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Abstract—**New results on high gain observer design for networked control systems via an emulation-like approach are presented. By using a general framework and a Lyapunov approach, we derive some explicit conditions on the maximum allowable transmission interval that ensure an exponential convergence of the observation error for a large class of network protocols.**

Index Terms—**High gain observer, networked control systems (NCSs), time-delay systems.**

I. Introduction

The control and observation of systems called *Networked control systems* (NCSs) are currently attracting a lot of attention in the control community. In many applications, the interest for NCSs is motivated by many advantages they offer such as the ease of maintenance and installation, the greater flexibility and the low cost. For these reasons, many industrial control applications use a serial communication channel to connect sensors and controllers. In NCSs, the serial communication channel has many nodes (sensors and actuators) but the signals of these nodes cannot be transmitted at the same time. The rule that selects which node will use the network to transmit its data, is called scheduling network protocol. In the present letter, only communication constraints on sensors are considered. It is well known that the stability of these systems is largely determined by the transmission protocol used and by the so-called maximum allowable transfer interval (MATI), i.e., the maximum allowable time between any two successive transmissions in the network. There exist several families of protocols in the literature, but in this study, only Uniformly Globally Exponentially Stable (UGES) protocols are considered. This class includes several protocols, such Round Robin (RR), Try-Once-Discard (TOD) and sampled data systems. For further details on NCSs, protocols and their properties, see the overview papers [1]–[4].

The present note is focused on the implementation of a class of nonlinear observers on NCSs. Although the control of these systems has received considerable attention in the recent years, only few papers which deal with the observer design problem exist in the literature. For example, in deterministic case, the authors in [5] derived some conditions in order to maintain the observability of discrete-time linear systems under network communications. In [6], the authors proposed a method for the mutual design of protocols and observers for linear systems. The considered protocol gives the priority to the node where the error, between the measure and the estimate is the biggest, according to some weighted norms. Sufficient conditions for quadratic stability properties are then given using matrix inequalities. In [7], the authors consider the design of a linear observer for NCSs in the presence of time delay and sampling phenomenons without considering the constraints introduced by the protocols. They give sufficient conditions guaranteing asymptotic stability, by using a *Lyapunov-Krasovskii* approach. Recently, a small gain approach was used in [8] and [9] for the observers design via an emulation-like approach for some classes of networked control systems. The results contained in [8] and [9] state that if the observer has some robustness properties with respect to measurements errors then, for sufficiently small MATI, the stability in an appropriate sense of the observation error is guaranteed for several classes of network protocols. On the other hand, even without disturbances, the framework developed in [9] does not allow asymptotic stability of the observation error when the network is in zero order hold (ZOH) fashion. In the present study, we focus on a class of nonlinear triangular systems without considering disturbances. More precisely, we provide a framework and use a Lyapunov approach to guarantee an exponential stability of the observation error and improve the bounds of MATI compared to small gain approaches. We specially show that by using the framework presented below and a suitable *Lyapunov-Krasovskii functional*, we can derive an exponential stability of the observation error when the network is in ZOH fashion. Note that the results presented in the present letter can be easily extended to several classes of observers such, for example, linear observers and the class of observers described in [10]. The present note is organized as follows:

In Section II, we present the notations which will be used throughout the letter. The framework that we propose is formulated in Section III. In Section IV, we present our results on the emulation of high gain observers on a class of networks.

II. Notations and Preliminaries

First some mathematical notations are introduced. Let $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = [0, \infty)$ and $\mathbf{N}$ is the set of natural numbers. The notation $\mathbf{N}^*$ denotes the set of strictly positive integers. The usual euclidian norm of any vector v will be noted by $|v|$ and the identity matrix of size p is denoted by I_p. The matrix 0_p is a square matrix of size p with all values equal to zero. The notations $\lambda_{min}(S)$ and $\lambda_{max}(S)$ denote respectively the minimum and maximum eigenvalues of the square matrix S. A continuous function $\Gamma : \mathbf{R}_+ \rightarrow \mathbf{R}_+$

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is said to be of class $\mathcal{K}$, if it is zero at zero, strictly increasing. It is of class $\mathcal{K}_\infty$, if it is of class $\mathcal{K}$ and unbounded. A continuous function $\varphi : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ is said to be of $class - \mathcal{KL}$, if $\varphi(.,t)$ is of class $\mathcal{K}$ for each $t \geq 0$, and $\varphi(s,.)$ is non increasing and satisfies $lim_{t\to\infty}\varphi(s,t) = 0$. The notation $e_y(a^+)$ represents $\lim_{t\to a, t>a} e_y(t)$ when it exists. The term $W_{[t_0,t)}$ represents $\mathrm{ess.\,sup}_{s\in[t_0,t)} |W(s)|$. Throughout this letter the vectors $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^p$ represent respectively the state vector and the outputs of the considered systems.

We consider the following class of nonlinear systems:

$$\begin{cases} \dot{x} = Ax + f(x) \\ y = Cx = x^1 \end{cases} \tag{1}$$

where $x = (x^1, x^2, \ldots, x^q)^T \in \mathbf{R}^n$ with $x^k \in \mathbf{R}^p$ and $p.q = n$

$$A = \begin{pmatrix} 0_p & I_p & 0_p & \ldots & 0_p \\ 0_p & 0_p & I_p & 0_p & \vdots \\ \vdots & 0_p & \ldots & I_p & 0_p \\ 0_p & \vdots & \ldots & 0_p & I_p \\ 0_p & \ldots & \ldots & \ldots & 0_p \end{pmatrix} \tag{2}$$

and

$$C = (I_p \quad 0_p \quad \ldots \quad 0_p). \tag{3}$$

Throughout this letter, we assume that the following hypotheses are satisfied:

Hypothesis 1: The vector $f(x)$ has the following form:

$$f(x) = \left(f_1(x^1), \ldots, f_s(x^1, \ldots, x^s), \ldots, f_q(x)\right)^T. \tag{4}$$

Hypothesis 2: The functions f_s are globally Lipschitz, i.e:

$$\begin{gathered} \exists \beta_0 > 0 \text{ such that } \forall (x_1, x_2) \in \mathbf{R^{p.s}} \times \mathbf{R^{p.s}} \\ |f_s(x_1) - f_s(x_2)| \leq \beta_0 |x_1 - x_2|, \ (s = 1, \ldots, q). \end{gathered} \tag{5}$$

Using these hypotheses, the following observer has been proposed in [11] for the class of systems (1) without considering network:

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) - \theta\Delta^{-1}S^{-1}C^T\left(C\hat{x}(t) - y\right) \tag{6}$$

where θ is a positive constant satisfying $\theta \geq 1$. S is a symmetric positive definite matrix, solution of the algebraic Lyapunov equation

$$SA + A^TS - C^TC = -S \tag{7}$$

and Δ is a diagonal matrix which has the following form:

$$\Delta = Diag\left(I_p, , \frac{1}{\theta}I_p, \ldots, \frac{1}{\theta^{q-1}}I_p\right). \tag{8}$$

The exponential convergence of this observer has been derived by using the Lyapunov function $V = \bar{x}^T S \bar{x}$, where $\bar{x} = \Delta\tilde{x}$. The authors prove in [11] that its time derivative satisfies

$$\dot{V} \leq -(\theta - \beta)V - \theta\bar{x}^TC^TC\bar{x} \quad \text{where} \quad \beta = 2\sqrt{q}\frac{\lambda_{max}(S)}{\lambda_{min}(S)}\beta_0. \tag{9}$$

III. Observers for Networked Control Systems

In this section, we present a framework dedicated to implementation of a class of nonlinear observers on a class of networks. Let us consider the nonlinear systems

$$\begin{cases} \dot{x} = f_M(x) \\ y = h_M(x) \end{cases} \tag{10}$$

and suppose that there exists an observer described by the dynamics

$$\dot{\hat{x}} = f_o\left(\hat{x}, (h_M(\hat{x}) - y)\right) \tag{11}$$

observation error $\tilde{x} = \hat{x} - x$ which is represented by the system

$$\dot{\tilde{x}} = f_o\left(\hat{x}, (h_M(\hat{x}) - y)\right) - f_M(x) \tag{12}$$

is globally asymptotically stable. Now, let us consider the case where all outputs of system (10) are connected to observer (11) via a serial communication channel. We suppose that the signals of these sensors cannot be transmitted to observer (11) at the same time, then this problem can be modeled by the following framework:

$$\begin{cases} \dot{x} = f_M(x) \quad t \in [t_{i-1}, t_i] \\ x\left(t_i^+\right) = x(t_i) \\ y = h_M(x) \\ \dot{\hat{x}} = f_o(\hat{x}, \hat{w} - w) \quad t \in [t_{i-1}, t_i] \\ \hat{x}\left(t_i^+\right) = \hat{x}(t_i) \\ \dot{w} = g_w(w, \hat{w}, \hat{x}) \quad t \in [t_{i-1}, t_i] \\ w\left(t_i^+\right) = y_s(t_i) + h_y\left(i, e_y(t_i)\right) \\ \dot{\hat{w}} = \hat{g}_{\hat{w}}(\hat{w}, w, \hat{x}) \quad t \in [t_{i-1}, t_i] \\ \hat{w}\left(t_i^+\right) = h_M\left(\hat{x}(t_i)\right) \\ \dot{y}_s = g_{y_s}(y_s, x) \quad t \in [t_{i-1}, t_i) \\ y_s\left(t_i\right) = y(t_i) \quad i \in \mathbf{N}^* \\ e_y = w - y_s \end{cases} \tag{13}$$

The vector $\hat{x}$ is the continuous-time estimate of the system state x. The monotonically increasing sequence $t_j, j \in \mathbf{N}$ represents the transmission instants. The vectors w and $\hat{x}$ (which are re-initialized at each instant t_j) represent respectively most recently transmitted output values via network and the prediction of the observer output. The vector y_s (which is also re-initialized at t_j) allows us the possibility to describe the sampling phenomena. More precisely, y_s can be used to represent the output of the zero order hold (ZOH) device. This variable is specially useful when the network operates in ZOH fashion. In this case we replace conjointly the outputs of the system and the observer by their sampled signals before considering the constraints introduced by the network. As, we will see in the sequel, this framework allows us to derive an asymptotic stability of the observation error when the network operates in ZOH fashion. The functions g_{y_s}, g_w and $\hat{g}_{\hat{x}}$ represent prediction functions between two transmission instants. The error induced by the network is represented by the vector e_y. The protocol h_y is the algorithm by which the access to network of each node is determined. At each t_j, the protocol h_y selects which nodes $k \in \{1, \ldots, l\}$ can transmit its data throughout the network. This algorithm is represented by the discrete time system: $e_y(t_j^+) = h_y(j, e_y(t_j))\ j \in \mathbf{N}$. Throughout this note, the variable $\tau = \max(t_{j+1} - t_j)$ represents the maximum allowable transfer interval (MATI) and τ_{MATI} represents an upper bound of τ. To prevent zeno solution we suppose that the MATI satisfies $\tau > t_{j+1} - t_j > \mu$, where μ is an arbitrary positive constant. Note that the sampled-data systems are a special case of NCSs since all sensor and control signals are transmitted at each transmission instant. In this case, the protocol h_y is equal to zero, and the maximum allowable time between any two successive transmissions is called maximum allowable sampling period MASP. Compared to [12] where a framework based on discrete-time approximate models is developed for the observer design for sampled-data systems, we can say that the framework (13) is an alternative approach to [12].

Hypothesis 3: We suppose that the protocol h_y is *UGES*. This means that there exists a positive function, $W : \mathbf{N} \times \mathbf{R}^{n_{e_y}} \to \mathbf{R}_+$ and some positive constants $\lambda \in [0,1)$, a_1, a_2 such that, for all $j \in \mathbf{N}$ and for all $e_y \in \mathbf{R}^{n_{e_y}}$

$$a_1|e_y| \le W(j,e_y) \le a_2|e_y| \tag{14}$$

$$W(j+1, h_y(j,e_y)) \le \lambda W(j,e_y). \tag{15}$$

Hypothesis 4: We also assume that there exists a positive constant M_0 so that for all $j \in \mathbf{N}$ and for all $e_y \in \mathbf{R}^{n_{e_y}}$

$$\left|\frac{\partial W(j,e_y)}{\partial e_y}\right| \le M_0. \tag{16}$$

This condition is satisfied by many protocols such Round Robin (RR) and Try-Once-Discard (TOD).

IV. High Gain Observer for NCSs

In this section we focus on the implementation of high gain observer (6) over a network with UGES protocols. We present two examples: The first one is an implementation in ZOH fashion, whereas the second one, is based on the introduction of an output predictor between two transmission instants. In both cases we prove exponential convergence of the high gain observer for sufficiently small MATI.

A. High Gain Observer in ZOH Fashion

When the network operates in ZOH fashion, we will consider that all prediction functions in (13) are equal to zero. This means that the corrector term $\hat{w} - w$ is held constant between two transmission instants. Following framework (13), we propose this observer:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + f(\hat{x}) - \theta\Delta^{-1}S^{-1}C^T(\hat{w} - w) \\ \dot{w} = 0 \quad t \in [t_{i-1}, t_i] \\ w\left(t_i^+\right) = y_s(t_i) + h_y\left(i, e_y(t_i)\right) \\ \dot{\hat{w}} = 0 \quad t \in [t_{i-1}, t_i] \\ \hat{w}\left(t_i^+\right) = C\hat{x}(t_i) \\ \dot{y}_s = 0 \quad t \in [t_{i-1}, t_i) \\ y_s\left(t_i\right) = y(t_i) \\ e_y = w - y_s \end{cases} \tag{17}$$

if we consider the observation error $\tilde{x} = \hat{x} - x$, then we have

$$\begin{cases} \dot{\tilde{x}} = (A - \theta\Delta^{-1}S^{-1}C^TC)\tilde{x} + f(\tilde{x} + x) - f(x) \\ -\theta\Delta^{-1}S^{-1}C^T\left(C\tilde{x}\left(t_{i-1}^+\right) - C\tilde{x}(t) - e_y\right) \quad t \in [t_{i-1}, t_i] \\ \dot{e_y} = 0 \quad t \in [t_{i-1}, t_i] \\ \tilde{x}\left(t_i^+\right) = \tilde{x}(t_i) \\ e_y\left(t_i^+\right) = h_y\left(i, e_y(t_i)\right) \end{cases} \tag{18}$$

As we can see, system (18) is disturbed by the term $C\tilde{x}(t_{i-1}^+) - C\tilde{x}(t)$. The Leibniz–Newton formula Provides

$$C\tilde{x}\left(t_{i-1}^+\right) - C\tilde{x}(t) = -C\int_{t-\tau_1(t)}^{t} \dot{\tilde{x}}(s_1)ds_1 \tag{19}$$

with $\tau_1(t) = t - t_{i-1}^+$ for all $t \in (t_{i-1}, t_i]$. Note that we have $\tau_1(t) \in [0, \tau]$ and $\dot{\tau}_1 = 1$. This means that system (18) can be viewed as a time delay system with bounded time varying delay. This allows us to extend the approach developed in [4], [13] and [14] to the problem considered in the present note.

Theorem 1: Let us consider system (1) and suppose that hypotheses 1, 2, 3, 4 hold. Then, there exists a positive constant θ_0, such that for all $\theta > \theta_0$ system (17) is a global exponential observer for system (1) if $\tau \in [\mu, \tau_{MATI})$ where$\tau_{MATI} = min(\lambda_{min}(S)/6sup\{[|A - S^{-1}C^TC|+\sqrt{q}\beta_0], \lambda_{min}(S)\lambda_{max}(S^{-1})\}^2\theta, 1-\lambda)$.

Proof: Consider the change of coordinates $\bar{x} = \Delta\tilde{x}$, and from the fact that $\Delta A\Delta^{-1} = \theta A$ and $C^TC\Delta^{-1} = C^TC$, then (18) will be

$$\begin{cases} \dot{\bar{x}} = \theta(A - S^{-1}C^TC)\bar{x} + \Delta\left(f(\tilde{x} + x) - f(x)\right) \\ +\theta S^{-1}C^T\left(e_y + C\int_{t-\tau_1(t)}^{t} \dot{\bar{x}}(s_1)ds_1\right) \quad t \in [t_{i-1}, t_i] \\ \dot{e_y} = 0 \quad t \in [t_{i-1}, t_i] \\ e_y\left(t_i^+\right) = h_y\left(i, e_y(t_i)\right) \end{cases} \tag{20}$$

In order to prove this theorem, let us consider the following candidate *Lyapunov-Krasovskii functional*:

$$U_1 = \bar{x}^TS\bar{x} + \int_{t-\tau_{MATI}}^{t}\int_{s}^{t}\left|\dot{\bar{x}}(s_1)\right|^2 ds_1 ds + \gamma_1\phi_1(t)W^2(j,e_y) \tag{21}$$

where $\gamma_1 > 0$, $j \in \mathbf{N}$ and $\phi_1(t)$ is a bounded, positive function and decreasing between $(t_{i-1}^+, t_i]$, with appropriate fixed values at instants t_{i-1}^+ and t_i. The functional U_1 has been used in [4], [13] and [14] for observer design with sampled and delayed measurements without network constraints communications by considering $W^2(j,e_y) = 0$. We will show below that this approach can be used here. More precisely, to prove exponential stability, it is sufficient to find τ_{MATI} and θ_0 so that for $\theta > \theta_0$ the following inequalities hold:

$$\begin{cases} \dot{U}_1 + \varepsilon_1 U_1 \le 0 \quad t \in (t_{i-1}, t_i] \\ U_1\left(t_i^+\right) < U_1(t_i) \quad \forall i \in \mathbf{N}^* \end{cases} \tag{22}$$

where $\varepsilon_1 > 0$. We can easily see that the second inequality of (22) can be satisfied by choosing $\phi_1(t_{i-1}^+) = 1$ and $\phi_1(t_{i-1}^+ + \tau_{MATI}) = \lambda$ $\forall i \in \mathbf{N}^*$ where $\lambda \in (0,1)$.

Let us compute the derivative of U_1 for $t \in (t_{i-1}, t_i]$, then we have

$$\dot{U}_1 = 2\bar{x}^TS\dot{\bar{x}} + \tau_{MATI}|\dot{\bar{x}}|^2 - \int_{t-\tau_{MATI}}^{t}\left|\dot{\bar{x}}(s)\right|^2 ds + \gamma_1\dot{\phi}_1(t)W^2(i-1,e_y). \tag{23}$$

Using (20), and Hölder inequality we can deduce that

$$|\dot{\bar{x}}|^2 \le 2k_2\theta^2\left[V + |I|^2 + \frac{W^2(i-1,e_y)}{a_1^2}\right] \tag{24}$$

where

$$k_2 = \frac{3sup\left\{\left[|A - S^{-1}C^TC| + \sqrt{q}\beta_0\right], \lambda_{min}(S)\lambda_{max}(S^{-1})\right\}^2}{2\lambda_{min}(S)}. \tag{25}$$

Now, let us set $V = \bar{x}^TS\bar{x}$, after some computations we deduce that for $t \in (t_{i-1}, t_i]$, we have

$$\begin{aligned} \dot{U}_1 + \varepsilon_1 U_1 \le & -\theta V + \beta V + \varepsilon_1 V + 2\theta\left(|I|^2 + \frac{W^2(i-1,e_y)}{a_1^2}\right) \\ & + 2\tau_{MATI}\theta^2k_2\left[V + |I|^2 + \frac{W^2(i-1,e_y)}{a_1^2}\right] \\ & + \varepsilon_1\tau_{MATI}\int_{t-\tau_{MATI}}^{t}\left|\dot{\bar{x}}(s)\right|^2 ds \\ & - \int_{t-\tau_{MATI}}^{t}\left|\dot{\bar{x}}(s)\right|^2 ds \\ & + \left[\gamma_1\dot{\phi}_1(t) + \varepsilon_1\gamma_1\phi_1(t)\right]W^2(i-1,e_y) \end{aligned}$$

where $I = \int_{t-\tau_1(t)}^{t} \dot{\tilde{x}}(s)ds$. From *Jensen's* inequality, we can derive

$$|I|^2 \leq \tau_{MATI} \int_{t-\tau_{MATI}}^{t} \left|\dot{\tilde{x}}(s)\right|^2 ds. \tag{26}$$

By choosing

$$\dot{\phi}_1 = -\varepsilon_1\phi_1 - 1 \; \forall t \in (t_{i-1}, t_i] \text{ and } \gamma_1 = \frac{a_1^2}{2\theta + 2\tau_{MATI}\theta^2 k_2} \tag{27}$$

we can say that the inequality $\dot{U}_1 + \varepsilon_1 U_1 \leq 0$ is ensured, if τ_{MATI} satisfies both (27) and the following inequalities:

$$\begin{cases} \theta - \beta - \epsilon_1 - 2\tau_{MATI}\theta^2 k_2 > 0 \\ 1 - 2\theta^2 k_2 \tau_{MATI}^2 - \epsilon_1\tau_{MATI} - 2\tau_{MATI}\theta > 0 \end{cases}. \tag{28}$$

From this, and by integrating (27) on $(t_{i-1}, t_{i-1} + \tau_{MATI}]$ and setting $\epsilon_1 \to 0$, we derive the following conditions:

$$\begin{cases} \tau_{MATI} = min\left\{\frac{1}{4k_2\theta}, 1 - \lambda\right\} \\ \theta > sup\{2\beta, 1\} \end{cases} \tag{29}$$

using the fact that $|\bar{x}| \leq (1/\sqrt{\lambda_{min}(S)})\sqrt{U_1(t)}$ and from (22) then we deduce that under conditions (29) there exist $c_1 > 0$ and $c_2 > 0$ such that $|\bar{x}| \leq c_1 exp^{-c_2(t-t_0)} \; \forall t \geq t_0$. From this we can say that $|\bar{x}|$ converges exponentially towards zero.

B. *High Gain Observer With an Output Predictor*

The idea of using a predictor of the output was introduced in [15] for the design of sampled-data observers. More precisely, this predictor is used by the observer between two transmission instants and its model is a copy of the output system model. This gives this differential equation $\dot{w} = (\partial h_M(\hat{x})/\partial\hat{x}) f_M(\hat{x})$ where the initial condition $w(t_0)$ is arbitrary and at each transmission instant t_i the value of w is reseated like this: $w(t_i^+) = y(t_i)$. This idea has been extended to networked systems in [8] and [9] and some bounds of MATI have been derived using small gain approach. The aim of this section is to improve the bound of MATI for high gain observer when an output predictor is used. Following the framework (13), we can write the following observer:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + f(\hat{x}) - \theta\Delta^{-1}S^{-1}C^T(\hat{w} - w) \\ \dot{w} = \hat{x}^2 + f_1(\hat{x}^1) \quad t \in [t_{i-1}, t_i] \\ w\left(t_i^+\right) = y_s(t_i) + h_y\left(i, e_y(t_i)\right) \\ y_s = y \\ \hat{w} = C\hat{x}(t) \end{cases} \tag{30}$$

Let us consider the observation error $\tilde{x} = \hat{x} - x$, then we have

$$\begin{cases} \dot{\tilde{x}} = (A - \theta\Delta^{-1}S^{-1}C^TC)\tilde{x} + f(\tilde{x} + x) - f(x) \\ +\theta\Delta^{-1}S^{-1}C^T e_y \quad t \in [t_{i-1}, t_i] \\ \dot{e_y} = \tilde{x}^2 + f_1(\tilde{x}^1 + x^1) - f_1(x^1) \quad t \in [t_{i-1}, t_i] \\ \tilde{x}\left(t_i^+\right) = \tilde{x}(t_i) \\ e_y\left(t_i^+\right) = h_y\left(i, e_y(t_i)\right) \end{cases} \tag{31}$$

Theorem 2: Let us consider system (1) and suppose that hypotheses 1, 2, 3, 4 hold. Then for all $\theta > \max\{\beta, 1\}$, system (30) is a global exponential observer for system (1) if $\tau \in [\mu, \tau_{MATI})$ where $\tau_{MATI} = a_1((\pi/2) - 2atan(\lambda))\sqrt{\lambda_{min}(S)(\theta - \beta)}/M_0(\theta + \beta_0)\sqrt{\theta}$.

Proof: As in the above section, using the same change of coordinates then (31) will be

$$\begin{cases} \dot{\bar{x}} = \theta(A - S^{-1}C^TC)\bar{x} + \Delta\left(f(\tilde{x} + x) - f(x)\right) \\ +\theta S^{-1}C^T e_y \quad t \in [t_{i-1}, t_i] \\ \dot{e_y} = \theta\bar{x}^2 + f_1(\tilde{x}^1 + x^1) - f_1(x^1) \quad t \in [t_{i-1}, t_i] \\ e_y\left(t_i^+\right) = h_y\left(i, e_y(t_i)\right) \end{cases} \tag{32}$$

Let us consider the following candidate Lyapunov function as in [16]:

$$U(t) = \sigma\bar{x}^T(t)S\bar{x}(t) + \gamma\phi(t)W^2\left(j, e_y(t)\right) \tag{33}$$

where $j \in \mathbf{N}$, σ and γ are two positive constants which will be computed below. The positive function $\phi(t)$ is bounded and decreasing on the transmission interval $(t_{i-1}, t_i]$ $i \in \mathbf{N}^*$. This function satisfies the following conditions:

$$\phi\left(t_{i-1}^+\right) = \lambda^{-1} \quad \text{and} \quad \phi\left(t_{i-1}^+ + \tau_{MATI}\right) = \lambda \quad \forall i \in \mathrm{N}^*$$

where$\lambda \in (0, 1)$, to guarantee that $U(t_i^+) \leq U(t_i)$. Now, let us compute the time derivative of U between $(t_{i-1}, t_i]$, then we have

$$\dot{U} = 2\sigma\bar{x}^T S\dot{\bar{x}} + \gamma\dot{\phi}W^2(i-1, e_y) + 2\gamma\phi W(i-1, e_y)\dot{W}(i-1, e_y) \tag{34}$$

after some computations, we deduce that

$$\begin{aligned} \dot{U} \leq& -\sigma(\theta - \beta)\lambda_{min}(S)|\bar{x}|^2 + \sigma\theta|e_y|^2 \\ &+\gamma\dot{\phi}W^2(i-1, e_y) + 2\gamma\phi W(i-1, e_y)\left|\frac{\partial W(i-1, e_y)}{\partial e_y}\right||\dot{e}_y|. \end{aligned}$$

Let us remark that

$$|\dot{e}_y| \leq \theta|\bar{x}^2| + \left|f_1(\hat{x}^1) - f_1(x^1)\right| \leq (\theta + \beta_0)|\bar{x}| \tag{35}$$

using the property (16), we derive the following inequalities $|\partial W(i - 1, e_y)/\partial e_y||\dot{e}_y| \leq M_0(\theta + \beta_0)|\bar{x}|$ and

$$\begin{aligned} \dot{U} \leq& -\sigma(\theta - \beta)\lambda_{min}(S)|\bar{x}|^2 + \sigma\theta|e_y|^2 \\ &+\gamma\dot{\phi}W^2(i-1, e_y) + 2\gamma\phi W(i-1, e_y)M_0(\theta + \beta_0)|\bar{x}|. \end{aligned}$$

Using the fact that the protocol h_y is UGES, and by adding and subtracting the term $\epsilon_0 W^2(i-1, e_y)$, where $\epsilon_0 > 0$, thus we have

$$\begin{aligned} \dot{U} \leq& -\sigma(\theta - \beta)\lambda_{min}(S)|\bar{x}|^2 - \epsilon_0 W^2(i-1, e_y) \\ &+ \frac{\sigma\theta}{a_1^2}W^2(i-1, e_y) + \gamma\dot{\phi}W^2(i-1, e_y) \\ &+ 2\gamma\phi W(i-1, e_y)M_0(\theta + \beta_0)|\bar{x}| + \epsilon_0 W^2(i-1, e_y) \end{aligned}$$

If we choose

$$\dot{\phi} = -\gamma(\phi^2 + 1) \quad \forall t \in (t_{i-1}, t_i] \tag{36}$$

thus, we will have

$$\begin{aligned} \dot{U} \leq& -\sigma(\theta - \beta)\lambda_{min}(S)|\bar{x}|^2 + M_0^2(\theta + \beta_0)^2|\bar{x}|^2 \\ &-\epsilon_0 W^2(i-1, e_y) - \left[\gamma^2 - \epsilon_0 - \frac{\sigma\theta}{a_1^2}\right]W^2(i-1, e_y) \end{aligned}$$

where $\epsilon_0 > 0$. Note that the term $\epsilon_0 W^2(i-1, e_y)$ is introduced to guarantee the exponential convergence. Choosing

$$\begin{cases} \sigma = \frac{M_0^2(\theta+\beta_0)^2}{(\theta-\beta)\lambda_{min}(S)} + \frac{\epsilon_0}{(\theta-\beta)\lambda_{min}(S)} \\ \gamma = \sqrt{\frac{\sigma\theta}{a_1^2} + \epsilon_0} \end{cases} \tag{37}$$

thus we can say that there exists a positive constant ξ, so that $\dot{U}(t) \leq -\xi U(t)$ $t \in (t_{i-1}, t_i]$ this means that for all $t \in (t_{i-1}, t_i]$

$$U(t) \leq U\left(t_{i-1}^+\right) exp^{-\xi\left(t-t_{i-1}^+\right)} \tag{38}$$

and

$$U(t_i) \leq U\left(t_{i-1}^+\right) exp^{-\xi\left(t_i-t_{i-1}^+\right)} \tag{39}$$

on the other hand, we have

$$U\left(t_i^+\right) = V\left(\bar{x}\left(t_i^+\right)\right) + \gamma\lambda^{-1}W^2\left(e_y\left(t_i^+\right)\right)$$

using the fact that the protocol is UGES, and $\bar{x}(t_i^+) = \bar{x}(t_i)$, then we can write

$$U\left(t_i^+\right) \leq V\left(\bar{x}(t_i)\right) + \gamma\lambda W^2\left(e_y(t_i)\right) \leq U(t_i) \quad i \in \mathbf{N}^* \tag{40}$$

thus, we derive

$$\begin{aligned} U(t) &\leq U\left(t_{i-1}^+\right) exp^{-\xi\left(t-t_{i-1}^+\right)} \quad \forall t \in (t_{i-1}, t_i] \\ U\left(t_i^+\right) &\leq U\left(t_{i-1}^+\right) exp^{-\xi\left(t_i-t_{i-1}^+\right)} \quad i \in \mathbf{N}^*. \end{aligned} \tag{41}$$

Using the fact that $|\bar{x}| \leq (1/\sqrt{\sigma\lambda_{min}(S)})\sqrt{U(t)}$ and from (41), then we deduce that there exist $\mu_1 > 0$ and $\mu_2 > 0$ such that $|\bar{x}| \leq \mu_1 exp^{-\mu_2(t-t_0)}$ $\forall t \geq t_0$. This means that $\bar{x}$ converges exponentially towards zero, and the value of τ_{MATI} is then derived by integrating (36) on $(t_{i-1}, t_{i-1} + \tau_{MATI}]$, with $\epsilon_0 \to 0$. This leads to

$$\begin{aligned} \tau_{MATI} &= lim_{\epsilon_0 \to 0}\left(\frac{1}{\gamma}\left[atan(\lambda^{-1}) - atan(\lambda)\right]\right. \\ &= \frac{a_1\left(\frac{\pi}{2} - 2atan(\lambda)\right)\sqrt{\lambda_{min}(S)(\theta-\beta)}}{M_0(\theta+\beta_0)\sqrt{\theta}}. \end{aligned} \tag{42}$$

1) Comparison With Small Gain Approach: Let us consider the function $V = \bar{x}^T S \bar{x}$, then after simple computations, we can easily derive the following inequality:

$$\dot{V} \leq -(\theta-\beta)V + \theta\frac{W^2}{a_1^2} \tag{43}$$

using the comparison lemma, then we will have

$$|\bar{x}| \leq \sqrt{\frac{V(t_0)}{\lambda_{min}(S)}} exp^{\frac{-(\theta-\beta)}{2}(t-t_0)} + \sqrt{\frac{\theta}{\lambda_{min}(S)(\theta-\beta)}}\frac{W_{[t_0,t)}}{a_1} \tag{44}$$

where $\sqrt{V(t_0)/\lambda_{min}(S)} exp^{((-(\theta-\beta))/2)}(t-t_0)$ is a function of class $\mathcal{KL}$.

Now, let us define, $\tilde{y} = M_0(\theta+\beta_0)\bar{x}$, then we derive the gain γ_0 from W to $\tilde{y}$

$$\gamma_0 = \frac{M_0(\theta+\beta_0)}{a_1}\sqrt{\frac{\theta}{\lambda_{min}(S)(\theta-\beta)}}. \tag{45}$$

Using (44) and the fact that

$$\left|\frac{\partial W(i-1, e_y)}{\partial e_y}\right| |\dot{e}_y| \leq M_0(\theta+\beta_0)|\bar{x}| \tag{46}$$

then from Theorem 4 in [2] we can say that τ_{MATI} is given by

$$\tau_{MATI} = \frac{1-\lambda}{\gamma_0} = \frac{a_1(1-\lambda)\sqrt{\lambda_{min}(S)(\theta-\beta)}}{M_0(\theta+\beta_0)\sqrt{\theta}}. \tag{47}$$

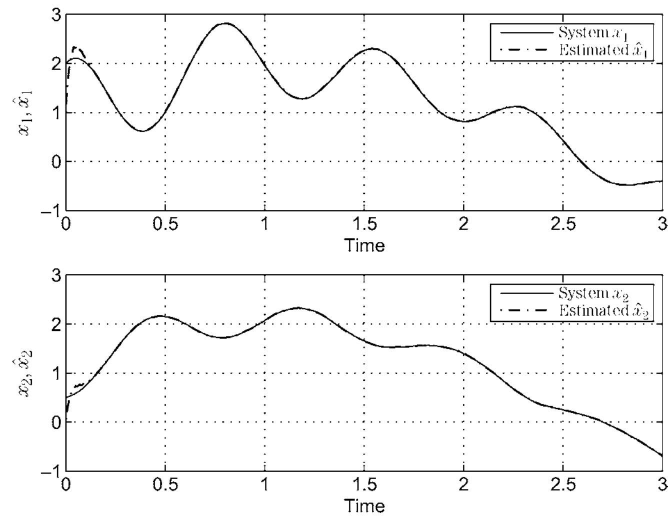

Fig. 1. Evolution of the first and second states for $\tau_{MATI} = 0.01$.

Now let us compute the ration R between (42) and (47), then we have $R = ((\pi/2) - 2atan(\lambda))/(1-\lambda)$. Since $1-\lambda \leq (\pi/2) - 2atan(\lambda)$ for all $\lambda \in [0, 1)$, then we deduce that $R \geq 1$. This means that the bound derived from Lyapunov approach is larger than the one derived from small gain approach. This result means that we have also enlarged the interval of admissible values of the gain θ for any fixed τ_{MATI} and therefore we have also improved the speediness of convergence. Note that in all formulas of τ_{MATI}, it is clear that τ_{MATI} depends inversely on θ. This obviously leads to small values of τ_{MATI} for high values of the gain θ.

V. Simulations

In this part, we present some simulations on the flexible joint robotic arm used in [13]

$$\begin{cases} \dot{x}_1 = x_3 + \delta(t) \\ \dot{x}_2 = x_4 + \delta(t) \\ \dot{x}_3 = -48.6x_1 + 48.6x_2 - 1.25x_3 + 21.6u + \delta(t) \\ \dot{x}_4 = 19.5x_1 - 19.5x_2 - 3.3sin(x_3) + \delta(t) \\ y_1 = x_1 \\ y_2 = x_2 \end{cases} \tag{48}$$

Without unknown uncertainties $\delta(t)$, system (48) satisfies hypotheses 1, 2, 3, 4 with $p = q = 2$ and $x = (x^1, x^2)$ where $x^1 = (x_1, x_2)$ and $x^2 = (x_3, x_4)$.

Constrained by the size of the present note, we focused on the simulations of observer (30) with Round-Robin (RR) protocols, that consists in granting access to each compenent of the vector ouputs after p transmission instants where p is the number of ouputs [2]. In the simulations presented below, we also suppose that unknown uncertainties $\delta(t)$ are represented by random function of Matlab with $\mathrm{variance} = 0.8$. The following simulations are performed with $u = 0.1sin(t)$, $\theta = 40$ and $\tau_{MATI} = 0.01$ (see Figs. 1 and 2).

As we can see, even with some uncertainties, the results remain good with $\theta = 40$ and $\tau_{MATI} = 0.01$. This is not a general conclusion. Indeed the uncertainties can cause robustness problems especially for high values of θ.

VI. Conclusion

In this letter, some results on the design of high gain observer for networked systems with UGES protocols were presented. These results can be easily extended to other observers. In future work, we will present some results concerning other classes of protocols.

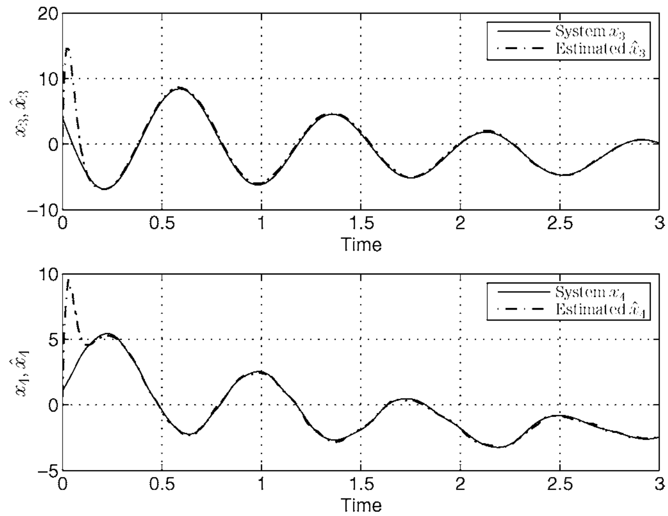

Fig. 2. Evolution of the third and fourth states for $\tau_{MATI} = 0.01$.

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