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Inapproximability proof of DSTLB and USTLB in planar graphs

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This document proves the problem of finding a minimum cost Steiner Tree covering k terminals with at most p branching nodes (with outdegree greater than 1), in a directed or an undirected planar graph with n nodes, is hard to approximate within a better ratio than n , even when the parameter p is fixed.

1 Theorem

Definition 1. In a undirected (resp. directed) tree, a *branching node* is a node whose degree (resp. outdegree) is strictly greater than 2 (resp. 1).

Problem 1. min- $(*, p)$ -USTLB: Given an undirected graph $G = (V, E)$ with n nodes and a non negative cost function ω on its edges, an integer k and a set $X \subset V$ of k terminals, determine, if it exists, a minimum cost tree T^* spanning all the nodes of X and containing at most p branching nodes.

Problem 2. min- $(*, p)$ -DSTLB: Given a directed graph $G = (V, E)$ with n nodes and a non negative cost function ω on its arcs, a node r , an integer k and a set $X \subset V$ of k terminals, determine, if it exists, a minimum cost directed tree T^* rooted at r , spanning all the nodes of X and containing at most p branching nodes.

Theorem 1. *Let $\epsilon < 1$ be a real number. If $P \neq NP$, the min- $(*, p)$ -DSTLB and the min- $(*, p)$ -USTLB problems with unit costs cannot be approximated within a factor of \mathcal{N}^ϵ where \mathcal{N} is the number of nodes in the instance, even if there is a trivial feasible solution.*

2 Proof of the theorem

2.1 Reduction

We prove the theorem in the directed case. The proof is similar in the undirected case.

Finding a hamiltonian path starting at a specified node v in a directed planar graph is a NP-complete problem [1].

Let $\mathcal{I} = (G = (V, A), v)$ be an instance of the hamiltonian path problem in a directed planar graph G . We construct a $\min-(*, p)$ -DSTLB instance $\mathcal{I}'_v = (G'_v, r, X, \omega)$ where G'_v is a directed planar graph.

The main idea is that G'_v is divided in three parts. An example is shown in Figure 1. Firstly, a graph $G' = (V' = V \cup W, A')$ built from G where each arc of A is divided in two or more arcs. Secondly, a binary tree \mathcal{B} rooted at r with p branching nodes and $p + 1$ leaves. We link one of the leaves of \mathcal{B} to v with an arc a_v . We define X as the leaves of \mathcal{B} and V . Finally, a graph H and an integer h which ensures the three following properties:

Property 1. Let n , $n_{G'}$ and n_H be the number of nodes in G , G' and H . $n_{G'} + n_H - n$ is no more than $4 \cdot n^3 \cdot h$.

Property 2. There exists an elementary path P in $G' \cup H$ going through each node of G starting at v .

Property 3. Any elementary path in $G' \cup H$ going through each node of G starting at v using a node of H not as endpoint contains at least h nodes of H .

Property 2 ensures the existence of a feasible solution. Properties 1, and 3 ensure an inapproximability gap, described in section 2.2. If h is long enough, Property 3 ensures that any node of H will not be allowed in any approximated solution. We will fix G' , H and the value of the parameter h later.

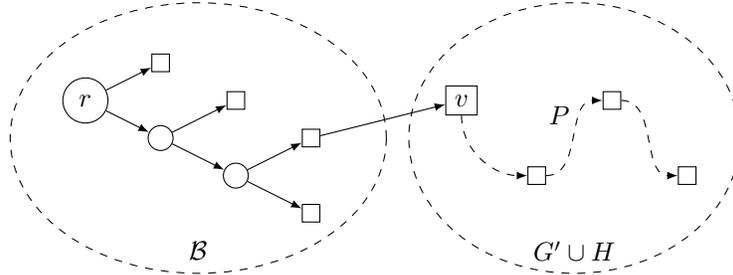


Figure 1: Example of reduction from a graph G with 4 nodes, and $p = 3$. Nodes of $W (= V' \setminus V)$ and H , and arcs of G' and H do not appear on that figure.

The number of nodes \mathcal{N} in G'_v is $n_{G'} + n_H + p + (p + 1)$.

2.2 Inapproximability gap

In this part, we fix the parameter h and show the approximability hardness of $(*, p)$ -DSTLB.

Let T^* be an optimal solution of \mathcal{I}'_v . It exists because $\mathcal{B} \cup P \cup a_v$ is a feasible solution by Property 2. Let $\epsilon < 1$, and suppose it exists a polynomial \mathcal{N}^ϵ -approximation algorithm for $\min-(*, p)$ -DSTLB in a planar graph. We will show that in that case, we could use this algorithm to decide whether G has a hamiltonian path starting at v .

If there exists a hamiltonian path in \mathcal{I} starting at v , T^* contains at most $n + 2p + 1$ nodes (the n nodes of G , the $2p + 1$ nodes of \mathcal{B}), thus it contains at most $n + 2p$ arcs. So the approximate solution has a cost $c_{\text{YES}} \leq (n + 2p) \cdot \mathcal{N}^\epsilon$.

We now discuss the case where there is no hamiltonian path starting at v in \mathcal{I} . Then, without H , we cannot build an elementary path going through each node of G .

Lemma 1. *Any feasible solution of \mathcal{I}'_v contains an elementary path going through each node of G starting at v .*

Proof. Let T be a feasible solution. T covers every leaf of \mathcal{B} , as a consequence it covers \mathcal{B} entirely. Because \mathcal{B} contains p branching nodes, all other terminals are covered with elementary paths connected to \mathcal{B} . T covers every nodes of G and a_v is the only arc linking \mathcal{B} to a node G . So T contains an elementary path going through each node of G starting at v . \square

By Lemma 1, without H , we cannot build a feasible solution of \mathcal{I}'_v . So the approximate solution uses at least one node of H . On of those node is not an endpoint. Indeed, in this case, we can remove them to get a hamiltonian path in G . By Property 3, it uses at least h nodes of H . So it has a cost $c_{\text{NO}} > h$.

If $c_{\text{NO}} > h > c_{\text{YES}}$, then the approximation algorithm can decide whether there is a hamiltonian path starting at v .

Lemma 2. *Let h satisfies $h = 5^{\frac{\epsilon}{1-\epsilon}}(n^3 + 2 \cdot p + 1)^{\frac{1+\epsilon}{1-\epsilon}} + 1$. Then $c_{\text{NO}} > h > c_{\text{YES}}$.*

Proof. Notice that $h > 1$ for all $\epsilon < 1$ and $n \geq 1$. Line 11 is proven by Line 10 and Property 1.

$$h > 5^{\frac{\epsilon}{1-\epsilon}}(n^3 + 2p + 1)^{\frac{1+\epsilon}{1-\epsilon}} \quad (1)$$

$$h^{1-\epsilon} > 5^\epsilon(n^3 + 2p + 1)^{1+\epsilon} \quad (2)$$

$$h > 5^\epsilon(n^3 + 2p + 1)^{1+\epsilon}h^\epsilon \quad (3)$$

$$h > (n^3 + 2p + 1)^{1+\epsilon}(5h)^\epsilon \quad (4)$$

$$h > (n^3 + 2p + 1)^{1+\epsilon}(1 + 4h)^\epsilon \quad (5)$$

$$h > (n^3 + 2p + 1) \cdot (1 + 4h)^\epsilon(n^3 + 2p + 1)^\epsilon \quad (6)$$

$$h > (n + 2p) \cdot (1 + 4h)^\epsilon(n^3 + 2p + 1)^\epsilon \quad (7)$$

$$h > (n + 2p) \cdot ((n^3 + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon \quad (8)$$

$$h > (n + 2p) \cdot ((n + 2p + 1) + 4 \cdot (n^3 + 2p + 1) \cdot h)^\epsilon \quad (9)$$

$$h > (n + 2p) \cdot ((n + 2p + 1) + 4 \cdot n^3 \cdot h)^\epsilon \quad (10)$$

$$h > (n + 2p) \cdot ((n + 2p + 1) + n_{G'} + n_H - n)^\epsilon \quad (11)$$

$$c_{\text{NO}} > h > c_{\text{YES}} \quad (12)$$

\square

As a consequence, if $P \neq NP$, such an algorithm does not exist.

2.3 Existence of G' and H

In this section, we explain how to build the graphs G' and H .

2.3.1 Construction of $G' = (V \cup W, A')$

G' is built from G where each arc of a is divided into several arcs of A' and nodes of W .

We first embed G in \mathbb{R}^2 such that v is on the outer face of G . For a node $w \in V$, we define its coordinates as x_w and y_w .

Lemma 3. *It exists an angle α such that the rotation $r_\alpha(G)$, of angle α and center v , rotates G so that each node $w \in V$ has a unique x -coordinate x_w with $x_v \leq x_w$ (v is 'on the left').*

Proof. Let α_m and α_M be two angles in $[0; 2\pi]$ such that for each $\alpha \in [\alpha_m; \alpha_M]$, $r_\alpha(G)$ places v on the left.

If there is no angle where, after G rotates, each node $w \in V$ has a unique x -coordinate x_w , for each $\alpha \in [\alpha_m; \alpha_M]$, there are two nodes (u, w) with $x_u = x_w$ and $y_u < y_w$. There are at most n^2 such couples. Let α_i , $i \in [1..(n^2 + 1)]$, be distinct angles in $[\alpha_m; \alpha_M]$, there are two distinct angles for which the same couple of nodes (u, w) verified, after G rotates, $x_u = x_w$ and $y_u < y_w$, which implies a contradiction. \square

We then sort the list of nodes v_i by its x coordinate : $x_v = x_{v_1} < x_{v_2} < x_{v_3} < \dots < x_{v_n}$.

We define D_i for $i \in [2..n]$ as the vertical strait lines of abscissa $x_i = \frac{x_{v_{i-1}} + x_{v_i}}{2}$. For each arc $a = (t, u)$ of G crossing a line D_i , we add a node w to W at the intersection of a and D_i and replace a in A' by the two arcs (t, w) and (w, u) . An example is shown in Figure 2.

As no new arc cross, G' is planar.

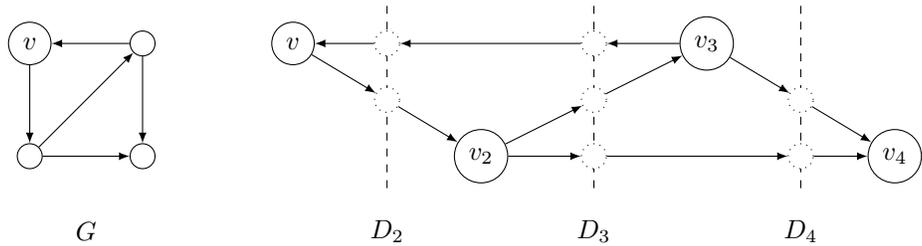


Figure 2: Example of graph $G' = (V \cup W, A')$ built from a graph G with 4 nodes. W is the set of dashed nodes.

2.3.2 Construction of H

We first prove three intermediate lemmas :

Lemma 4. *Any arc of G' starting at a vertical strait line D_i goes to the left, or goes above, below, from or to v_i .*

Proof. Let a be an arc of G' crossing a vertical strait line D_i at a node u . If a goes to the left, the lemma is verified. Else, if a do not go above, below, from and to v_i , there is a node $t \in V'$ with $a = (u, t)$ or $a = (t, u)$ and $x_t \in [x_i, x_{v_i}[$. If $t \in V$, by definition of D_i , $x_i > x_t$ which implies a contradiction. If $t \in W$, there is a strait line D_j with $x_t = x_j \in [x_i, x_{v_i}[$ which also implies a contradiction. \square

We can similarly prove the following lemma :

Lemma 5. *Any arc of G' going above, below, from or to v_i goes to the right of v_i or cross D_i .*

Lemma 6. *For each node $v_i \in V$, $i \in [2..n]$, we can add to H a node $v_{i,l}$ on D_i and an arc $(v_{i,l}, v_i)$ such that the graph $G' \cup H$ remains planar.*

Proof. Let a_m and a_M be respectively the lowest arc of G' going above v_i and the highest arc going below or to v_i , going from or to the left of v_i . An example is shown in figure 3.

If a_m and a_M do not exist, by Lemma 4, there is no arc crossing D_i going to or from the right (the graph is then disconnected). We can add $v_{i,l}$ on D_i anywhere there is no node of W .

If only a_m exists, the arc cross D_i at a node t_m by Lemma 5. We can add $v_{i,l}$ on D_i anywhere below t_m where there is no node of W .

If only a_M exists, the arc cross D_i at a node t_M by Lemma 5. We can add $v_{i,l}$ on D_i anywhere above t_M where there is no node of W .

If a_m and a_M exists, they cannot cross at a point of abscissa $x \in [x_i; x_{v_i}]$. If they do, either G' is not planar which is not true, or G' contains a node $t \in [x_i; x_{v_i}]$. Like in the proof of lemma 1, this would imply a contradiction. So we can add $v_{i,l}$ on D_i anywhere above t_M and below t_m where there is no node of W . \square

Similarly, for each node $v_i \in V$, $i \in [2..n]$ we can add to H a node $v_{i,r}$ on D_{i+1} and an arc $(v_i, v_{i,r})$ such that the graph $G' \cup H$ remains planar.

Finally, for $i \in [2..n]$, we sort the nodes of abscissa x_i by increasing y -coordinate (those nodes are nodes of G' or nodes of H). For each couple (u, t) of consecutive nodes we add to H a path of h nodes going from u to t if $v_{i-1,r}$ is before u in the list, from t to u otherwise. An example is shown in Figure 4.

Lemma 7. *G' , H and h verify Properties 1, 2 and 3.*

Proof. $n'_G + n_H - n$ is the number of arcs in H and W , in other words, the nodes of all the lines D_i . For each vertical line D_i , we create at most m nodes of G' , and $2 + h \cdot (1 + m)$ nodes of H . So $n'_G + n_H - n \leq n \cdot (1 + m) \cdot (h + 1)$. Thus $n'_G + n_H - n < n \cdot (2m) \cdot (2h) < 4n^3h$. Property 1 is verified.

The path P starting at v , going to $v_{1,r}$, from $v_{i-1,r}$ to $v_{i,l}$ through D_i and to v_i for $i \in [2..n]$ goes through each node of G . Property 2 is verified.

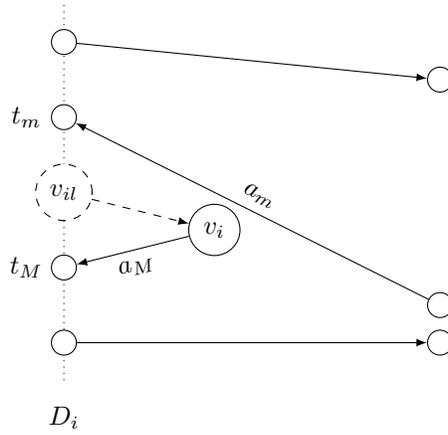


Figure 3: Example of insertion of v_{il}

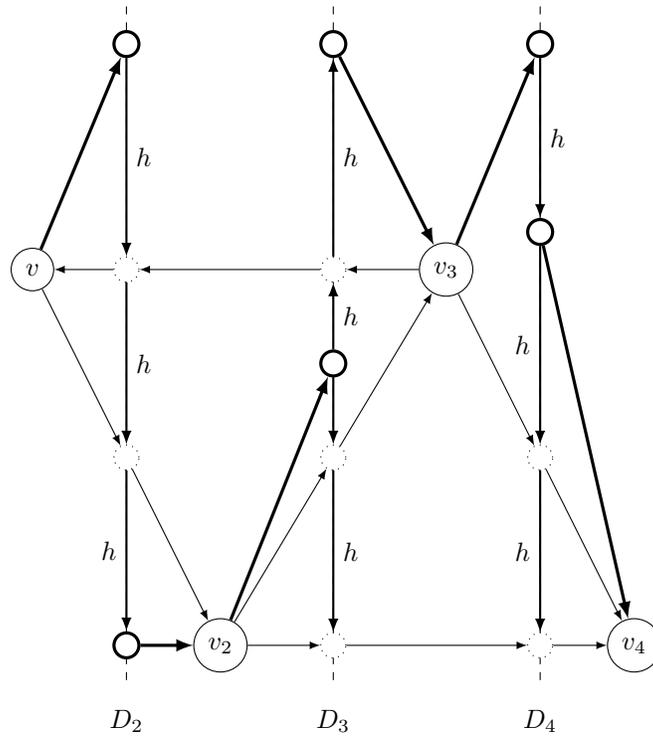


Figure 4: Example of graph $G' \cup H$ built from a graph G with 4 nodes. Thick nodes are v_{ir} and v_{il} . Dashed nodes are W . Each vertical *arc* is actually a path with h nodes.

Let P be an elementary path going through every nodes of G and one node of H not as endpoint. As only the nodes $v_{i-1,r}$ and $v_{i,l}$, $i \in [2..n]$ are linked to a node of G . If P contains a node of H , it exists a node t and $i \in [2..n]$ such that $t = v_{i-1,r}$ or $t = v_{i,l}$ is in P . As t is linked to only one node not in D_i , P goes out of D_i (or enters D_i) through an other node of D_i and P contains at least h nodes of D_i . Property 3 is verified. \square

References

- [1] GAREY, M. R., AND JOHNSON, D. S. *Computers and Intractability: A Guide to the Theory of NP-completeness*. WH Freeman and Company, New York, 1979.