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Rate-distortion bounds for Wyner-Ziv coding with Gaussian scale mixture correlation noise

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***Abstract*—The objective of this work is the characterization of the Wyner-Ziv rate-distortion function for memoryless continuous sources, when the correlation between the sources is modeled via an additive noise channel. Modeling the distribution of the correlation noise via a Gaussian mixture, with discrete or continuous mixing variable, provides a unified signal model able to describe a wide class of distributions, useful in the context of practical applications. The Wyner-Ziv rate-distortion function associated with this signal model cannot, in general, be obtained in analytical form. This work contributes a method for its analysis, by providing computable upper and lower bounds.**

***Index Terms*—Gaussian scale mixture noise, rate-distortion bounds, Wyner-Ziv.**

I. Introduction

THE Wyner-Ziv problem refers to lossy compression of a source, when a correlated side information is available at the decoder, but not at the encoder [1]. For jointly Gaussian sources and quadratic distortion measure the Wyner-Ziv rate-distortion function is known in closed form [2], and can be used as comparison in the assessment of the performance in the design of coding schemes, see, *e.g.*, [3]–[5]. The quadratic Gaussian setup, however, fails to capture the features of the real signals involved in practical applications (*e.g.*, distributed video coding [6], data compression in sensor networks [7]), where the choice of an accurate model is crucial to ensure good performance [8]–[10]. This work contributes tools for the characterization of the Wyner-Ziv rate-distortion function for a class of distributions of interest in this context.

The sources are assumed continuous, memoryless, and the quadratic distortion measure is considered. The correlation is modeled via an additive noise channel. When the variances of the source and of the side information are fixed, it is established that the Wyner-Ziv rate-distortion function for Gaussian distribution of the sources provides an upper bound to the achievable performance of any coding scheme.

The main limitation of the quadratic Gaussian setup is the assumption that the correlation between the sources is constant and known during the entire transmission. A more realistic model should, at least in first approximation, allow the correlation to vary in time, in a non-transparent way for the encoder and the decoder. The correlation channel is hence assumed governed by a hidden discrete state variable, whose realization determines the additive noise variance. The state variable follows a memoryless distribution law, known *a priori*, and the resulting correlation noise follows a Gaussian mixture law. Strictly related models are considered in [11], where the Wyner-Ziv correlation channel is controlled by a state variable, whose realization is assumed to vary slowly in time, and in [12], in relation to the Heegard-Berger problem, with the state variable representing the fading state observed by the decoder, but unknown at the encoder. The correlation noise model is extended, by allowing the state variable to be defined on continuous support. The correlation noise results distributed according to a Gaussian scale mixture law [13], which accounts for a wider class of distributions (*e.g.*, generalized Gaussian, Student-t, exponential, Laplace distributions), including those habitually targeted in the design of practical applications [9].

In order to characterize the Wyner-Ziv rate-distortion function for the mixture correlation models, we derive computable lower and upper bounds. The lower bound is established considering the performance of a genie-aided setup, where the encoder and the decoder are informed of the realizations of the hidden state variable. The upper bound is derived using test-channel characterization, and provides, in most cases, a refinement with respect to the Gaussian-equivalent upper bound.

The paper is organized as follows. Section II formally defines the considered signal model, and states the main results. The proofs to the lower bound and to the upper bound are detailed in Section III and Section IV respectively. Section V provides numerical examples.

II. Bounds to the Wyner-Ziv rate-distortion function

A. The signal model

The statistical dependence between the source and the side information symbols X and Y is captured by the additive channel

$$Y = X + Z,$$

where X and Z are independent, and $X \sim \mathcal{N}(0, \sigma_X^2)$. The correlation noise symbol Z is given by $Z = \sqrt{S}V$, where S and V are independent, and $V \sim \mathcal{N}(0, 1)$. The hidden state variable S has distribution law $\mathbb{P}_S$ and support in $\mathbb{R}^+$. The probability density function (pdf) of Z is referred to as a scale

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mixture of normal distributions [13]

$$p_Z(z) = \int \frac{1}{(2s\pi)^{1/2}} \exp\left(-\frac{z^2}{2s}\right) \mathrm{d}\mathbb{P}_S(s). \quad (1)$$

The second moment of Z is $\sigma_Z^2 = \mathbb{E}[S]$. The pdf (1) assumes different forms, according to the choice of $\mathbb{P}_S$. A few examples are the Gaussian mixture density, obtained for S defined over a finite support; the Student-t density, obtained for $S \sim \text{Inv Gamma}(\alpha, \alpha)$; the Laplacian density, obtained for S following an exponential distribution [14].

B. Notation

Several functions will be used in the following. Let X be the source symbol, and denote by W_e and W_d the side information symbols available at the encoder and at the decoder, respectively. The distortion measure $d : \mathcal{X} \times \widehat{\mathcal{X}} \to \mathbb{R}^+$ is the quadratic error $d(X, \widehat{X}) = \| X - \widehat{X} \|^2$.

The conditional rate-distortion function [15] describes the asymptotically achievable performance of the setup where identical side information is available at the encoder and at the decoder, *i.e.*, $W_\mathrm{e} = W_\mathrm{d} = W$:

$$R_{X|W}(D) = \inf_{\widehat{X} \,:\, \mathbb{E}[d(X,\widehat{X})] \leq D} I(X; \widehat{X}|W). \quad (2)$$

The Wyner-Ziv rate-distortion function [2] describes the achievable performance of the setup where the side information is available only at the decoder, *i.e.*, $W_\mathrm{e} = \emptyset$, $W_\mathrm{d} \neq \emptyset$:

$$R^{\mathrm{wz}}_{X|W_\mathrm{d}}(D) = \inf_{U \in \mathcal{M}(D)} \big(I(X;U) - I(U;W_\mathrm{d}) \big). \quad (3)$$

In (3), $\mathcal{M}(D)$ is the set of auxiliary random variables U satisfying the constraint $U \leftrightarrow X \leftrightarrow W_\mathrm{d}$, for which it exists a reconstruction function $\widehat{X} : \mathcal{U} \times \mathcal{W}_d \to \widehat{\mathcal{X}}$ such that $\mathbb{E}\big[d(X, \widehat{X})\big] \leq D$.

The mixed-side information rate-distortion function [16] describes the achievable performance of the setup where the encoder and the decoder have access to different side information[1], *i.e.*, $W_\mathrm{e} \neq W_\mathrm{d}$:

$$R^{\mathrm{wz}}_{X,W_\mathrm{e}|W_\mathrm{d}}(D) = \inf_{U \in \mathcal{N}(D)} \big(I(X, W_\mathrm{e}; U) - I(U; W_\mathrm{d}) \big). \quad (4)$$

In (4), $\mathcal{N}(D)$ is the set of auxiliary random variables U satisfying the constraint $U \leftrightarrow (X, W_\mathrm{e}) \leftrightarrow W_\mathrm{d}$, and for which it exists a reconstruction function $\widehat{X} : \mathcal{U} \times \mathcal{W}_d \to \widehat{\mathcal{X}}$ such that $\mathbb{E}\big[d(X, \widehat{X})\big] \leq D$.

C. Main result

This work focusses on the Wyner-Ziv rate distortion function $R^{\mathrm{wz}}_{X|Y}(D)$, defined in (3), with $W_\mathrm{d} = Y$, for the signal model introduced in Section II-A. The difference $I(X;U) - I(U;Y)$ in (3) cannot, in general, be expressed in analytical form, thus making the minimization problem (3) hard to solve explicitly[2]. A further obstacle to the computability of (3) is given by the dependence on the auxiliary variable U. As a consequence of Theorem 1 (stated below), a first approximation of the behavior of $R^{\mathrm{wz}}_{X|Y}(D)$ is deduced from the rate-distortion function $R^{\mathrm{wz}}_{X|\underline{Y}}(D)$, given by (3) for $W_\mathrm{d} = \underline{Y} = X + \underline{Z}$, and $\underline{Z} \sim \mathcal{N}(0, \mathbb{E}[S])$, which describes its upper bound. The analysis of $R^{\mathrm{wz}}_{X|Y}(D)$ can nevertheless be refined by characterization of computable lower and upper bounds dependent on the statistical distribution of the hidden state variable S.

As discussed in [18], the natural lower bound to $R^{\mathrm{wz}}_{X|Y}(D)$ is the corresponding conditional rate-distortion function $R_{X|Y}(D)$, given by (2) with $W = Y$. A remarkable result in [18] is the characterization of the difference $R^{\mathrm{wz}}_{X|Y}(D) - R_{X|Y}(D)$, upper bounded by 0.5 bits regardless of the distribution of the sources. For Gaussian scale mixture correlation noise, however, the conditional rate-distortion $R_{X|Y}(D)$ is non computable, due to the difficulty of identifying the MMSE estimator of a non-Gaussian variable. The results in [18], hence, do not allow to derive analytical lower and upper bounds to $R^{wz}_{X|Y}(D)$ for the considered signal model. An alternative lower bound is identified here in the rate-distortion function $R^{\mathrm{wz}}_{X,S|Y,S}(D)$ for the mixed side information setup, given in (4) with $W_\mathrm{e} = S$ and $W_\mathrm{d} = (Y, S)$, which can be expressed in analytical form. It corresponds to the performance of the genie-aided setup where the realization of S is instantaneously known to the system. The upper bound is obtained via a test channel characterization. It describes the maximum rate penalty incurred with respect to $R^{\mathrm{wz}}_{X,S|Y,S}(D)$, when S is undisclosed. The lower and upper bounds, generalizing the results presented in [19] for S with discrete support, are formally stated in Theorem 2 and Theorem 3, and proven in Section III and Section IV, respectively.

Theorem 1. *Let the correlation between the source symbol X and the side information symbol Y be modeled via an additive noise channel,* i.e.*, $Y = X + Z$ or $X = Y + Z$. Let σ_X^2 and σ_Z^2 be the variances of the source and correlation noise symbols, respectively. The Wyner-Ziv rate-distortion function $R^{\mathrm{wz}}_{X|Y}(D)$ is upper bounded as*

$$R^{\mathrm{wz}}_{X|Y}(D) \leq R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(D), \quad (5)$$

where $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(D)$ is the Wyner-Ziv rate-distortion function for source $\underline{X} \sim \mathcal{N}(0, \sigma_X^2)$ and correlation noise $\underline{Z} \sim \mathcal{N}(0, \sigma_Z^2)$.

Proof: Consider first the correlation channel $Y = X + Z$. Denote $\mathcal{M}_{X|Y}(D)$ the minimization set relative to $R^{\mathrm{wz}}_{X|Y}(D)$ and $\mathcal{M}_{\underline{X}|\underline{Y}}(D)$ the minimization set relative to $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(D)$. Let $(U, \widehat{X}) \in \mathcal{M}_{X|Y}(D)$ be the test channel defined by letting $U = X + \Psi$, with $\Psi \sim \mathcal{N}(0, \sigma_\Psi^2)$, independent on X, and by letting $\widehat{X}$ be the linear MMSE estimate of X from the observations (U, Y) [20]

$$\widehat{X} = \left(\frac{1}{\sigma_X^2} + \frac{1}{\sigma_Z^2} + \frac{1}{\sigma_\Psi^2} \right)^{-1} \left(\frac{Y}{\sigma_Z^2} + \frac{U}{\sigma_\Psi^2} \right). \quad (6)$$

[1]The Wyner-Ziv rate-distortion function for mixed side information $R^{\mathrm{wz}}_{X,W_\mathrm{e}|W_\mathrm{d}}(D)$ is derived in [16] employing the following argument [17, Sec. 2]: the system can be seen as a standard Wyner-Ziv setup, where (X, W_e) is the source to be encoded, W_d is the side information at the decoder, and the distortion constraint $\mathbb{E}[d(X, \widehat{X})] \leq D$ is imposed only on the reconstruction of X.

[2]An obvious exception is when the support of S is a singleton. In this case the signal model reduces to the quadratic Gaussian, for which the optimum test channel is known [2, Sec. 3], and $R^{\mathrm{wz}}_{X|Y}(D)$ is found in closed form.

Let the test channel $(\underline{U}, \widehat{\underline{X}}) \in \mathcal{M}_{\underline{X}|\underline{Y}}(\underline{D})$ be defined analogously. The MSE on the reconstruction depends only on the first and second moments of the source and observation noise [20], hence $D = \underline{D} = \left(1/\sigma_X^2 + 1/\sigma_\Psi^2 + 1/\sigma_Z^2\right)^{-1}$. The test channel $(\underline{U}, \widehat{\underline{X}}) \in \mathcal{M}_{\underline{X}|\underline{Y}}(D)$ is the optimum test channel, achieving the rate-distortion function $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(D)$ [2, Sec. 3]. Comparison with the rate $r^{\mathrm{wz}}_{X|Y}(D)$ achieved by the test channel $(U, \widehat{X}) \in \mathcal{M}_{X|Y}(D)$, and evaluated as the difference of mutual informations in (3), yields

$$\begin{aligned} R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(D) - r^{\mathrm{wz}}_{X|Y}(D) & \qquad (7)\\ &= h(\underline{X}+\Psi|\underline{X}+\underline{Z}) - h(X+\Psi|X+Z) \\ &\quad + h(X+\Psi|X) - h(\underline{X}+\Psi|\underline{X}) \\ &= h\big((1-\alpha)\underline{X} + \Psi - \alpha\underline{Z}\big) \\ &\quad - h\big((1-\alpha)X + \Psi - \alpha Z \big| X+Z\big) \qquad (8)\\ &\geq h\big((1-\alpha)\underline{X} + \Psi - \alpha\underline{Z}\big) \\ &\quad - h\big((1-\alpha)X + \Psi - \alpha Z\big) \geq 0, \qquad (9) \end{aligned}$$

with $\alpha = \sigma_X^2/(\sigma_X^2+\sigma_Z^2)$, and where $\alpha(\underline{X}+\underline{Z})$ and $\alpha(X+Z)$ are the linear MMSE estimates of $\underline{X}$ and X, respectively. Both the source $\underline{X}$ and the observation $(\underline{X}+\underline{Z})$ are Gaussian, and the linear estimate $\alpha(\underline{X}+\underline{Z})$ is optimal in the MMSE sense, with error $X - \alpha(\underline{X}+\underline{Z})$ independent on the observation $(\underline{X}+\underline{Z})$ [20]. This is used to obtain the first term in (8). The last inequality in (9) is obtained observing that the first member is the differential entropy of a Gaussian random variable. This shows that the test channel $(U, \widehat{X}) \in \mathcal{M}_{X|Y}(D)$ provides an achievable rate-distortion performance at least as good as $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(D)$, hence (5).

Now turn to the correlation channel $X = Y + Z$. Define the test channel $(U, \widehat{X}) \in \mathcal{M}_{X|Y}(D)$ such that $U = X + \Psi$, with $\Psi \sim \mathcal{N}(0, \sigma_\Psi^2)$, independent on X. The reconstruction function is given by

$$\widehat{X} = \alpha(U - Y) + Y, \qquad (10)$$

where $\alpha = \sigma_Z^2/(\sigma_Z^2 + \sigma_\Psi^2)$, and $\alpha(U-Y) = \alpha(Z+\Psi)$ is the linear MMSE estimate of Z from the observation $(Z+\Psi)$. The distortion on the reconstruction of X is given by $D = \sigma_Z^2\sigma_\Psi^2/(\sigma_Z^2+\sigma_\Psi^2)$. Similarly, define the test channel $(\underline{U}, \widehat{\underline{X}}) \in \mathcal{M}_{\underline{X}|\underline{Y}}(\underline{D})$, by letting $\underline{U} = \underline{X} + \Phi$, with $\Phi \sim \mathcal{N}(0, \sigma_\Phi^2)$, independent on $\underline{X}$. Let the reconstruction function $\widehat{\underline{X}}$ be defined via the linear MMSE estimator of $\underline{Z}$, analogously to (10). The distortion on the reconstruction of $\underline{X}$ is given by $\underline{D} = \sigma_Z^2\sigma_\Phi^2/(\sigma_Z^2+\sigma_\Phi^2)$. The test channel $(\underline{U}, \widehat{\underline{X}}) \in \mathcal{M}_{\underline{X}|\underline{Y}}(\underline{D})$ achieves the Wyner-Ziv rate-distortion function $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(\underline{D})$ [2, Sec. 3]. Impose the rate $r^{\mathrm{wz}}_{X|Y}(D)$ achieved with the test channel $(U, \widehat{X}) \in \mathcal{M}_{X|Y}(D)$ to be equal to $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(\underline{D})$. Developing the difference of mutual informations in (3) yields $r^{\mathrm{wz}}_{X|Y}(D) = I(X;U|Y) = I(Z;Z+\Psi)$ and, similarly, $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(\underline{D}) = I(\underline{Z};\underline{Z}+\Phi)$. A necessary condition to $r^{\mathrm{wz}}_{X|Y}(D) = R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(\underline{D})$ is given by

$$h(\underline{Z}) - h(Z) = h(\underline{Z}|\underline{Z}+\Phi) - h(Z|Z+\Psi). \qquad (11)$$

Exploiting the symmetry of the Gaussian distribution, (11) is reformulated as

$$h(\underline{Z}) - h(Z) = h(\Phi) - h(\Psi). \qquad (12)$$

The first term in (12) is non-negative, due to the fact that $\underline{Z}$ is Gaussian. Since both Ψ and Φ are normally distributed, the condition (12) implies $\sigma_\Phi^2 \geq \sigma_\Psi^2$. This, in consequence, implies $\underline{D} \geq D$. The test channel $(U, \widehat{X}) \in \mathcal{M}_{X|Y}(D)$, then, provides rate-distortion performance at least as good as $R^{\mathrm{wz}}_{\underline{X}|\underline{Y}}(\underline{D})$, hence (5). ∎

Theorem 2. *Consider the signal model defined in Section II-A. The Wyner-Ziv rate-distortion function $R^{\mathrm{wz}}_{X|Y}(D)$ is lower bounded as*

$$R^{\mathrm{wz}}_{X,S|Y,S}(D) \leq R^{\mathrm{wz}}_{X|Y}(D). \qquad (13)$$

The lower bound $R^{\mathrm{wz}}_{X,S|Y,S}(D)$ has analytic form

$$R^{\mathrm{wz}}_{X,S|Y,S}(D) = \frac{1}{2}\,\mathbb{E}\left[\log_2\left(\frac{\varphi(S)}{\Delta^*(S)}\right)\right], \qquad (14)$$

with

$$\varphi(S) = \frac{\sigma_X^2 S}{(\sigma_X^2 + S)} \qquad (15)$$

and

$$\Delta^*(s) = \begin{cases} \lambda & \text{if } \varphi(s) > \lambda \\ \varphi(s) & \text{else,} \end{cases} \qquad (16)$$

where $\lambda = \frac{1}{(2\ln 2)\mu}$ and such that $\mathbb{E}[\Delta^*(S)] = D$.

Theorem 3. *Consider the signal model defined in Section II-A. The Wyner-Ziv rate-distortion function $R^{\mathrm{wz}}_{X|Y}(D)$ is upper bounded as*

$$\begin{aligned} &R^{\mathrm{wz}}_{X|Y}(D) \qquad (17)\\ &\quad \leq \min\left\{R^{\mathrm{wz}}_{X,S|Y,S}(D) + L(D) + I(S;Z)\ ,\ R^{\mathrm{wz}}_{X|\underline{Y}}(D)\right\}. \end{aligned}$$

The term $L(D)$ in (17) is given by

$$L(D) = \frac{1}{2}\,\mathbb{E}\left[\log_2\left(D\left(\frac{1}{\varphi(S)} - \frac{1}{\sigma_X^2}\right) + 1\right)\right], \qquad (18)$$

and $R^{\mathrm{wz}}_{X|\underline{Y}}(D)$ refers to the Wyner-Ziv rate-distortion function (3), for $\underline{Y} = X + \underline{Z}$ and $\underline{Z} \sim \mathcal{N}(0, \mathbb{E}(S))$.

Remark 1. *The upper bound (17) is reminiscent of the results in [18], where the rate loss between the Wyner-Ziv and the conditional rate-distortion functions is characterized. Similarly, (17) expresses an upper bound to the rate loss between $R^{\mathrm{wz}}_{X|Y}(D)$ and $R^{\mathrm{wz}}_{X,S|Y,S}(D)$. The term $L(D)$ given in (18) is monotonic decreasing in D, and vanishes as $D \to 0$. In the high rate – low distortion regime, thus, the upper bound to the rate loss reduces to the constant $I(S;Z)$. This expresses the additional information available in the genie-aided setup, where the state variable S is observed by the encoder and the decoder.*

III. The lower bound

This section is devoted to the proof of Theorem 2. Proposition 1 provides the analytical expression of the conditional rate-distortion function $R_{X|Y,S}(D)$, given by (2) for $W = (Y,S)$. Proposition 2 shows that the Wyner-Ziv setup with mixed side information (4), for $W_{\mathrm{e}} = S$, $W_{\mathrm{d}} = (Y,S)$, suffers no loss with respect to the conditional setup, and hence $R_{X|Y,S}(D) = R^{\mathrm{wz}}_{X,S|Y,S}(D)$. Finally, Proposition 3 shows that

$R^{\mathrm{WZ}}_{X,S|Y,S}(D)$ is a legitimate lower bound to $R^{\mathrm{WZ}}_{X|Y}(D)$. This completes the proof.

Proposition 1. *Consider the signal model defined in Section II-A. The analytical form of the conditional rate-distortion function $R_{X|Y,S}(D)$ is given by*

$$R_{X|Y,S}(D) = \frac{1}{2}\ \mathbb{E}\left[\log_2\left(\frac{\varphi(S)}{\Delta^*(S)}\right)\right], \tag{19}$$

where $\varphi(S)$ and $\Delta^*(S)$ are the functions defined in (15) *and* (16)*, respectively.*

Proof: The rate-distortion function $R_{X|Y,S}(D)$ is defined by the minimization problem (2), which can be marginalized with respect to the state variable[3] S. The mutual information takes form

$$I(X;\widehat{X}|Y,S) = \int I(X;\widehat{X}|Y,s)\ \mathrm{d}\mathbb{P}_S(s). \tag{20}$$

Express the distortion constraint as

$$\mathbb{E}[d(X,\widehat{X})] = \mathbb{E}_S\big[\mathbb{E}_{X,\widehat{X}|S}[d(X,\widehat{X})|S]\big] = \mathbb{E}[\Delta(S)], \tag{21}$$

where $\Delta(S) \in L^1(\mathrm{d}\mathbb{P}_S)$ is a measurable function such that $\mathbb{E}[\Delta(S)] \le D$. Using (20) and (21) the minimization problem (2) is recast as

$$R_{X|Y,S}(D) = \tag{22}$$
$$\inf_{\Delta\in L^1:\mathbb{E}[\Delta]\le D}\left(\int \inf_{\widehat{X}:\mathbb{E}[d(X,\widehat{X})|s]\le\Delta(s)} I(X;\widehat{X}|Y,s)\ \mathrm{d}\mathbb{P}_S(s)\right).$$

where $L^1 = L^1(\mathrm{d}\mathbb{P}_S)$. The minimization problem in the integrand in (22) defines the Wyner-Ziv rate-distortion function for $Y = X + Z$, $Z \sim \mathcal{N}(0,s)$, whose analytical expression is known [15]. Hence (22) is rewritten as

$$R_{X|Y,S}(D) = \tag{23}$$
$$\inf_{\Delta\in L^1(\mathrm{d}\mathbb{P}_S):\mathbb{E}[\Delta]\le D}\frac{1}{2}\ \mathbb{E}\left[\log_2\left(\frac{\varphi(S)}{\Delta(S)}\right)\mathbb{1}_{\{S:\Delta(S)\le\varphi(S)\}}\right],$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function. The explicit form of the rate-distortion (23) is found solving the optimization problem

$$\Delta^*(S) = \underset{\Delta\in L^1(\mathrm{d}\mathbb{P}_S):\Delta\le\varphi,\mathbb{E}(\Delta)\le D}{\arg\min}\ \frac{1}{2}\ \mathbb{E}\left[\log_2\left(\frac{\varphi(S)}{\Delta(S)}\right)\right], \tag{24}$$

where the functional is reformulated with the constraint that Δ has to satisfy $\Delta(S) \le \varphi(S)$ for every $S \in \mathbb{R}^+$. Notice that $\varphi(S)$, defined in (15), is a continuous increasing function from $\mathbb{R}^+$ to $(0,\sigma_X^2)$. Hence, if $D \ge \sigma_X^2$, the optimal function is given by $\Delta^*(S) = \varphi(S)$, $\forall S \in \mathbb{R}^+$, and only the solutions for $D < \sigma_X^2$ need to be found.

When S has finite support, the optimization problem (24) is equivalent to the reverse water-filling problem in [21, Sec. 10.3.3]. The system is supposed to operate in a time-division regime, where the state variable S takes the value s for the fraction of the transmission time corresponding to $\mathbb{P}(S=s)$. The optimum allocation of the distortion constraints to be satisfied in each fraction of time, expressed by the function $\Delta^*(S)$, determines the optimum allocation of the rate.

Assume now that S has a continuous distribution, and let $\mathrm{d}\mathbb{P}_S = \phi_S\ \mathrm{d}s$ denote its pdf. Now (24) expresses a constrained differentiable convex infinite-dimensional optimization problem defined in the space $L^1(\mathrm{d}\mathbb{P}_S)$. The generalized Karush-Kuhn-Tucker theorem (see [22], for instance) ensures that solving (24) is equivalent to finding the stationary point of the Lagrangian functional

$$\begin{cases} \exists\Lambda\in L^\infty(\mathrm{d}\mathbb{P}_S),\ \exists\mu\in\mathbb{R}^+ \\ \qquad L(\Delta,\mu) = \mathbb{E}\left[\frac{1}{2}\log_2\left(\frac{\varphi(S)}{\Delta(S)}\right) + \mu\Delta(S) + \Lambda(S)\Delta(S)\right], \\ \forall s\in\mathbb{R}^+\ \ \Lambda(s)(\Delta(s)-\varphi(s)) = 0, \\ \text{and } \mu\mathbb{E}[\Delta(S)-D] = 0. \end{cases} \tag{25}$$

The Gateaux derivative at Δ of the functional in (25) is defined by the unique linear application $dL_\Delta : L^1(\mathrm{d}\mathbb{P}_S) \to \mathbb{R}$ such that

$$\forall h\in L^1(\mathrm{d}\mathbb{P}_S) \qquad \lim_{t\to 0}\frac{|L(\Delta+th,\mu) - L(\Delta,\mu) - tdL_\Delta(h)|}{t} = 0.$$

This gives

$$\begin{aligned} &L(\Delta+th,\mu) - L(\Delta,\mu) \\ &\quad= \mathbb{E}\left[\frac{1}{2}\log_2\left(\frac{\varphi(S)}{\Delta(S)+th(S)}\right) - \frac{1}{2}\log_2\left(\frac{\varphi(S)}{\Delta(S)}\right)\right. \\ &\qquad\qquad \left. + th(S)\mu + th(S)\Lambda(S)\right] \\ &\quad= \mathbb{E}\left[-\frac{1}{2\ln 2}t\frac{h(S)}{\Delta(S)} + th(S)(\mu+\Lambda(S)) + o(t^2)\right]. \end{aligned}$$

By setting

$$dL_\Delta(h) = \mathbb{E}\left[-\frac{1}{2\ln 2}\frac{h(S)}{\Delta(S)} + h(S)(\mu+\Lambda(S))\right]$$

one obtains

$$\frac{|L(\Delta+th,\mu) - L(\Delta,\mu) - tdL_\Delta(h)|}{t} = o(t),$$

which tends to zero as $t\to 0$. Hence the solution to (24) is the function Δ such that, for any $h\in L^1(\mathrm{d}\mathbb{P}_S)$,

$$\begin{cases} \displaystyle\int_{\mathbb{R}^+}\left(-\frac{1}{2\ln 2}\frac{1}{\Delta(s)}\mathbb{1}_{\{s:\Delta\le\varphi\}} + \mu + \Lambda(s)\right)h(s)\phi_S(s)\ \mathrm{d}s = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall h\in L^1(\mathrm{d}\mathbb{P}_S), \\ \forall s\in\mathbb{R}^+\ \ \Lambda(s)(\Delta(s)-\varphi(s)) = 0, \\ \text{and } \mu\mathbb{E}[\Delta(S)-D] = 0. \end{cases} \tag{26}$$

The second condition in (26) defines two cases. If $s\in\mathbb{R}^+$ is such that $\Delta(s) = \varphi(s)$ then $\Lambda(s) > 0$ and $\Delta(s) < \frac{1}{(2\ln 2)\mu}$. Else, $\Lambda(s) = 0$ and $\Delta(s) = \frac{1}{(2\ln 2)\mu}$ is a constant such that $\frac{1}{(2\ln 2)\mu} < \varphi(s)$. This yields (16), and completes the proof. ■

Remark 2. *Alternatively, Proposition 1 can be proven following the strategy in [23, Sec. 4.5]. If S has discrete support, the rate-distortion function $R_{X|Y,S}(D)$ can be seen as the*

[3]This is reminiscent of [15, Thm. 5], with the difference that the marginalization is done with respect to only one component of the side information (Y,S).

optimum rate allocation for the compression of parallel, independent Gaussian sources of variances $\varphi(s_i)^{p(s_i)}$. Through the analysis of the parametric representation of $R_{X|Y,S}(D)$ it is determined that the optimum $\Delta^*(\cdot)$ takes the form (16)*. The validity of* (16) *for continuous support of S is then verified binning $\mathbb{R}^+$, and taking the limit of the solution as the volume of each bin vanishes.*

Proposition 2. *For the signal model defined in Section II-A, the mixed side information rate-distortion function for $W_{\mathrm{e}} = S$, $W_{\mathrm{d}} = (Y, S)$ suffers no rate loss with respect to the conditional rate-distortion function*

$$R_{X|Y,S}(D) = R^{\mathrm{WZ}}_{X,S|Y,S}(D). \tag{27}$$

Proof: Applying the same marginalization argument used to obtain (2), the rate-distortion $R^{\mathrm{WZ}}_{X,S|Y,S}(D)$ is expressed as

$$R^{\mathrm{WZ}}_{X,S|Y,S}(D) = \tag{28}$$
$$\inf_{\Delta\in L^1:\mathbb{E}[\Delta]\le D} \int \inf_{U\in\mathcal{N}(\Delta(s))} \big(I(X,s;U) - I(U;Y,s)\big)\ \mathrm{d}\mathbb{P}_S(s),$$

where $\mathcal{N}(\Delta(s))$ is the minimization set of the mixed side information setup under the distortion constraint $\mathbb{E}[d(X,\widehat{X})|s] \le \Delta(s)$. The innermost minimization problem in (28) corresponds to the definition (3) of the Wyner-Ziv rate-distortion function $R^{\mathrm{WZ}}_{X|Y}(\Delta(s))$, for $Y = X + Z$ and $Z \sim \mathcal{N}(0, s)$. The analytical form of the optimum test channel achieving $R^{\mathrm{WZ}}_{X|Y,s}(\Delta(s))$ is provided in [2, Sec. 3]. Therefore (28) can be written as

$$R^{\mathrm{WZ}}_{X,S|Y,S}(D) = \inf_{\Delta\in L^1(\mathrm{d}\mathbb{P}_S):\mathbb{E}[\Delta]\le D} \frac{1}{2}\ \mathbb{E}\left[\log_2\left(\frac{\varphi(S)}{\Delta(S)}\right)_+\right]. \tag{29}$$

Comparison of (29) with (23) completes the proof. ∎

Proposition 3. *For the signal model defined in Section II-A, the mixed side information rate-distortion function $R^{\mathrm{WZ}}_{X,S|Y,S}(D)$ lower bounds the Wyner-Ziv rate-distortion function $R^{\mathrm{WZ}}_{X|Y}(D)$*

$$R^{\mathrm{WZ}}_{X,S|Y,S}(D) \le R^{\mathrm{WZ}}_{X|Y}(D). \tag{30}$$

Proof: The presence of the common side information S in the mixed side information setup is, at worst, non beneficial, hence inequality (29). ∎

IV. The upper bound

This section is devoted to the proof to Theorem 3. Theorem 1 and Proposition 4 provide the two upper bounds.

Proposition 4. *For the signal model defined in Section II-A, the Wyner-Ziv rate-distortion function $R^{\mathrm{WZ}}_{X|Y}(D)$ is upper bounded by*

$$R^{\mathrm{WZ}}_{X|Y}(D) \le R^{\mathrm{WZ}}_{X,S|Y,S}(D) + L(D) + I(S;Z), \tag{31}$$

where $L(D)$ is defined in (18)*.*

Proof: Let $\Psi \sim \mathcal{N}(0,\sigma^2_\Psi)$, independent on X. Define $U = X + \Psi$, and let the reconstruction function $\widehat{X} = \mathbb{E}[X|U,Y]$ be the MMSE estimate of X from the observations U and Y. Marginalization with respect to S yields

$$\begin{aligned}\widehat{X} &= \mathbb{E}_{S|Y,U}\big[\mathbb{E}_{X|U,Y,S}[X|U,Y,S]\big] \\ &= \mathbb{E}_{S|Y,U}\left[\frac{\sigma^2_X(SU+\sigma^2_\Psi Y)}{\sigma^2_X(S+\sigma^2_\Psi)+\sigma^2_\Psi S}\right].\end{aligned} \tag{32}$$

The variance $\sigma^2_\Psi = \sigma^2_\Psi(D)$ in (32) is chosen to satisfy the distortion constraint $\mathbb{E}[d(X,\widehat{X})] \le D$. The test channel $(U,\widehat{X})$ provides the rate-distortion performance $r^{\mathrm{WZ}}_{X|Y}(D)$, which is evaluated as the difference of mutual informations in (3)

$$\begin{aligned}&r^{\mathrm{WZ}}_{X|Y}(D) = \\ &\quad h(U) - h(U|X) - h(Y|S) + h(Y|U,S) + I(S;U|Y).\end{aligned} \tag{33}$$

For $\alpha = \sigma^2_X/(\sigma^2_X+\sigma^2_\Psi)$, the expression αU is the MMSE estimate of X from the observation $U = X + \Psi$. The term $h(Y|U,S)$ in (33) is developed as

$$\begin{aligned}h(Y|U,S) &= h(X+Z-\alpha(X+\Psi)|X+\Psi,S) &(34)\\ &= h(X+Z-\alpha(X+\Psi)|S), &(35)\end{aligned}$$

where (35) is obtained noticing that Z is independent on X and Ψ, and that the estimation error $X-\alpha(X+\Psi)$ is independent on the observation $X+\Psi$. Since Ψ, U, $(Y|S)$, $(X+Z-\alpha(X+\Psi)|S)$ are Gaussian random variables, (33) becomes

$$\begin{aligned}&r^{\mathrm{WZ}}_{X|Y}(D) = \\ &\quad \frac{1}{2}\int \log_2\left(\frac{\varphi(s)+\sigma^2_\Psi(D)}{\sigma^2_\Psi(D)}\right)\ \mathrm{d}\mathbb{P}_S(s) + I(S;U|Y).\end{aligned} \tag{36}$$

Comparison of (36) with the lower bound (14) yields the following inequality chain

$$\begin{aligned}&r^{\mathrm{WZ}}_{X|Y}(D)\\ &\quad= R^{\mathrm{WZ}}_{X,S|Y,S}(D) + I(S;U|Y)\\ &\qquad+\frac{1}{2}\int\log_2\left(\Delta^*(s)\left(\frac{1}{\varphi(s)}+\frac{1}{\sigma^2_\Psi(D)}\right)\right)\mathrm{d}\mathbb{P}_S(s)\\ &\quad\le R^{\mathrm{WZ}}_{X,S|Y,S}(D) + I(S;U|Y) &(37)\\ &\qquad+\frac{1}{2}\int\log_2\left(\Delta^*(s)\left(\frac{\sigma^2_X-D}{\sigma^2_X\ D}+\frac{1}{\varphi(s)}\right)\right)\mathrm{d}\mathbb{P}_S(s)\\ &\quad\le R^{\mathrm{WZ}}_{X,S|Y,S}(D) + I(S;U|Y) &(38)\\ &\qquad+\frac{1}{2}\int\log_2\left(D\left(\frac{1}{\varphi(s)}-\frac{1}{\sigma^2_X}\right)+1\right)\mathrm{d}\mathbb{P}_S(s)\\ &\quad\le R^{\mathrm{WZ}}_{X,S|Y,S}(D) + I(S;Z) &(39)\\ &\qquad+\frac{1}{2}\int\log_2\left(D\left(\frac{1}{\varphi(s)}-\frac{1}{\sigma^2_X}\right)+1\right)\mathrm{d}\mathbb{P}_S(s).\end{aligned}$$

The inequality $\sigma^2_\Psi(D) \ge \sigma^2_X\ D/(\sigma^2_X - D)$ used to obtain (37) is justified noticing that the distortion D obtained by estimating X using both Y and U (32) does not exceed the distortion $\sigma^2_X\sigma^2_\Psi(D)/(\sigma^2_X+\sigma^2_\Psi(D))$ obtained by estimation from U alone. The inequality (38) is obtained applying Jensen's inequality $\mathbb{E}\big[\log_2(\Delta^*(S))\big] \le \log_2\big(\mathbb{E}[\Delta^*(S)]\big) = \log_2(D)$. The inequality (39) is obtained using the relations

$$\begin{aligned}I(S;U|Y) &= h(S|Y) - h(S|Y,U)\\ &\le h(S|Y) - h(S|Y,U,Z)\\ &\le h(S) - h(S|Z) = I(S;Z),\end{aligned} \tag{40}$$

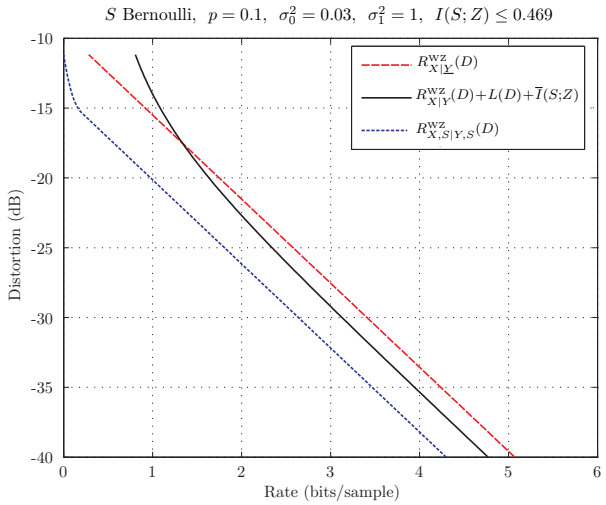


Fig. 1. The lower and upper bounds to the rate-distortion function $R^{\mathrm{WZ}}_{X|Y}(D)$ for Bernoulli-Gaussian distribution of the correlation noise.

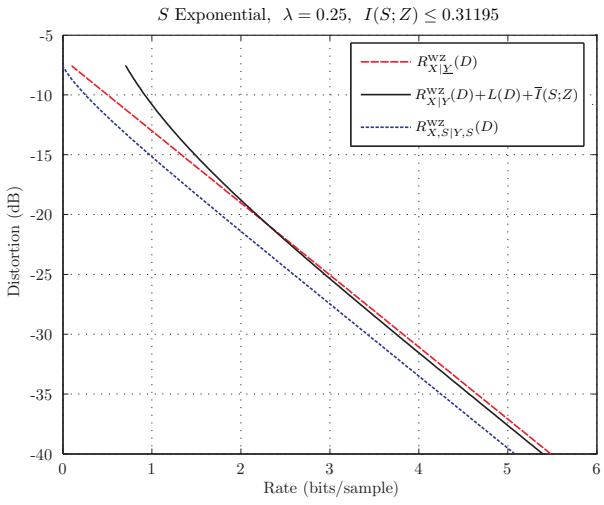


Fig. 2. The lower and upper bounds to the rate-distortion function $R^{\mathrm{WZ}}_{X|Y}(D)$ for Laplacian distribution of the correlation noise.

where (40) follows from the fact that (Y, U) depends on S only through Z. Substitution of (18) in (39) allows the inequality chain

$$R^{\mathrm{WZ}}_{X|Y}(D) \leq r^{\mathrm{WZ}}_{X|Y}(D) \leq R^{\mathrm{WZ}}_{X,S|Y,S}(D) + L(D) + I(S;Z), \tag{41}$$

which provides (31). ■

V. Illustrations

This section considers, in three examples, the numerical evaluation of the bounds to $R^{\mathrm{WZ}}_{X|Y}(D)$ given in Theorem 2 and Theorem 3, for the signal model introduced in Section II-A. The lower bound is given by (14): for each D, λ is chosen such that $\mathbb{E}\left[\Delta^*(S)\right] = D$, and $R^{\mathrm{WZ}}_{X,S|Y,S}(D)$ is obtained via numerical integration. The upper bound (17) requires the evaluation of $L(D)$, which is obtained via numerical integration of (37). The term $I(S;Z)$ is approximated by its upper bound $\overline{I}(S;Z)$, defined as follows

$$\begin{aligned} I(S;Z) &= h(Z) - h(Z|S) \\ &\leq h(\underline{Z}) - h(Z|S) \triangleq \overline{I}(S;Z), \end{aligned} \tag{42}$$

with $\underline{Z} \sim \mathcal{N}(0, \mathbb{E}[S])$.

A. Bernoulli-Gaussian distribution of the correlation noise

Let S be a random variable following a Bernoulli distribution of parameter p, such that $\mathbb{P}(S = 0) = 1 - p$ and $\mathbb{P}(S = 1) = p$. Define σ_0^2 and σ_1^2 as the variances of the correlation noise $(Z|S)$ associated with the events $(S = 0)$ and $(S = 1)$ respectively. The correlation noise Z is distributed according to a Bernoulli-Gaussian mixture with density $p_Z(z) = (1-p)\,\mathcal{N}(0, \sigma_0^2) + p\,\mathcal{N}(0, \sigma_1^2)$. Figure 1 depicts the lower bound (14) and the upper bounds (17) to $R^{\mathrm{WZ}}_{X|Y}(D)$, for $p = 0.1$, $\sigma_0^2 = 0.03$ and $\sigma_1^2 = 1$. In this case, $I(S;Z) \leq 0.469$ bit/symbol.

B. Laplace distribution of the correlation noise

Let S be distributed according to the single-sided exponential distribution with parameter λ

$$\phi_S(s) = \begin{cases} \frac{1}{\lambda}\exp\left(-\frac{s}{\lambda}\right) & s \geq 0, \\ 0 & s < 0. \end{cases} \tag{43}$$

The correlation noise Z results distributed according to the Laplace density

$$p_Z(z) = \frac{1}{2}\sqrt{\frac{2}{\lambda}}\exp\left(-\sqrt{\frac{2}{\lambda}}z\right).$$

Figure 2 depicts the lower bound (14) and the upper bounds (17) to $R^{\mathrm{WZ}}_{X|Y}(D)$, for $\lambda = 0.25$. In this case, $I(S;Z) \leq 0.312$ bit/symbol.

C. Student-t distribution of the correlation noise

Now, S is distributed according to an inverse Gamma distribution with parameters α and β

$$\phi_S(s) = \frac{\beta^\alpha}{\Gamma(\alpha)} s^{-\alpha-1}\exp\left(-\frac{\beta}{s}\right). \tag{44}$$

When $\alpha = \beta = \frac{\nu}{2}$, Z is distributed according to the Student-t density

$$p_Z(z) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\nu}\,\Gamma\left(\frac{\nu}{2}\right)}\left(1 + \frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}.$$

Figure 3 depicts the lower bound (14) and the upper bounds (17) to $R^{\mathrm{WZ}}_{X|Y}(D)$, for $\alpha = \beta = 2$. In this case, $I(S;Z) \leq 0.178$ bit/symbol.

VI. Conclusions

This work addresses the problem of the characterization of the Wyner-Ziv rate-distortion function, when the correlation channel between the sources is represented using an additive noise channel. It is shown that the Wyner-Ziv rate-distortion

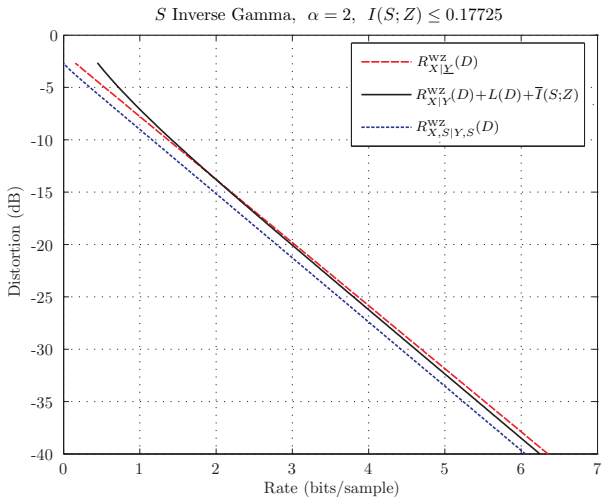


Fig. 3. The lower and upper bounds to the rate-distortion function $R^{\mathrm{WZ}}_{X|Y}(D)$ for Student-t distribution of the correlation noise.

function relative to the quadratic-Gaussian setup upper bounds, when the variance of the sources is fixed, the achievable performance of any coding scheme, independently on the distribution of the sources. A wide class of correlation noise distributions can be described via Gaussian mixture modeling, employing discrete or continuous mixing variables. A unified method of analysis allows to derive computable upper and lower bounds to the corresponding Wyner-Ziv rate-distortion functions, which can be useful as comparison in the assessment of the performance of practical coding schemes.

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