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Constrained Control Design for Linear Systems with Geometric Adversary Constraints ⋆

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Abstract: This paper is concerned with the optimal control of linear dynamical systems in the presence of a set of adversary constraints. One of the novelties is the type of constraints introduced in the receding horizon optimization problem. These constraints can be considered as adversary by their non convex characteristics which make the convergence of the systems' dynamics towards the "natural" equilibrium position an impossible task. In this case, the default equilibrium point has to be replaced by a set of equilibrium points or even to accept the existence of limit cycles. The present paper proposes a *dual-mode control strategy* which builds on an optimization based controller and a fixed constrained control law whenever the adversary constraints are activated. Furthermore, the method which exhibits effective performance builds on invariance concepts and proves to be related to the classical eigenstructure assignment problems. In order to illustrate the benefits of the proposed method, typical applications involving the control of Multi-Agent Systems are considered.

Keywords: equilibrium point, positive invariance, non convex constraints, constrained MPC, Multi-Agent Systems

1. INTRODUCTION

In many control engineering problems *collision avoidance* represents a fundamental issue that needs to be integrated in the design strategy (see, for instance, Grundel and Pardalos [2004] and the references therein). This problem turns out to be difficult, one of the features being the non convexity of the associated constraints (see, for details Prodan et al. [2012]).

Various control methods for solving the collision avoidance problem are related to the potential field approach Tanner et al. [2007], graph theory Lafferriere et al. [2005] or other optimization-based approaches which handle indirectly the constraints by penalty terms in the cost function. One shortcoming of all these methods is that the constraints activation is denied. Furthermore, there are methods based on *Mixed Integer Programming* (MIP) (see the comprehensive monography Jünger et al. [2009]) which have the ability to include *explicitly* non convex constraints and discrete decisions in the optimization problem. A novel approach for reducing the number of binary variables used in MIP formulation, together with an application in the obstacles avoidance problem is detailed in Stoican et al. [2011a] and Stoican et al. [2011b].

The problem of the avoidance of convex fixed obstacles is examined in Rakovic et al. [2007] from a set-theoretic point of view. The authors propose to deal with the non convex control problem by considering approximation procedures which "inner and outer convexify" the exact capture sets. Furthermore, in Raković and Mayne [2007] the same problem is tackled by using set computations and polyhedral algebra. However, in all of these papers the origin is part of the feasible region. To the best of the authors knowledge, there does not exists any results treating the case of constraints which are not satisfied by the origin (understood as the equilibrium point of the dynamical system to be controlled).

The goal of the present paper is twofold. In a first stage, we perform a detailed analysis of the limit behavior for a *linear* dynamical system in the presence of *adversary constraints*. More precisely, we need to define the fixed points and the invariance properties for the system state trajectory while avoiding a convex region containing the origin in its strict interior. In the context of multi-agent systems, this region can, in fact, represent an obstacle (static constraints) but can be extended to the safety region of a different agent (leading to a parametrization of the set of constraints with respect to the global current state).

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In a second stage, our interest is to ensure the stability over the feasible region of the state space using a dual-mode strategy. The basic principles are those of Model Predictive Control (MPC) technique (see, for instance, Maciejowski [2002] for basic notions) including avoidance constraints. There is a fundamental difference to the classical MPC which rely on the assumption that the origin is in the relative interior of the feasible region (see, for example, Mayne et al. [2005], **?**, Bemporad et al. [2002]) or on the frontier of the feasible region Pannocchia et al. [2003]. In the present paper, we provide *necessary and sufficient conditions* for the existence of a stable equilibrium point having the entire feasible region as a basin of attraction.

The rest of the paper is organized as follows: In Section II the constrained predictive control problem is formulated. Section III presents the local constrained control problem based on invariance concepts, while the designed problem is developed in Section IV. Discussions based on the simulation results are presented in Section V and the conclusions are drawn in Section VI.

The following notations will be used throughout the paper. The spectrum of a matrix $M \in \mathbb{R}^{n\times n}$ is the set of its eigenvalues, denoted by $\Lambda(M) = \{\lambda_i : i = 1 : n\}$. A point x_e is a fixed point of a function f if and only if $f(x_e) = x_e$ (i.e. a point identical to its own image). Let $x_{t+1|t}$ denote the value of x at time instant $t+1$, predicted upon the information available at time $t \in \mathbb{N}$. We write $Q \succeq 0$ to denote that Q is a positive semidefinite matrix.

2. PRELIMINARIES

Consider the discrete time linear time-invariant system:

$$x_{t+1} = Ax_t + Bu_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the input signal and A, B are state matrices of appropriate dimensions. It is assumed that the pair (A, B) is *stabilizable*.

The minimization of infinite horizon cost function (usually a quadratic function involving states and inputs) leads to the linear state-feedback control law characterizing the optimal *unconstrained* control:

$$u_t = K_{LQ}x_t, \tag{2}$$

with K_{LQ} computed from the solution of the discrete algebraic Riccati equation.

Consider now the problem of optimal control of the system state (1) towards the origin while its trajectory *avoids* a polyhedral region[1] defined by:

$$S = \left\{x \in \mathbb{R}^n : h_i^T x < k_i, \quad i = 1 : n_h\right\}, \tag{3}$$

with $(h_i, k_i) \in \mathbb{R}^n \times \mathbb{R}$ and n_h being the number of half-spaces. In this paper we focus on the case where $k_i > 0$ for all $i = 1 \dots n_h$, meaning that the origin is contained in the strict interior of the polytopic region, i.e. $0 \in S$. Note that, the feasible region is a non convex set defined as the complement of (3), namely $\mathbb{R}^n \setminus S$.

Whenever, (2) is infeasible a corrective control action needs to be design such that the system's trajectories evolve outside the interdicted region (3):

$$x_t \notin S. \tag{4}$$

[1] Such limitations often arise in control applications and we will discuss later in the paper the collision or obstacle avoidance problems.

A tractable approach is the recursive construction of an optimal control sequence $\mathbf{u} = \{u_{t|t}, u_{t+1|t}, \cdots, u_{t+N-1|t}\}$ over a *finite* constrained receding horizon, which leads to a predictive control policy:

$$\mathbf{u}^* = \arg\min_{\mathbf{u}} (x_{t+N|t}^T P x_{t+N|t} + \sum_{i=1}^{N-1} x_{t+i|t}^T Q x_{t+i|t} + \\ + \sum_{i=0}^{N-1} u_{t+i|t}^T R u_{t+i|t}), \tag{5}$$

subject to the set of constraints:

$$\begin{cases} x_{t+i+1|t} = Ax_{t+i|t} + Bu_{t+i|t}, \\ x_{t+i|t} \in \mathbb{R}^n \setminus S, \quad i = 1 \dots N. \end{cases} \tag{6}$$

Here $Q = Q^T \succeq 0$, $R \succ 0$ are weighting matrices, $P = P^T \succeq 0$ defines the terminal cost and N denotes the length of the prediction horizon.

In the present paper, we propose a dual-mode control strategy based on a local feedback guaranteeing the stability of an equilibrium point outside the interdicted region (3) and an outer controller design to handle the transitory behavior.

Besides satisfying the constrains (6), additionally we would like that the systems' state approaches a unique equilibrium point and avoids the existence of multiple basins of attraction, cyclic or chaotic behavior. In the general case the periodic solutions can be considered as optimal candidates for the limit behavior. In the present paper, the control objective is to avoid limit cycles and concentrate on the convergence to a unique fixed point with basin of attraction $\mathbb{R}^n \setminus S$.

Remark 1. Usual MPC concerns regarding *feasibility* or *recursive feasibility* are not critical for the problem (5)–(6) as long as the feasible set is actually unbounded. Such discussion becomes relevant if additional *convex* state/input constraints are to be handled. We refer to reachability studies in order to deal with these problems which are out of the scope of the present paper. □

3. LOCAL CONSTRAINED CONTROL

In this section, we first establish conditions for an affine state-feedback control law to render a half-space positively invariant. Second, we associate the half-space to an equilibrium state lying on its boundary. Then, these conditions are used for the derivation of a control law that transfers the system's state as close as possible to the origin, all by avoiding the interdicted region.

3.1 Equilibrium states

Consider an affine control law of the form:

$$u_t = K(x_t - x_e) + u_e, \tag{7}$$

with $x_e \in \mathbb{R}^n$ the desired equilibrium state, $K \in \mathbb{R}^{m\times n}$ the feedback-gain matrix and $u_e \in \mathbb{R}^m$ the feed-forward parameter. The resulting closed-loop system is described by the state equation

$$x_{t+1} = Ax_t + BK(x_t - x_e) + Bu_e, \tag{8}$$

and $x_t - x_e$ defines its transient behavior.

A state x_e is an equilibrium state for the closed-loop system (1) if:

$$x_e = Ax_e + Bu_e. \tag{9}$$

Therefore, only the points x_e belonging to the preimage through the linear map $(I_n - A)$, of the linear subspace, spanned by the columns of matrix B, can represent equilibrium states.

Remark 2. The geometrical locus of the equilibrium states is independent of K. Consequently, the states that can be equilibria are defined by the dynamics of the unforced system and are completely specified by u_e. □

Illustrative example: As previously mentioned, the set of equilibrium points $x_e \in \mathbb{R}^n$ is the image of matrix $(I_n - A)^{-1}B$ in the case when $(I_n - A)$ is non-singular. The particular characteristics of the dynamics (state and input dimensions) define the shape of the subspace of states that can be equilibria. Figure 1 depicts a 2-dimensional system with a scalar input. It can be seen that the geometrical locus of the fixed points is in fact, a straight line which trespasses the origin.

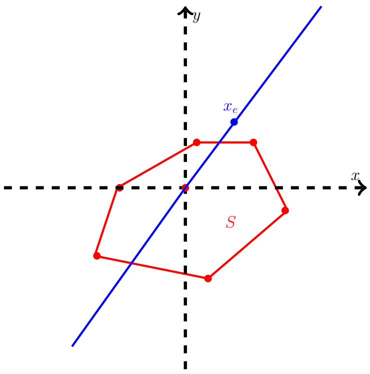

Fig. 1. Interdicted region and geometrical locus of equilibrium states.

3.2 Positive invariance conditions

We can further concentrate on one of the key issues for the control design: the controlled invariance with respect to an affine control law (7) and subsequently, the closed-loop stability. For solving this problem, the following lemma provides algebraic invariance conditions. Note that this result is a particular case of a more general result established in Bitsoris and Truffet [2011].

Lemma 3. The half-space defined by the inequality $v^T x \leq \gamma$ is a positively invariant set of the affine system $x_{t+1} = Mx_t + c$, if and only if there exists a positive real number g such that:

$$v^T M = gv^T, \tag{10}$$

and

$$g\gamma + v^T c \leq \gamma. \tag{11}$$

□

Remark 4. From Lemma 3, it is clear that

$$v^T M = gv^T \tag{12}$$

and

$$-g\gamma - v^T c \leq -\gamma \tag{13}$$

are necessary and sufficient conditions for the opposite half-spaces defined by inequality $v^T x \geq \gamma$ to be positively invariant with respect to system $x_{t+1} = Mx_t + c$. Note however that (10)–(11) is not equivalent to (12)–(13). □

The next result exploits the invariance properties and relates the algebraic conditions to the equilibrium states.

Theorem 5. If $x_e \in \mathbb{R}^n$ is an equilibrium state of the closed-loop system (1) lying on the hyperplane $v^T x = \gamma$, then a necessary and sufficient condition for this hyperplane to partition the state-space into two positively invariant half-spaces is that $v^T \in \mathbb{R}^{1\times n}$ to be *left* eigenvector of the closed-loop matrix $A + BK \in \mathbb{R}^{n\times n}$ associated to a positive eigenvalue $\lambda \neq 1$. □

Proof: See Appendix.

Illustrative example: The above results can be depicted for a 2-dimensional system $x_{t+1} = Mx_t + c$ with scalar input. In Figure 2 the straight line which separates the state space into two invariant half-spaces and the trajectories converging to the equilibrium point are shown.

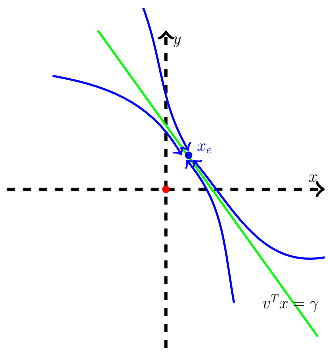

Fig. 2. Invariant half-spaces for an affine system

3.3 Eigenstructure assignment analysis

As seen above, the eigenvector of a closed-loop matrix can be seen as the normal to a hyperplane. Under mild assumptions, this hyperplane can partition the space into two complementary and invariant half-spaces. In the context of control design we are interested in the converse problem: *Given a hyperplane, does there exist a certain structural constraint on the gain matrix K which makes the resulting closed-loop matrix to have the normal to the hyperplane as an eigenvector? If not, which is the closest approximation possible (in the sense of the infinity norm)?*

These questions lead to an eigenstructure assignment analysis. Starting with the set of hyperplanes defining the polyhedral interdicted region (3), we search for the gain matrices of the control laws which assure stability and assign a left eigenvector as *close* as possible to the normal to a frontier of S in (3). Additionally, we show this to be optimal for some a priori given cost matrices (e.g., as in a Riccati equation setting).

In the following, we will derive necessary and sufficient conditions for the existence of a gain matrix K which assigns a prescribed eigenvector and the associated eigenvalue.

Proposition 6. Given a controllable pair (A, B) and the pair $(\lambda, v) \in \mathbb{R} \times \mathbb{R}^n$ the following relations hold:

1. If (λ, v^T) is the eigenvalue/*left* eigenvector pair associated to the matrix $A + BK$, then $K \in \mathbb{R}^{m\times n}$ satisfies
$$K^T z = w, \tag{14}$$
with $w = (\lambda I_n - A)^T v \in \mathbb{R}^n$ and $z = B^T v \in \mathbb{R}^m$.
2. If (λ, v) is the eigenvalue/*right* eigenvector pair associated to the matrix $A + BK$, then $K \in \mathbb{R}^{m\times n}$ satisfies
$$Kv = \tilde{w}, \tag{15}$$
with $\tilde{w} \in \mathbb{R}^m$ the solution of $(A - \lambda I_n)v = -B\tilde{w}$.

□

Proof: See Appendix.

Remark 7. Note that Theorem 6 provides a constraint on the gain K which depends on the placement of a *single pair* (eigenvalue/eigenvector). No assumption or particular implications are made with regards to the rest of the closed-loop matrix A+BK eigenstructure. Rather, the structural constraints (14) or (15) will be added further in a design procedure, to construct a gain which ensures, additionally to a particular invariance property, the stability of the unconstrained closed-loop system. □

The aforementioned theorem offers the framework for the following design procedure.

Description: An optimization problem can be formulated in order to find a *stable* eigenvalue associated to a *left/right* eigenvector which corresponds to a normal of a given hyperplane v. Concomitantly, a vector of parameters $w \in \mathbb{R}^n$ or $\tilde{w} \in \mathbb{R}^m$ and implicitly the linear structural constraint on the feedback gain K as in (14) or (15) is obtained.

Input: The controllable pair (A, B) describing the system (1) and the normal vector $v \in \mathbb{R}^n$ to a given hyperplane.

Output: A structural (linear) constraint (14) or (15) on the feedback gain ensuring the invariance property and the stability of the respective eigenvalue.

1.
$$\begin{aligned} &\min_{\delta,\lambda,w} \delta \\ \text{s.t.:}\quad & -\mathbf{1}\delta \leq v^T(A - \lambda I_n) + w^T \leq \mathbf{1}\delta \\ &\delta \geq 0, \quad 0 < \lambda < 1; \end{aligned} \tag{16}$$

2.
$$\begin{aligned} &\min_{\epsilon,\lambda,\tilde{w}} \epsilon \\ \text{s.t.:}\quad & -\mathbf{1}\epsilon \leq (A - \lambda I_n)v + B\tilde{w} \leq \mathbf{1}\epsilon, \\ &\epsilon \geq 0, \quad 0 < \lambda < 1. \end{aligned} \tag{17}$$

In the case when the optimal solution of the optimization problem (16) is $\delta^* = 0$, the vector v^T can be used for the separation of invariant half-spaces (as detailed in Lemma 3 and Theorem 5) with respect to the closed-loop dynamics. Moreover, the conditions imposed on the associated eigenvalue assure the contractiveness of the respective eigenvector.

The optimal argument $w^* \in \mathbb{R}^n$ of the LP problem (16) or $\tilde{w}^* \in \mathbb{R}^m$ of the LP problem (17), will be instrumental in the control design problem through a structural constraint on the fixed gain matrix as in (14) or (15).

3.4 Affine parametrization of the feedback policies

As it can be seen from the eigenstructure assignment approach described above, the main difficulty for proving the stability in the neighborhood of x_e is imposed by the structural constraint on the gain matrix inherited from the invariance desideratum. This imposes a reformulation of the local control problem in order to identify the design parameters. In the following, we will derive an affine parametrization of the feedback policies such that a fixed gain matrix K can be used for feedback, while respecting the constraint (14) or (15).

Theorem 8. Consider the stabilization of system (1).

1. A stabilizing feedback gain K satisfying (14) exists if and only if the pair $(A + B\Gamma^T, B\Psi^T)$ is stabilizable with $\Gamma \in \mathbb{R}^{n\times m}$ and $\Psi \in \mathbb{R}^{(m-1)\times n}$ defined as:
$$\Gamma = \begin{bmatrix} \mathbf{0}_{n\times(m-1)} & w\tilde{z}^{-1} \end{bmatrix}, \; \Psi = \begin{bmatrix} \mathbf{I}_{(m-1)} & -\hat{z}\tilde{z}^{-1} \end{bmatrix}, \tag{18}$$
with $z = [\hat{z}\ \tilde{z}]$, $\tilde{z} \in \mathbb{R}^*$, $\hat{z} \in \mathbb{R}^{m-1}$ and $w \in \mathbb{R}^n$ as in (14).
2. A stabilizing feedback gain K satisfying (15) exists if and only if the following system is output stabilizable[2] (through $u_t = \breve{K}y_t$):
$$\begin{aligned} x_{t+1} &= \left(A + B\tilde{\Gamma}\right) x_t + Bu_t, \\ y_t &= \tilde{\Psi}x_t, \end{aligned} \tag{19}$$
with $\tilde{\Gamma} \in \mathbb{R}^{m\times n}$ and $\tilde{\Psi} \in \mathbb{R}^{(n-1)\times n}$ defined as:
$$\tilde{\Gamma} = \begin{bmatrix} \mathbf{0}_{m\times(n-1)} & \tilde{w}\tilde{v}^{-1} \end{bmatrix}, \; \tilde{\Psi} = \begin{bmatrix} \mathbf{I}_{(n-1)} & -\hat{v}\tilde{v}^{-1} \end{bmatrix}, \tag{20}$$
with $v = [\hat{v}\ \tilde{v}]$, $\tilde{v} \in \mathbb{R}^*$, $\hat{v} \in \mathbb{R}^{n-1}$ and $\tilde{w} \in \mathbb{R}^m$ as in (15).

□

Proof: See Appendix.

Remark 9. Note that for $m = 1$ in (39), the gain matrix is directly imposed by $\Gamma = w\tilde{z}^{-1}$ since for this particular case the subspace defining $\breve{K}$ is null. The same remark can be extended for $n = 1$ in (42), where the gain matrix is given by $\tilde{\Gamma} = \tilde{w}\tilde{v}^{-1}$. □

Illustrative example: We propose here an illustrative example of the reasoning leading to equations (39)–(18) (note that is similar in the case of (42)–(20)) and the subsequent values of the matrices involved. In this sense, let us consider a matrix $K \in \mathbb{R}^{2\times 2}$ which respects the constraint (14). Then, we can write:
$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$
which is equivalent to:
$$\underbrace{\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}}_{K^T\in\mathbb{R}^{2\times 2}} = \underbrace{\begin{bmatrix} 0 & w_1 z_2^{-1} \\ 0 & w_2 z_2^{-1} \end{bmatrix}}_{\Gamma\in\mathbb{R}^{2\times 2}} + \underbrace{\begin{bmatrix} k_{11} \\ k_{21} \end{bmatrix}}_{\breve{K}^T\in\mathbb{R}^{2}} \underbrace{\begin{bmatrix} 1 & -z_1 z_2^{-1} \end{bmatrix}}_{\Psi\in\mathbb{R}^{1\times 2}}.$$

---

[2] The system (19) is *output stabilizable* if the control law $u_t = \breve{K}y_t$ is stable, that is, $x_{t+1} = \left(A + B\tilde{\Gamma} + B\breve{K}\tilde{\Psi}\right) x_t$ is stable.

where each of the vectors/matrices corresponds with the notation in (39)–(18). A similar decomposition can be applied to another particular case, i.e. for $K \in \mathbb{R}^{3\times 3}$:

$$\underbrace{\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}}_{K^T \in \mathbb{R}^{3\times 3}} = \underbrace{\begin{bmatrix} 0 & 0 & w_1 z_3^{-1} \\ 0 & 0 & w_2 z_3^{-1} \\ 0 & 0 & w_3 z_3^{-1} \end{bmatrix}}_{\Gamma \in \mathbb{R}^{3\times 3}} + \underbrace{\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \\ k_{31} & k_{32} \end{bmatrix}}_{\tilde{K}^T \in \mathbb{R}^{3\times 2}} \underbrace{\begin{bmatrix} 1 & 0 & -z_1 z_3^{-1} \\ 0 & 1 & -z_2 z_3^{-1} \end{bmatrix}}_{\Psi \in \mathbb{R}^{2\times 3}}.$$

3.5 Local controller synthesis

Theorem 8 states that the controllability of system (1) with the gain matrix subject to a condition of type (14), (15), respectively, is equivalent with the controllability of a reduced order dynamical system. Specifically, in the case 1 of Theorem 8, the usual controllability tests (e.g. the gramian of controllability, controllability matrix) and design of gain matrix apply (e.g., pole placement, Linear Quadratic Regulator (LQR) or solving a Riccati equation). In the present paper we choose to construct the controller $\tilde{K}$ in (41) with a LQ design using the solution of the discrete algebraic Riccati equation

$$\tilde{A}^T P + P\tilde{A} - P\tilde{B}R^{-1}\tilde{B}^T P + Q = 0, \qquad (21)$$

where we denote $\tilde{A} = A + B\Gamma$ and $\tilde{B} = B\Psi^T$. Furthermore, assuming the system (41) is controllable and a suitable gain matrix $\tilde{K} = R^{-1}B^T P$ is obtained, it is simple to introduce it in (39) and to obtain the stabilizing K for the original dynamics (1) which additionally provide certain invariance properties with respect to a given hyperplane.

In the case 2 of Theorem 8, several design approaches can be proposed. Here we propose a solution close to the ideas in Crusius and Trofino [1999], based on a LMI conditions.

Proposition 10. Given the matrices $\tilde{\Gamma}$,B, $\tilde{\Psi}$, with B full column rank, any feasible solution in terms of matrices $G = G^T \succ 0$, M, N for

$$\begin{cases} \begin{bmatrix} -G & \left(A + B\tilde{\Gamma}\right)^T G + \tilde{\Psi}^T N^T B^T \\ G\left(A + B\tilde{\Gamma}\right) + BN\tilde{\Psi} & -G \end{bmatrix} \prec 0, \\ BM = GB. \end{cases} \qquad (22)$$

provides a gain matrix $\breve{K} = M^{-1}N\tilde{\Psi}$ stabilizing the system (19). □

Proof: See Appendix.

Subsequently, the matrix $\breve{K}$ can be replaced in (42) to obtain the desired state feedback gain matrix K.

Illustrative example: Figure 3 resumes the theoretical details discussed in this section by a graphical illustration. Therefore, $S \in \mathbb{R}^2$ is the interdicted region defined as in (3). All the equilibrium states lie on the hyperplane which trespass the origin (see, (9) and Remark 2). Solving the optimization problem (16), we find an eigenvector which approximates the normal to one of the frontiers of the interdicted region. Moreover, the hyperplane partition the space into invariant half-spaces (see Lemma 3).

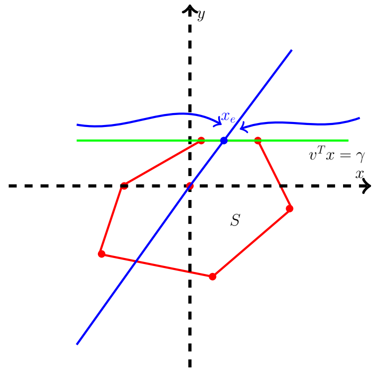

Fig. 3. Interdicted region and equilibrium point lying on the boundary of the feasible region

4. THE GLOBAL DESIGN PROBLEM

The goal of this paper is, ultimately, to define a control law which, given the system dynamics described by (1) and the interdicted region (3), transfers all the possible trajectories asymptotically to an equilibrium point lying *as close as possible* to the origin while respecting the constraints (6). With the results obtained in the previous sections, a local linear control feedback gain is available such that x_e, a point on the frontier of S, is an attractor for the closed-loop unconstraint trajectories. In the constrained case, the condition $x_t \notin S$ is assured only for a half-space, described by one of the supporting hyperplanes of S.

In the present section we will describe the procedure to ensure the stability of x_e by the use of a receding horizon optimal control procedure. Its design principles are related to the dual-mode control:

- a generic optimization-based control integrating collision avoidance constraints;
- its equivalence with the unconstrained feedback law (7) over an invariant region containing x_e;
- guarantees of convergence in finite time towards this invariant region.

Consider the system (1), the receding horizon optimization problem to be solved is formulated as:

$$\begin{aligned} \mathbf{u}^* = & \arg\min_{\mathbf{u}}((x_{t+N|t} - x_e)^T P (x_{t+N|t} - x_e) + \qquad (23) \\ & + \sum_{i=1}^{N-1} (x_{t+i|t} - x_e)^T Q (x_{t+i|t} - x_e) + \\ & + \sum_{i=0}^{N-1} (u_{t+i|t}^T - Kx_e - u_e) R (u_{t+i|t} - Kx_e - u_e)), \end{aligned}$$

$$\text{s.t.:} \begin{cases} x_{t+i+1|t} = Ax_{t+i|t} + Bu_{t+i|t}, \\ x_{t+i|t} \in \mathbb{R}^n \setminus S, \quad i = 1:N, \end{cases} \qquad (24)$$

with $\mathbf{u} = \{u_{t|t}, \ldots, u_{t+N-1|t}\}$. The parameters x_e, u_e and K, are determined in the previous section (see (7)–(9), (14)–(41)). Applying the first component of the optimal formulation (23)–(24) and reiterating the optimization using the new state x_k, considered measurable, we dispose of a global control law with the following properties (formulated here without the formal proofs which can be derived without difficulties based on the classical results in Mayne et al. [2000] and Chmielewski and Manousiouthakis [1996]):

- the optimization problem is recursively feasible (as consequence of the unbounded feasible domain);
- it is tractable (finite number of constraints);
- the matrices P, Q, R can be tuned upon inverse optimality principles to ensure the equivalence between the unconstrained optimum and the feedback control action (7), $u_t = K(x_t - x_e) + u_e$;
- reachability analysis can be used to determine the minimal horizon N such that the predicted state trajectory
$$v^T x_{t+N} \geq \gamma, \forall x_t \in (R^n \setminus S).$$

In order to avoid the difficulties of the reachability analysis in the choice of the prediction horizon N, one can use the following receding horizon formulation:

$$\begin{aligned}\mathbf{u}^* =& \arg\min_{\mathbf{u}}((x_{t+N|t} - x_e)^T P(x_{t+N|t} - x_e)+ \\ &+ \sum_{i=1}^{N-1}(x_{t+i|t} - x_e)^T Q(x_{t+i|t} - x_e)+ \\ &+ \sum_{i=0}^{N-1}(u_{t+i|t}^T - Kx_e - u_e)R(u_{t+i|t} - Kx_e - u_e)),\end{aligned} \quad (25)$$

$$\text{s.t.:} \begin{cases} x_{t+i+1|t} = Ax_{t+i|t} + Bu_{t+i|t}, \\ x_{t+i|t} \in \mathbb{R}^n \setminus S, \quad i = 1:N, \\ x_{t+N|t} \in v^T x_t \leq \gamma. \end{cases} \quad (26)$$

Note that the last constraint ensures the invariance and the contractivity of the domain of attraction of the corresponding closed-loop feasible states for (26). Furthermore, in this formulation the adding of constraints on the input can be handled, the direct consequence being the reduction of the feasible domain.

The former construction depends explicitly upon cost matrices P, Q, R. In usual designs, the same matrices are used to provide (via the Ricatti equation) the optimal gain K. Here however, we already have a proposed value for the gain, based upon invariance assumptions. Thus, an inverse optimality reasoning becomes necessary. Having the control law $u_t = Kx_t + u_e$, we deduce (see, Larin [2003] for further details) a triplet of cost matrices P, Q, R for which the matrix K would be the solution of the corresponding Ricatti equation.

On a more general note, a last aspect which need to be pointed out is that, under some reasonable assumptions it is always possible to define a so-called "viability kernel" (the interested reader is referred to Definition 4.4.1, p. 140, Aubin et al. [2011]) to which a trajectory can be steered in a finite time whilst satisfying the constraints (see, for extensive details Aubin et al. [2011]). In practical terms, this means that there exists a finite value of the prediction horizon such that the trajectory can be guaranteed to reach a terminal region and thus assure the feasibility of the scheme.

5. COLLISION AVOIDANCE EXAMPLES

A number of commonly found situations in the control related to Multi-Agent Systems imply a cost function that has to be minimized, while in the same time, the agent avoids collision with obstacles and other agents. To solve this problem, there exists various methods. Arguably, they can be gathered in methods which penalize through the cost function as the violation of the constraints (e.g. Potential Field Method Tanner et al. [2007], Navigation Functions Rimon and Koditschek [1992], and methods which impose hard constraints that may not be broken. The latter group usually employs receding horizon techniques as they naturally take into account constraints Richards and How [2005].

Example 1: In the first illustrative example, we will describe the limit behavior of an agent in the presence of adversary constraints. More precisely, the convergence to the relative position "zero" is impossible for the agent since a fixed convex obstacle contains the equilibrium position.

As a practical application, we consider a linear system (vehicle, pedestrian or agent in general form), whose dynamics is described by:

$$A = \begin{bmatrix} -0.78 & 0.33 \\ -0.85 & 1.08 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix} \quad (27)$$

The components of the state are the position coordinates of the agent. Note that the pair (A, B) is stabilizable. The state constraints as described in (3) are illustrated in Figure 4 by the red polytope. Solving the optimization problem (16), we obtained an affine parametrization of the gain matrix $K = \begin{bmatrix} -0.17 & -0.09 \\ 0.74 & -0.38 \end{bmatrix}$ as in (39) with $\tilde{K} = [-0.17 \ -0.09]$, $\Gamma = \begin{bmatrix} 0 & 0.86 \\ 0 & -0.31 \end{bmatrix}$ and $\Psi = [1 \ 0.70]$. This makes the closed-loop matrix to have a hyperplane of the interdicted region as an eigenvector. Furthermore, we obtained $u_e = [0.09 \ \ 1.29]$ and the equilibrium point $x_e = [0 \ \ 10.1]$, illustrated as a green dot in Figure 4. The tuning parameters of the optimization problem (23) are: $P = \begin{bmatrix} 0.59 & -0.04 \\ -0.04 & 0.50 \end{bmatrix}$, $Q = \begin{bmatrix} 0.11 & 0.30 \\ 0.30 & 0.21 \end{bmatrix}$, $R = \begin{bmatrix} 0.54 & -0.30 \\ -0.30 & 0.65 \end{bmatrix}$ and the prediction horizon $N = 2$. Finally, Figure 4 depicts three different state trajectories converging to a unique equilibrium point when the predictive control law (23) is applied.

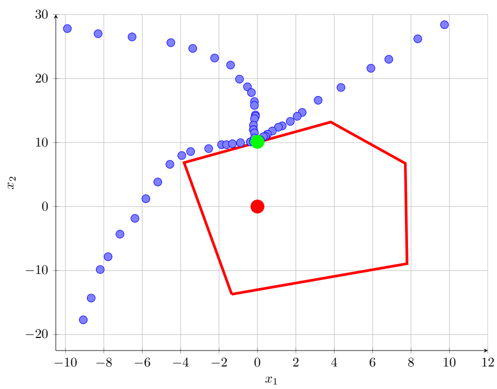

Fig. 4. The interdicted region and different agent state trajectories which converge to a fixed point.

Example 2: The method presented in the present paper can also be extended to solve collision or obstacles avoidance problems for multiple agents. This implies at the modeling stage a compact representation of the obstacles and/or a safety region for an agent in terms of (3). Consequently, a safety region can be associated to each agent and imposes that the inter-agent dynamics do not overlap each individual restriction. It is important to assure that a control action will not lead to a cycling behavior which implies energy consumption. Formally, the fact that a set of agent remains in (or arrives to) a unique configuration, as a result of some suitable control strategy, is equivalent with saying that in an extended space, there exists and it is unique a point which can be made fixed through the same control strategy.

Without entering into an exhaustive presentation, we will make use of the techniques presented in Prodan et al. [2011] and Stoican et al. [2011b], where the collision avoidance problem in the context of multi-agent formations is studied in detail. Consequently, here we will only illustrate that, by using the proposed method the agents converge to a unique configuration (i.e., to a unique equilibrium point in an extended state space). Figure 5 depicts the evolution of two heterogeneous agents with different associated safety regions (the blue and the red polytopes described as in (3)) and different initial positions.

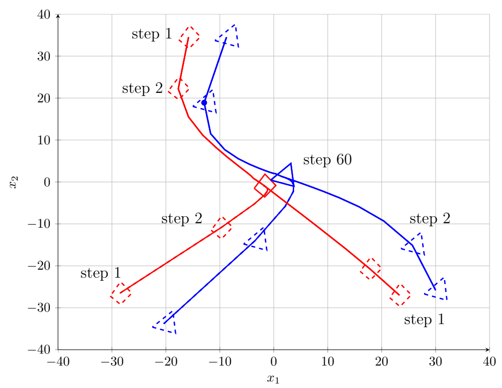

Fig. 5. The evolution of the agents with different initial positions at three different time steps.

6. CONCLUSIONS

A finite horizon predictive optimization problem formulation was proposed in order to describe the evolution of a linear system in the presence of a set of adversary constraints. This type of constraints are particular, as they make the convergence of the system trajectory to origin an infeasible task. We propose a dual-mode control law which switches between an unconstrained optimum controller and a local solution which handles the constraints activation, when necessary. Simple algebraic conditions for the existence and uniqueness of a stable fixed point on the boundary of the feasible region represent the main result of this paper, completed with an optimization based control for the global attractivity. The analyzed cases are presented through some illustrative examples and collision avoidance simulation results. Future work will focus on collision avoidance in the context of multi-agent formations and the stabilization of multiple agents around a limit cycle.

REFERENCES

Aubin, J.P. and Bayen, A. and Saint-Pierre, P. (2011): Viability Theory: New Directions. Springer Verlag.

Bemporad A., Morari M., Dua V., and Pistikopoulos E.N. (2002): The explicit linear quadratic regulator for constrained systems. Automatica, vol. 38(1):3–20.

Bitsoris G. and Truffet L. (2011): Positive invariance, monotonicity and comparison of nonlinear systems. Systems & Control Letters, vol. 60(12):960–966.

Chmielewski D. and Manousiouthakis V. (1996): On constrained infinite-time linear quadratic optimal control. Systems and Control Letters, vol. 29(3):121–129.

Crusius C.A.R. and Trofino A. (1999): Sufficient lmi conditions for output feedback control problems. *IEEE TAC*, 44(5):1053–1057.

Grundel D. and Pardalos P.M. (2004): Theory and algorithms for cooperative systems, vol. 4. World Scientific Pub Co Inc.

Jünger M., Liebling T.M., Naddef D., Nemhauser G., and Pulleyblank W.R. (2009): 50 Years of Integer Programming 1958-2008: From the Early Years to the State-of-the-Art. Springer Verlag.

Lafferriere G., Caughman J., and Williams A. (2005): Graph theoretic methods in the stability of vehicle formations. In Proceedings of the American Control Conference, vol. 4, pp. 3729–3734.

Larin V.B. (2003): About the inverse problem of optimal control. *Appl. Comput. Math*, vol. 2(2), pp. 90–97.

Maciejowski J.M. (2002): Predictive control: with constraints. Pearson education.

Mayne D.Q., Rawlings J.B., Rao C.V., and Scokaert P.O. (2000): Constrained model predictive control: Stability and optimality. Automatica, vol. 36, pp. 789–814.

Mayne D.Q., Seron M.M., and Raković S.V. (2005): Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2), pp. 219–224.

Pannocchia G., Wright S.J., and Rawlings J.B. (2003): Existence and computation of infinite horizon model predictive control with active steady-state input constraints. In IEEE Transactions on Automatic Control, vol. 48(6), pp. 1002–1006.

Prodan I., Stoican F., Olaru S., and Niculescu S.I. (2012): Enhancements on the Hyperplanes Arrangements in Mixed-Integer Techniques. *Journal of Optimization Theory and Applications*, vol. 154(2), pp. 549–572.

Prodan I., Olaru S., Stoica C., and Niculescu S.I. (2011): Predictive control for tight group formation of multi-agent systems. In *Proceedings of the 18th IFAC World Congress*, pp. 138–143.

Raković S.V. and Mayne D.Q. (2007): Robust model predictive control for obstacle avoidance: discrete time case. *Assessment and Future Directions of Nonlinear Model Predictive Control*, pp. 617–627.

Rakovic S.V., Blanchini F., Cruck E., and Morari M. (2007): Robust obstacle avoidance for constrained linear discrete time systems: A set-theoretic approach. In *Decision and Control, 2007 46th IEEE Conference on*, pp. 188–193.

Richards A. and How J.P. (2005): Model predictive control of vehicle maneuvers with guaranteed completion time and robust feasibility. In *Proceedings of the 24th IEEE ACC*, vol. 5, pp. 4034–4040.

Rimon E. and Koditschek D.E. (1992): Exact robot navigation using artificial potential functions. *IEEE Transactions on robotics and automation*, vol. 8(5), pp. 501–518.

Seron M.M., De Dona J.A., and Goodwin G.C. (2002): Global analytical model predictive control with input constraints. vol. 1, pp. 154–159.

Stoican F., Prodan I., and Olaru S. (2011): On the hyperplanes arrangements in mixed-integer techniques. In Proceedings of the 30th American Control Conference, pp. 1898–1903.

Stoican F., Prodan I., and Olaru S. (2011): Enhancements on the hyperplane arrangements in mixed integer techniques. In Proceedings of the 50th IEEE CDC- ECC, pp. 3986-3991.

Tanner H.G., Jadbabaie A., and Pappas G.J. (2007): Flocking in fixed and switching networks. In IEEE Transactions on Automatic Control, vol. 52(5), pp. 863–868.

APPENDIX

Proof of Theorem 5:
Applying Lemma 3 to system (8) leads to:

$$x_{t+1} = (A + BK)x_t + B(u_e - Kx_e) \tag{28}$$

and considering Remark 4 the invariance yields the following algebraic conditions:

$$v^T(A + BK) = \lambda v^T, \tag{29}$$

$$\lambda\gamma + v^T B(u_e - Kx_e) \leq \gamma, \tag{30}$$

$$-\lambda\gamma - v^T B(u_e - Kx_e) \leq -\gamma, \tag{31}$$

$$\lambda \geq 0. \tag{32}$$

Equation (29) directly proves that (λ, v^T) is a eigenvalue/*left* eigenvector pair. From (30) and (31) we have

$$\lambda\gamma + v^T B(u_e - Kx_e) = \gamma, \tag{33}$$

which can be rewritten as:

$$\lambda\gamma + v^T((A+BK)x_e + B(u_e - Kx_e)) - v^T(A+BK)x_e = \gamma. \tag{34}$$

Then, relation (34) becomes $\lambda\gamma + v^T x_e - \lambda v^T x_e = \gamma$ or $(1-\lambda)v^T x_e = (1-\lambda)\gamma$. Considering the hypothesis $\lambda \neq 1$, we obtain $v^T x_e = \gamma$, thus proving the sufficiency.

Conversely, the invariance of the half-space $v^T x \leq \gamma$ is equivalent to:

$$v^T(A + BK) = \lambda v^T, \tag{35}$$

$$\lambda\gamma + v^T B(u_e - Kx_e) \leq \gamma. \tag{36}$$

Condition (35) is satisfied by the eigenstructure properties. In the same time, x_e is an equilibrium state of the closed-loop system (28) lying on the hyperplane $v^T x = \gamma$. Then, x_e satisfies relation (9) and $v^T x_e = \gamma$. Exploiting these facts, condition (36) becomes $\lambda\gamma + v^T x_e - v^T(A + BK)x_e \leq \gamma$ and is equivalent to $\lambda\gamma + \gamma - \lambda v^T x_e \leq \gamma$. Finally, we have that $v^T x_e \geq \gamma$, which is trivially verified as long as $v^T x_e = \gamma$. Similar manipulations provides the invariance properties for the opposite half-space $v^T x \geq \gamma$ thus, proving the necessity. □

Proof of Proposition 6:
1) For the dynamics described by (1) and a given vector $v \in \mathbb{R}^n$, under controllability assumptions, there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that the pair (λ, v^T) is an eigenvalue/*left* eigenvector of matrix $(A + BK)$ and K verifies the linear constraint:

$$v^T B \cdot K = w^T, \tag{37}$$

with $w \in \mathbb{R}^n$ and $w^T \triangleq v^T(\lambda I_n - A)$. The equation (14) is obtained from (37) by considering $z = B^T v$ under full-column rank hypothesis, concerning the matrix B.

2) Similarly, let $K \in \mathbb{R}^{m\times n}$ such that the pair (λ, v) is an eigenvalue/*right* eigenvector of matrix $(A + BK)$:

$$(A + BK)v = \lambda v. \tag{38}$$

If we rewrite (38) in the form $(A - \lambda I_n)v = -BKv$ we obtain the linear constraint (15) with $\tilde{w} \in \mathbb{R}^m$, a solution of the system of equations: $(A - \lambda I_n)v = -B\tilde{w}$. □

Proof of Theorem 8:
1) We start by decomposing $z \in \mathbb{R}^m$ in (14) into two elements $z = [\hat{z}\ \tilde{z}]$ such that the element $\tilde{z} \in \mathbb{R}^*$ (a non-zero scalar) and $\hat{z} \in \mathbb{R}^{m-1}$. Then, decomposing $K^T \in \mathbb{R}^{n\times m}$ similarly into $\hat{K}^T \in \mathbb{R}^{n\times 1}$ and $\tilde{K}^T \in \mathbb{R}^{n\times(m-1)}$ we can express after simple algebraic manipulations $\hat{K}^T$ as a function of w, $\hat{z}$ and $\tilde{z}^{-1}$. Furthermore, introducing this into the original equality (14) we obtain an affine relation with K^T:

$$K^T = \Gamma + \tilde{K}^T \cdot \Psi \tag{39}$$

with $\Gamma \in \mathbb{R}^{n\times m}$ and $\Psi \in \mathbb{R}^{(m-1)\times n}$ defined as in (18). Using the above parametrization, relation (39) can be introduced into the closed-loop matrix as follows:

$$A + BK = A + B\left(\Gamma^T + \Psi^T\tilde{K}\right) = \left(A + B\Gamma^T\right) + B\Psi^T\tilde{K}. \tag{40}$$

This leads to a reformulation of the original dynamics (1):

$$x_{t+1} = \left(A + B\Gamma^T\right)x_t + B\Psi^T\tilde{K}x_t \tag{41}$$

and complete the equivalence between the original constrained stabilization problem and the controllability of the pair $(A + B\Gamma^T, B\Psi^T)$.

2) Similarly, $v \in \mathbb{R}^n$ in (15) can be decomposed into two elements $v = [\hat{v}\ \tilde{v}]$ such that the element $\tilde{v} \in \mathbb{R}^*$ and $\hat{v} \in \mathbb{R}^{n-1}$. As in the previous case, we obtain an affine description of $K \in \mathbb{R}^{m\times n}$ using the independent parameters contained in $\breve{K} \in \mathbb{R}^{m\times(n-1)}$:

$$K = \tilde{\Gamma} + \breve{K} \cdot \tilde{\Psi}, \tag{42}$$

with $\tilde{\Gamma} \in \mathbb{R}^{m\times n}$ and $\tilde{\Psi} \in \mathbb{R}^{(n-1)\times n}$ defined in (20). Using the parametrization (42) we obtain:

$$A + BK = A + B\left(\tilde{\Gamma} + \breve{K}\cdot\tilde{\Psi}\right) = \left(A + B\tilde{\Gamma}\right) + B\breve{K}\tilde{\Psi}. \tag{43}$$

This leads to a reformulation of the original dynamics (1) into a novel formulation as described in (19). □

Proof of Proposition 10:
If B is full column rank, then it follows from $BM = GB$ that M is also full rank, and thus invertible. This allows to obtain $B = GBM^{-1}$. Furthermore, the desire control law has the structure $u_t = -M^{-1}N\tilde{\Psi}x_t$. Exploiting these facts, from the first condition we obtain that

$$\begin{bmatrix} -G & ((A + B\tilde{\Gamma}) + B\breve{K})^T G \\ G((A + B\tilde{\Gamma}) + B\breve{K}) & -G \end{bmatrix} \prec 0, \tag{44}$$

or equivalently using Schur complement:

$$-G + ((A + B\tilde{\Gamma}) + B\breve{K})^T G((A + B\tilde{\Gamma}) + B\breve{K}) \prec 0,$$

which proves that system (19) is stabilizable via the proposed state feedback. □