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# A new characterisation of exponential stability

Elena Panteley Antonio Loria

**Abstract**—We present a new characterization of exponential stability for nonlinear systems in the form of Lyapunov functions which may be upper and lower bounded by monotonic functions satisfying a growth order relationship rather than being polynomials of the state’s norm. In particular, one may allow for Lyapunov functions with arbitrary weakly homogeneous bounds.

## I. INTRODUCTION

Consider the ordinary differential equation

$$\dot{x} = f(t, x) \quad (1)$$

where  $f$  is continuous in  $t$  and locally Lipschitz in  $x$  uniformly in  $t$ . Let  $x = 0$  be an equilibrium point of the latter and denote the solutions with initial conditions  $t_o \in \mathbb{R}_+$  and  $x_o \in \mathbb{R}^n$ , by  $x(t, x_o, t_o)$ . We recall that the trivial solution  $x = 0$  is uniformly exponentially stable if there exist constants  $k, \lambda$  an  $r$  such that

$$|x_o| < r, t \in \mathbb{R}_+ \Rightarrow |x(t, x_o, t_o)| \leq k|x_o|e^{-\lambda(t-t_o)}. \quad (2)$$

We say that the origin is uniformly globally exponentially stable if  $r = \infty$ .

Exponential stability of nonlinear systems described by ordinary differential equations dates back at least to Krasovskii’s work in the late 1950s –cf. [4, Theorem 11.1]. The classical characterisation of uniform global exponential stability involves a Lyapunov function which satisfies upper and lower quadratic bounds of  $|x|$ . This has been extended to bounds that are polynomial of any order –cf. [2], [10], [3]:

**Theorem 1** *Let  $x = 0$  be an equilibrium point for  $\dot{x} = f(t, x)$ , where  $f$  is a locally Lipschitz function and  $D \subset \mathbb{R}^n$  be a domain containing  $x = 0$ . Then  $x = 0$  is uniformly exponentially stable if and only if there exist a continuously differentiable function  $V : [0, \infty) \times D \rightarrow \mathbb{R}_+$  such that*

$$k_1|x|^p \leq V(t, x) \leq k_2|x|^p, \quad (3a)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x) \leq -k_3|x|^p \quad (3b)$$

for all  $t \geq t_o \geq 0$  and all  $x \in D$ , where  $p$  and  $k_i, i = 1, 2, 3$  are positive constants. If the assumptions hold globally, then  $x = 0$  is uniformly globally exponentially stable.

Equivalent characterizations, in terms of a Lyapunov functions decreasing at sampling times have been reported in the context of adaptive control. See for instance [6], [3, Theorem 4.5] and [1]. For time-varying differential equations with

The authors are with the C.N.R.S. at LSS-SUPELEC, 3, Rue Joliot Curie, 91192 Gif s/Yvette, France E-mail: panteley[loria]@lss.supelec.fr

locally Lipschitz right-hand side, the following is essentially contained in [5]:

**[Integral characterization of UGES]** For the dynamical system  $\dot{x} = f(x, t)$  with  $f$  locally Lipschitz and  $\sup_{x \neq 0} |f(x, t)|/|x| < \infty$  the origin is uniformly globally exponentially stable if and only if there exists  $\gamma > 0$  and  $p \geq 1$  such that

$$\int_0^\infty |x(t, x_o, t_o)|^p dt \leq \gamma|x_o|^p. \quad (4)$$

In [9] exponential stability is characterized by integral conditions for systems described by differential inclusions (convex, upper semi-continuous). Such characterization is useful to establish UGES from inputoutput interconnection properties involving, e.g.  $L_2$  bounds. In this short note we give new *differential* characterisation of exponential stability equivalent to (3) but which does not rely on *explicit* polynomial bounds but functions satisfying a growth-order relation. Further results are established for weakly homogeneous systems.

## II. MAIN RESULTS

Consider the system

$$\dot{x} = f(t, x), \quad f(t, 0) = 0 \quad (5)$$

**Theorem 2** *Let  $B_r := \{x \in \mathbb{R}^n : |x| < r\}$  and suppose that  $f(t, \cdot)$  is Lipschitz on  $B_r$  uniformly in  $t$ . Let  $V : \mathbb{R}_+ \times B_r \rightarrow \mathbb{R}_+$  be a continuously differentiable function such that for all  $x \in B_r$ , all  $t \geq t_o$  and all  $t_o \in \mathbb{R}_+$*

$$\alpha_1(|x|) \leq V(t, x) \leq \alpha_2(|x|) \quad (6)$$

$$\dot{V}(t, x) \leq -\mu V(t, x), \quad (7)$$

where  $\mu > 0$  is a constant and  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  (functions  $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ , strictly increasing, zero at zero and proper). Then, the origin of (5) is exponentially stable on  $B_r$  if and only if there exist constants  $c > 0, c_1 > 0$  and  $c_2 \in (0, 1)$  such that the following inequalities hold for all  $s \geq 0$ :

$$\alpha_1^{-1} \circ \alpha_2(s) \leq cs \quad (8)$$

$$\alpha_1^{-1} \circ (c_1 \alpha_2(s)) \leq c_2 s. \quad (9)$$

If all the conditions of the theorem are satisfied globally (i.e. if  $r = +\infty$ ) then the origin is uniformly globally exponentiable.

### Proof of Theorem 2

I. Sufficiency.

Following the arguments of the proof of [3, Theorem 3.8] we can show that, for any given  $D$  and any constants  $\rho > 0$  and  $r > 0$  which satisfy  $B_r \subset D$  and  $\rho < \alpha_1(r)$ , all the

solutions of (5) starting from the set of initial conditions  $x_0 \in \{x \in B_{\alpha_2^{-1}(\rho)}\}$  are well defined and moreover  $x(x_0, t) \in B_r$  for all  $t \geq 0$ .

Now, let  $T = \frac{1-c_1}{\mu c_1}$  if  $c_1 < 1$  and  $T = 1$  otherwise. From (7) it follows that  $V(x(t)) \leq V(x(\tau))e^{-\mu(t-\tau)}$  for all  $t \geq \tau \geq 0$  hence,  $V(x(t)) \leq V(x(\tau))$  for all  $\tau \in [0, t)$ .

Integrating (6) from  $t$  to  $t + T$  (with  $T$  defined above) we obtain

$$\begin{aligned} V(x(t+T)) - V(x(t)) &\leq -\mu \int_t^{t+T} V(x(\tau))d\tau \\ &\leq -\mu \int_t^{t+T} V(x(t+T))d\tau \\ &\leq -\mu TV(x(t+T)) \quad \forall t \geq 0 \end{aligned}$$

where in the second step above we used the fact that  $V(x(\tau)) \geq V(x(t+T))$  for all  $\tau \in [t, t+T]$ . Hence,  $(1+\mu T)V(x(t+T)) \leq V(x(t))$  for all  $t \geq 0$ . From this we obtain

$$\begin{aligned} V(x(t+T)) &\leq \frac{1}{1+\mu T}V(x(t)) \\ &\leq c_1V(x(t)) \quad \text{if } c_1 \leq 1 \\ V(x(t+T)) &\leq \frac{1}{1+\mu T}V(x(t)) \\ &\leq V(x(t)) \leq c_1V(x(t)) \quad \text{if } c_1 \geq 1 \end{aligned} \tag{11}$$

Therefore,  $V(x(t+T)) \leq c_1V(x(t))$  for all  $t \geq 0$ . Using the bounds (11) and (6) we obtain

$$\alpha_1(|x(t+T)|) \leq c_1\alpha_2(|x(t)|) \quad \forall t \geq 0.$$

Then, from (9) it follows that for all  $t \geq 0$

$$|x(t+T)| \leq c_2|x(t)|. \tag{12}$$

The rest follows the proof guidelines of [3, Theorem 4.5]. For any  $t \geq 0$  let  $N = \lfloor \frac{t}{T} \rfloor$ , where  $\lfloor \cdot \rfloor$  stands for the lower integer part. Divide the interval  $[t - NT, t]$  into  $N$  equal subintervals then,

$$\begin{aligned} |x(t)| &\leq c_2|x(t-T)| \\ &\leq c_2^2|x(t-2T)| \\ &\vdots \\ &\leq c_2^N|x(t-NT)|. \end{aligned} \tag{13}$$

Since for all  $t \geq 0$  we have  $V(x(t)) \leq V(x_0)$  therefore,

$$|x(t)| \leq \alpha_1^{-1}\alpha_2(|x_0|) \quad \forall t \in [0, T] \tag{14}$$

consequently, from (8) it follows that  $|x(t)| \leq c|x_0|$  for all  $t \in [0, T]$ . Combining the last bound with (13) we obtain

$$|x(x_0, t)| \leq c_2^N c|x_0| \leq cc_2^{t/T}|x_0| = c|x_0|e^{-bt},$$

where  $b = \frac{1}{T} \ln \frac{1}{c_2}$ . In case conditions (11)-(9) are satisfied globally we obtain that the system is UGES.

## II. Necessity

Assume that the system (5) is (globally) exponentially stable for all  $x \in D$  ( $x \in R^n$ ) i.e., there exist  $k, \gamma > 0$  such that the trajectories of (5) satisfy

$$|x(t, x_0)| \leq k|x_0|e^{-\gamma t} \quad \forall t \geq 0, x_0 \in D \quad (x_0 \in R^n).$$

Then, from the converse theorem on exponential stability (see for example [3]) it follows that there exists a Lyapunov function  $V : D \rightarrow R_+$  ( $V : R^n \rightarrow R_+$ ) that satisfies the inequalities

$$\begin{aligned} a_1|x|^2 &\leq V(x) \leq a_2|x|^2 \\ \frac{dV}{dx}f(x) &\leq -a_3|x|^2 \end{aligned}$$

for some positive constants  $a_i, i = 1, 2, 3$ .

Hence, the functions  $\alpha_i$  ( $i = 1, 2, 3$ ) in (11), (7) are given by  $\alpha_i(s) = a_i s^2$ . Simple calculations show that inequalities (8), (9) are satisfied. Indeed,  $\alpha_1^{-1}(s) = \left(\frac{s}{a_1}\right)^{1/2}$  therefore, for arbitrary  $c > 0$  we have

$$\alpha_1^{-1}(c_2\alpha_2(s)) = \left(\frac{c\alpha_2(s)}{a_1}\right)^{1/2} = \left(\frac{ca_2s^2}{a_1}\right)^{1/2} = \sqrt{\frac{ca_2}{a_1}}s.$$

From this it follows trivially that (8) is satisfied with  $c = \sqrt{a_2/a_1}$  and (9) is satisfied for arbitrary  $c_2 \in (0, 1)$  with  $c_1 = \frac{a_1c_2^2}{a_2}$ . ■

## Remark 1

- If all conditions of the theorem 2 are satisfied except (8), then proceeding as before we obtain from (13) and (14) that

$$|x(t, x_0)| \leq \alpha(|x_0|)e^{-bt}$$

- The first part of the statement i.e., without requiring (8), (9) is equivalent to UGAS –cf. [8], [7].

## III. EXPONENTIAL STABILITY FROM WEAKLY HOMOGENEOUS BOUNDS

**Definition 1** A real function  $f : R^n \rightarrow R$  is said to be homogeneous of order  $k$  if for any constant  $\alpha \geq 0$  the following inequality holds:

$$f(\alpha x) = \alpha^k f(x).$$

Classical conditions imposed on the bounds of  $V(x)$  and its derivative to insure exponential stability are formulated in terms of powers of  $x$  which are evidently homogeneous functions. In this section we show that this classical result can be extended to the class of systems with weakly homogeneous bounds on  $V(x)$  and its derivative.

Our second theorem is stated in terms of Lyapunov functions satisfying weakly homogeneous bounds. We recall that real function  $f : R_+ \rightarrow R_+$  is weakly homogeneous if for some function  $L \in \mathcal{K}_\infty$  one has

$$f(\lambda x) \leq L(\lambda)f(x) \tag{15}$$

for all  $\lambda > 0$ .

**Theorem 3** *The origin of  $\dot{x} = f(x)$  is a uniformly exponentially stable equilibrium if and only if there exists continuously differentiable function  $V : B_r \rightarrow \mathbb{R}_+$ , a weakly homogeneous function  $\alpha \in \mathcal{K}_\infty$  such that for all  $x \in D$  and all  $t \geq 0$*

$$a_1\alpha(|x|) \leq V(x) \leq a_2\alpha(|x|) \quad (16)$$

$$\dot{V}(x) \leq -a_3V, \quad (17)$$

for some positive constants  $a_1, a_2, a_3$ . If all the conditions of the theorem are satisfied globally (with  $r = +\infty$ ) then the system (5) is uniformly globally exponentially stable.

We wrap up this note with an example for which uniform global exponential stability is difficult to conclude from Theorem 1 yet, it may be concluded from our main results.

*Example.* Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + \frac{x_2 g(x)}{1 + x_1^2} \\ \dot{x}_2 &= -x_2 + \frac{x_1 g(x)}{1 + x_2^2} \end{aligned}$$

where  $g$  is any locally Lipschitz function. It is easy to show that the origin is uniformly globally exponentially stable by invoking Theorem 3 and using the Lyapunov function

$$V(x) = \frac{1}{2}(x_1^4 + x_2^4) + x_1^2 + x_2^2.$$

Indeed the total time derivative of  $V$  yields  $\dot{V} \leq -2V$ . Note that this function does not have lower nor upper polynomial bounds however, the function  $\alpha_1(s) := \frac{1}{2}V(s)$  is of the same growth order as  $\alpha_2(s) := 2V(s)$ .  $\diamond$

*Proof.* I. Sufficiency.

From theorem 2 it follows that we need only to verify that conditions (8) and (9) are satisfied. To simplify the calculations let us take  $a_1 = 1$  (what can be always done just by choosing e.g.  $V_{new}(x) = V(x)/a_1$ ). Since  $\alpha(s)$  is a weakly homogeneous function, then there exists  $l \in \mathcal{K}_\infty$  such that  $\alpha(\lambda s) \geq l(\lambda)\alpha(s)$  for all  $\lambda > 0$ . Then, for any  $c_1 > 0$ , defining  $k = l^{-1}(c_1 a_2)$  we have

$$\alpha^{-1}(c_1 a_2 \alpha(s)) \leq \alpha^{-1}(l(k)\alpha(s)) \leq \alpha^{-1}(\alpha(ks)) = ks \quad (18)$$

Inequality (18) is valid for any  $c_1 > 0$ , therefore it is valid for  $c_1 = 1$ , hence (8) is satisfied. Moreover, we can always choose  $c_1$  so that  $k = l^{-1}(c_1 a_2) < 1$ , so that (9) is satisfied as well. Therefore all the conditions of the theorem 2 are satisfied and therefore the system (5) is exponentially stable.

II. Necessity

The “only if” part of the proof follows directly the steps of the proof of Theorem 2.  $\blacksquare$

#### IV. CONCLUSIONS

We have presented a new Lyapunov-like characterization of exponential stability for ordinary differential equations. Such characterization covers naturally sufficient and necessary conditions for particular cases such as  $\mathcal{K}$ -exponential stability and for homogeneous systems.

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