, Properties 1 and 2 are used. The complexity of this zonotope is limited by using Property 3. To compute the intersection set of step 3 an optimization problem will be detailed. As $\bar{X}_k$ is a zonotope and X_{y_k} is a strip, it is convenient to obtain an outer bound of the intersection of a zonotope and a strip.

The next property provides a family of zonotopes (parameterized by the vector λ) that contains the intersection of a zonotope and a strip.

Property 4: [14] Given the zonotope $X = p \oplus H\mathbf{B}^r \subset \mathbb{R}^n$, the strip $S = \{x \in \mathbb{R}^n : |c^T x - d| \leq \sigma\}$ and the vector $\lambda \in \mathbb{R}^n$, define a vector $\hat{p}(\lambda) = p + \lambda(d - c^T p) \in \mathbb{R}^n$ and a matrix $\hat{H}(\lambda) = [(I - \lambda c^T)H \ \sigma\lambda] \in \mathbb{R}^{n\times(m+1)}$. Then the following expression holds $X \cap S \subseteq \hat{X}(\lambda) = \hat{p}(\lambda) \oplus \hat{H}(\lambda)B^{r+1}$.

Proof:

Supposing an element $x \in X \cap S$, on one hand this means that $x \in X = p \oplus H\mathbf{B}^r$. Using the definition of a m-zonotope implies that there exists a vector $z \in \mathbf{B}^r$ such that

$$x = p + Hz \tag{2}$$

Adding and subtracting $\lambda c^T Hz$ to the previous equality leads to the following expression:

$$x = p + \lambda c^T Hz + (I - \lambda c^T)Hz \tag{3}$$

On the other hand, from $x \in X \cap S$ it is inferred that $x \in S = \{x \in \mathbb{R}^n : |c^T x - d| \leq \sigma\}$. Thus, there exists $\omega \in [-1; 1]$ such that $c^T x - d = \sigma\omega$. Taking into account the form of the vector x given by (2) leads to $c^T(p + Hz) - d = \sigma\omega$, which is equivalent to $c^T Hz = d - c^T p + \sigma\omega$. Substituting $c^T Hz$ in equation (3), the following expression is obtained:

$$\begin{aligned} x &= p + \lambda(d - c^T p + \sigma\omega) + (I - \lambda c^T)Hz \\ &= p + \lambda(d - c^T p) + \lambda\sigma\omega + (I - \lambda c^T)Hz \end{aligned} \tag{4}$$

After simple computations and using the notation defined in Property 4, the following form is obtained:

$$x = \hat{p}(\lambda) + \left[(I - \lambda c^T)H \quad \sigma\lambda\right] \begin{bmatrix} z \\ \omega \end{bmatrix} = \hat{p}(\lambda) \oplus \hat{H}(\lambda) \begin{bmatrix} z \\ \omega \end{bmatrix} \tag{5}$$

and the following inclusion holds:

$$x = \hat{p}(\lambda) \oplus \hat{H}(\lambda) \begin{bmatrix} z \\ \omega \end{bmatrix} \in \hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+1} = \hat{X}(\lambda). \tag{6}$$

□

In order to choose λ, two approaches were presented in [14]:

1) Minimizing the segments of the zonotope offers a fast computation but with a loss of performance for the estimation;
2) Minimizing the volume of the intersection leads to more accurate results, but at each sample time an optimization problem must be solved.

In this section, an approach that offers both good performance and a fast computation time is proposed. This corresponds in fact to a new method used during the correction step of the algorithm proposed in Section 3. A different criterion is presented to compute the vector λ in order to overcome the drawbacks of the two mentioned methods.

Suppose an outer approximation of state set $\hat{X}_k = p \oplus H\mathbf{B}^r$ at the time instant k and the measured output $d = y_{k+1}$ at the instant $k+1$. The predicted state set at the next instant $\bar{X}_{k+1}$ can be computed using (1), Property 1 and Property 2:

$$\bar{X}_{k+1} = Ap \oplus \begin{bmatrix} AH & F \end{bmatrix} \mathbf{B}^{r+n_\omega} \tag{7}$$

The results stated by Property 4 allow the computation of an outer approximation of the intersection (exact estimation set) between the predicted state set and the strip (which represents the measured output):

$$\hat{X}(\lambda) = \hat{p}(\lambda) \oplus \hat{H}(\lambda)\mathbf{B}^{r+n_\omega+1} \tag{8}$$

with $\hat{p}(\lambda) = Ap + \lambda(d - c^T Ap)$
and $\hat{H}(\lambda) = \left[(I - \lambda c^T)\begin{bmatrix} AH & F \end{bmatrix} \quad \sigma\lambda\right]$.

In order to compute the vector λ, the approach considered in this paper is the following: compute a symmetric definite positive matrix P and a vector λ such that at each sample time, the P-radius of this zonotopic state estimation set is decreased. This means that the zonotopic state estimation set is contracted in time. As F (included in $\hat{H}_{k+1}(\lambda)$) represents a fixed part of the zonotope which can not be decreased at each iteration, it is allowed to exclude F from the last condition. This condition can be expressed in a mathematical formulation as follows:

$$\max_{\tilde{z},\eta}(\|\tilde{H}\begin{bmatrix} \tilde{z} \\ \eta \end{bmatrix}\|_P^2) \leq \max_{z}(\beta\|Hz\|_P^2) + \sigma^2 \tag{9}$$

with $\tilde{z}, z \in \mathbf{B}^r$, $\eta \in \mathbf{B}^1$, $\beta \in [0, 1)$ and $\tilde{H} = \left[(I - \lambda c^T)AH \quad \sigma\lambda\right]$. This can be rewritten as:

$$\max_{\tilde{z},\eta}(\|\tilde{H}\begin{bmatrix} \tilde{z} \\ \eta \end{bmatrix}\|_P^2) - \max_{z}(\beta\|Hz\|_P^2) - \sigma^2 \leq 0 \tag{10}$$

If the following expression is true then the expression (10) is also true:

$$\max_{\tilde{z},\eta}(\|\tilde{H}\begin{bmatrix} \tilde{z} \\ \eta \end{bmatrix}\|_P^2 - \beta\|H\tilde{z}\|_P^2) - \sigma^2 \leq 0 \tag{11}$$

Using the definition $\eta \in \mathbf{B}^1$ so that $\|\eta\|_\infty \leq 1$, the following expression holds $\sigma^2(1 - \eta^2) \geq 0$. Adding this term to the left hand side of (11) leads to:

$$\max_{\tilde{z},\eta}(\|\tilde{H}\begin{bmatrix} \tilde{z} \\ \eta \end{bmatrix}\|_P^2 - \beta\|H\tilde{z}\|_P^2) - \sigma^2 + \sigma^2(1 - \eta^2) \leq 0 \tag{12}$$

or to a more compact form:

$$\begin{bmatrix} \tilde{z} \\ \eta \end{bmatrix}^T \tilde{H}^T P \tilde{H} \begin{bmatrix} \tilde{z} \\ \eta \end{bmatrix} - \beta\tilde{z}H^T PH\tilde{z} - \sigma^2\eta \leq 0, \ \forall \begin{bmatrix} \tilde{z} \\ \eta \end{bmatrix} \tag{13}$$

with $\tilde{H} = \begin{bmatrix}(I - \lambda c^T)AH & \sigma\lambda\end{bmatrix}$ as defined before. Denoting $\gamma = H\tilde{z}$ then the inequality (13) can be written in the matrix formulation:

$$\begin{bmatrix}\gamma\\ \eta\end{bmatrix}^T \begin{bmatrix}A_{11} & A_{12}\\ * & A_{22}\end{bmatrix} \begin{bmatrix}\gamma\\ \eta\end{bmatrix} \leq 0,\ \ \forall \begin{bmatrix}\gamma\\ \eta\end{bmatrix} \neq 0 \qquad (14)$$

with '*' denoting the terms required for the symmetry of the matrix and the following additional notations:

$$\begin{cases} A_{11} = ((I - \lambda c^T)A)^T P((I - \lambda c^T)A) - \beta P\\ A_{12} = ((I - \lambda c^T)A)^T P\sigma\lambda\\ A_{22} = \sigma^2\lambda^T P\lambda - \sigma^2 \end{cases} \qquad (15)$$

Using the definition of a positive-definite matrix allows to rewrite (14) as:

$$\begin{bmatrix}A_{11} & A_{12}\\ * & A_{22}\end{bmatrix} \preceq 0 \qquad (16)$$

which is further equivalent to:

$$\begin{bmatrix}-A_{11} & -A_{12}\\ * & -A_{22}\end{bmatrix} \succeq 0 \qquad (17)$$

Using the explicit notations (15) and doing some manipulations in (17) a BMI (Bilinear Matrix Inequality) is derived:

$$\begin{bmatrix}\beta P & 0\\ * & \sigma^2\end{bmatrix} - \\ - \begin{bmatrix}(A^T - A^T c\lambda^T)P\\ \lambda^T P\sigma\end{bmatrix} P^{-1} \begin{bmatrix}(A^T - A^T c\lambda^T)P\\ \lambda^T P\sigma\end{bmatrix}^T \succeq 0 \qquad (18)$$

Using the Schur complement [16], it is equivalent to the following BMI:

$$\begin{bmatrix}\beta P & 0 & (A^T - A^T c\lambda^T)P\\ * & \sigma^2 & \lambda^T P\sigma\\ * & * & P\end{bmatrix} \succeq 0 \qquad (19)$$

with β, P, λ as decision variables.

Denote the P-radius of the state estimation set at instant k as $d_k(x) = \max(\|x - p_k\|_P^2)$ where $x \in \hat{X}_k$. Then the condition (9) can be written as $d_{k+1}(x) \leq \beta d_k(x) + \sigma^2$. At infinity, this expression is equivalent to:

$$d_\infty(x) = \beta d_\infty(x) + \sigma^2 \qquad (20)$$

leading to

$$d_\infty(x) = \frac{\sigma^2}{1-\beta} \qquad (21)$$

Let us consider an ellipsoid $E = \{x : x^T P x \leq \frac{\sigma^2}{1-\beta}\}$ which is equivalent to $E = \{x : x^T \frac{(1-\beta)P}{\sigma^2} x \leq 1\}$. In order to minimize the radius (L_∞) of the zonotope, the ellipsoid of smallest diameter must be found [16]. The following EVP (eigenvalue problem) has to be solved to find the values of $P = P^T \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}^n$.

Algorithm 2

Find the smallest value of $\beta \in [0, 1)$ using the bisection algorithm [17] such that the following optimization is feasible:

$$\max_{\tau,P,Y} \tau$$

subject to the LMIs

$$\begin{cases} \frac{(1-\beta)P}{\sigma^2} \succeq \tau I, \quad \tau \geq 0\\ \begin{bmatrix}\beta P & 0 & A^T P - A^T c Y^T\\ * & \sigma^2 & Y^T\sigma\\ * & * & P\end{bmatrix} \succeq 0 \end{cases} \qquad (22)$$

with $Y = P\lambda$.

Remark: Using this algorithm, the size of the state estimation set is decreased in time. Denote $\hat{X}_\infty = p_\infty \oplus H_\infty B^q$ where q is chosen in accordance with the dimension of matrix H_∞ the state estimation set at infinity. If a control law is used to steer the center of the zonotopic estimation set to the origin then the state system is steered to the zonotope $H_\infty B^q$ containing the origin. The system is stable in the sense of input-to-state stability.

V. Tube-based output feedback MPC design

The output feedback MPC controller designed has the form $u_k = K\hat{x}_k + c_k$ [3], [4], [18]. Considering $\hat{x}_k$ as the center of the zonotopic estimation set at instant k and using the Property 4 and (8), the following state equation can be obtained:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \lambda(y_{k+1} - c^T(A\hat{x}_k + Bu_k)) \qquad (23)$$

Using (1) and doing some manipulations, the following expression is obtained:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \omega_k^{co} \qquad (24)$$

with $\omega_k^{co} = \lambda(c^T A(x_k - \hat{x}_k) + c^T F\omega_k + \sigma v_{k+1})$. Denoting $S_k^e = \hat{X}_k \ominus p_k$, the input disturbance ω_k^{co} is bounded by $\mathbb{W}_k^{co} = \lambda c^T A S_k^e \oplus \lambda c^T W \oplus \lambda V$. Note that for simplicity reason the same notations as in [4] have been used. Considering now the nominal system which is not affected by disturbances:

$$\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k \qquad (25)$$

To counteract the disturbances, the trajectory is desired to lie close to the nominal trajectory. If the nominal system is steered to the origin then the center of the zonotopic state estimation is bounded by a compact set and so is the real state of the system. This can be done by using the control u defined as:

$$u = \bar{u} + K(\hat{x} - \bar{x}) \qquad (26)$$

where $\bar{u}$ is the control law applied to the nominal system (25). The error between the estimation state and the nominal state $e_k = \hat{x}_k - \bar{x}_k$ satisfies the difference equation: $e_{k+1} = A_k e_k + w_k^{co}$, $A_k = A + BK$. The matrix K is chosen such that A_k is stable ($\rho(A_k) < 1$). Consequently, if at time k, e_k lies in the set S_k^{co}, then e_{k+1} lies in the set $S_{k+1}^{co} = A_k S_k^{co} \oplus \mathbb{W}_k^{co}$. Using the vector λ, the size of S_k^e is decreased in time and also the size of S_k^{co}. These sets converge respectively to S_∞^e and S_∞^{co}. Define $S_k = S_k^e \oplus S_k^{co}$, then S_k tends to $S_\infty = S_\infty^e \oplus S_\infty^{co}$.

The robust tube MPC can be summarized as follows. At time k a state estimation set is computed and a nominal optimal control problem is solved on-line. Define the cost function for the nominal system as:

$$V_N(\bar{x}, \bar{u}) = V_f(\bar{x}_N) + \sum_{i=0}^{N-1} l(\bar{x}_i, \bar{u}_i) \tag{27}$$

where N is the prediction horizon, $\bar{u} = \{\bar{u}_0, ..., \bar{u}_{N-1}\}$, $\bar{x}_{i+1} = A\bar{x}_i + B\bar{u}_i$. If the current time is k, $\bar{x}_i$ and $\bar{u}_i$ are the predicted state and control at time $k + i$. The stage cost function and the terminal cost function are defined by: $l(x, u) = 0.5(x^T Qx + u^T Ru)$, $V_f(x) = 0.5x^T P_f x$ where P_f, Q, R are positive definite matrices. With this notations, the time varying constraints at current time k are:

$$\begin{cases} \bar{u}_i \in \bar{U}_{k+i},\ i \in \mathbb{N}_{[0,N-1]} \\ \bar{x}_i \in \bar{X}_{k+i},\ i \in \mathbb{N}_{[0,N-1]} \\ \bar{x}_N \in \bar{X}_f \end{cases} \tag{28}$$

with $\bar{U}_{k+i} = U \ominus KS^{co}_{k+i}$, $\bar{X}_{k+i} = X \ominus S_{k+i}$. To ensure the feasibility and stability of this control law the following conditions are assumed [4].

Assumption 1: Consider $S_0 = S^{co}_0 \oplus S^e_0 \subset X$ and $KS^{co}_0 \subset U$.

Assumption 2: $\bar{X}_f$ is a proper C-set, is positively invariant for $\bar{x}_{k+1} = A_k\bar{x}_k$ and satisfies $\bar{X}_f \subseteq \bar{X}_N$ and $K\bar{X}_f \subseteq \bar{U}_N$.

Assumption 3: $V_f(.)$ is a local control Lyapunov function for $\bar{x}_{k+1} = A_k\bar{x}_k$ for all $\bar{x} \in \bar{X}_f$. There exist constant c_1, $c_2 > 0$ such that $c_1|\bar{x}|^2 \leq V_f(\bar{x}) \leq c_2|\bar{x}|^2$ and $V_f(A_k\bar{x}) + l(\bar{x}, K\bar{x}) \leq V_f(\bar{x})$. This means that the Lyapunov function is decreased at next sampling time.

Denote the set of admissible control sequences at instant k, with the nominal state $\bar{x}$:

$$\mathcal{U}_N(\bar{x}, k) = \{\bar{u}_i : \bar{u}_i \in \bar{U}_{k+i}, \bar{x}_i \in \bar{X}_{k+i}, \bar{x}_N \in \bar{X}_f, \\ i \in \mathbb{N}_{[0,N-1]}\} \tag{29}$$

The optimal control problem solved on-line is:

$$V^*_N(\hat{x}, k) = \min_{\bar{x},\bar{u}} \{V_N(\bar{x}, \bar{u}) : \bar{u} \in \mathcal{U}_N(\bar{x}, k), \hat{x} \in \bar{x} \oplus S^{co}_k\} \tag{30}$$

Consider:

$$\bar{x}^*(\hat{x}, k), \bar{u}^*(\hat{x}, k) = arg \min_{\bar{x},\bar{u}} \{V_N(\bar{x}, \bar{u}) : \bar{u} \in \mathcal{U}_N(\bar{x}), \\ \hat{x} \in \bar{x} \oplus S^{co}_k\} \tag{31}$$

then the control law applied to the system is obtained:

$$\kappa_N(\hat{x}, k) = \hat{u}^*(0, \hat{x}, k) + K(\hat{x}_k - \bar{x}^*(\hat{x}, k)) \tag{32}$$

with $\hat{u}^*(0, \hat{x}, k)$ the first element of the sequence $\bar{u}^*(\hat{x}, k)$.

Using this control law it can be proved that $(x, \hat{x})$ is robustly steered to $S_\infty \times S^{co}_\infty$ exponentially fast satisfying all constraints [4].

VI. Illustrative example

Consider a second-order system:

$$x_{k+1} = \begin{bmatrix} 1 & 1.1 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k + \omega_k$$

$$y_k = \begin{bmatrix} -2 & 1 \end{bmatrix} x_k + v_k$$

with the disturbances $(\omega, v) \in W \times V$ where $W = \{\omega \in \mathbb{R}^2 : \|\omega\|_\infty \leq 0.1\}$ and $V = \{v \in \mathbb{R} : |v| \leq 0.05\}$. The state and control constraints are $(x, u) \in X \times U$ where $X = \{x \in \mathbb{R}\}^2 : x_1 \in [-50, 3], x_2 \in [-50, 3]\}$ and $U = \{u \in \mathbb{R} : |u| \leq 9\}$. The feedback control matrix is $K = \begin{bmatrix} -0.618 & -1 \end{bmatrix}$. The weighting matrices in the cost function are $Q = I$, $R = 0.01$. The terminal cost $V_f(\bar{x})$ is the value function $\bar{x}^T P_f x$ for the unconstrained optimal control problem for the nominal system $\bar{x}_{k+1} = A\bar{x}_k + B\bar{u}_k$ and $\bar{u}_k = K_f\bar{x}_k$ is the associated LQR control. The initial sets S^e_0, S^{co}_0 are computed using the result in [19]. The terminal constraint set $\bar{X}_f$ (the black set depicted in Figure 3) is the maximal positively invariant set for the system $\bar{x}_{k+1} = (A + BK_f)\bar{x}_k$ under the tighter constraints $\bar{X}_N = X \ominus S_N$ and $\bar{U}_N = U \ominus KS^{co}_N$ [20], [21]. The prediction horizon is $N = 13$. The LMI optimization (22) is solved by using the LMI toolbox of $Matlab^{®}$. Figures 1 and 2 compare the state estimation sets of three approaches: the segment minimization, the volume minimization and the P-radius minimization. These figures show the advantage of proposed estimation method: the performance is better than the segment minimization method and as well as the volume minimization method. In addition, Table I offers a comparison of the computation time (using an Intel Core 2 Duo E8500 3.16 GHz). The conclusion is that the computation time of the proposed method is the same as the segment minimization method and faster than the volume minimization method.

Figure 3 shows the tube trajectory of the system. The largest zonotope (red) is the set $\bar{x}^*(\hat{x}_k, k) \oplus S_k$, the smaller zonotope (green) is the set $\bar{x}^*(\hat{x}_k, k) \oplus S^{co}_k$ and the smallest (blue) is the guaranteed state estimation set. Figure 4 shows the stability of this output feedback system respecting the constraints.

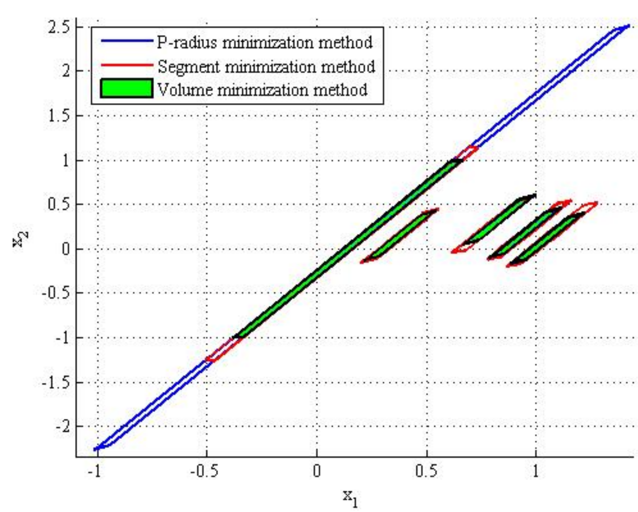

Fig. 1. Evolution of the state estimation set

VII. Conclusion

This paper presents a solution of robust output feedback MPC for linear systems subject to state and input constraints with bounded state disturbances and measurement noises. It

TABLE I

COMPUTATION TIME AFTER 50 SAMPLE TIMES

Algorithm	Time(second)
Segment minimization	0.0312
Presented algorithm (without LMI optimization)	0.0312
Presented algorithm (with LMI optimization)	0.7488
Volume minimization	7.8469

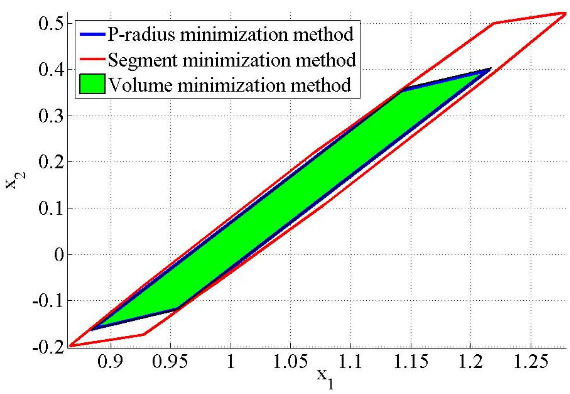

Fig. 2. Evolution of the state estimation set (zoom of Fig. 1)

combines an off-line zonotopic set-membership estimation with a robust tube-based constrained MPC controller. The state and input constraints are satisfied and the stability of the closed-loop is guaranteed. This method will be extended in future work for linear systems with interval uncertainties.

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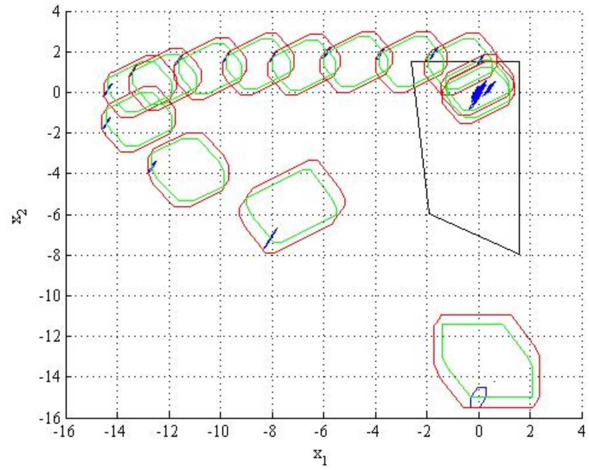

Fig. 3. Tube trajectory of the closed-loop response of the system

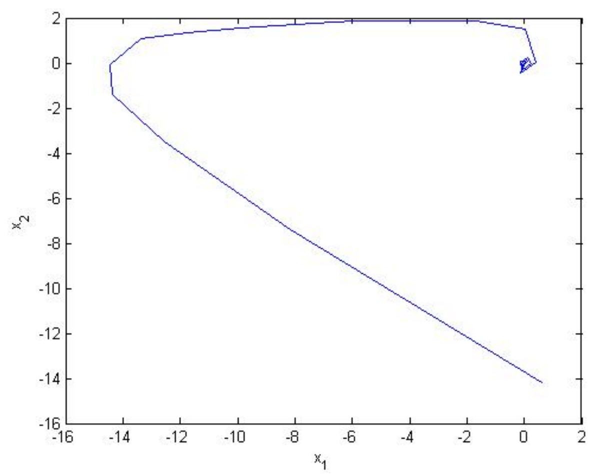

Fig. 4. Closed-loop response of the system

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