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***Abstract*—This letter investigates hybrid lower bounds on the mean square error in order to predict the so-called threshold effect. A new family of tighter hybrid large error bounds based on linear transformations (discrete or integral) of a mixture of the McAulay-Seidman bound and the Weiss-Weinstein bound is provided in multivariate parameters case with multiple test points. For use in applications, we give a closed-form expression of the proposed bound for a set of Gaussian observation models with parameterized mean, including tones estimation which exemplifies the threshold prediction capability of the proposed bound.**

***Index Terms*—Parameter estimation, mean-square-error bounds, threshold SNR, Hybrid bounds, MAPMLE.**

I. Introduction

Since its introduction in the context of array shape calibration [2], hybrid parameter estimation has given rise to a growing interest as both random and nonrandom parameters occur simultaneously in miscellaneous estimation problems [2][3][4][5][6][7][8][9][10]. However, the hybrid estimation framework is not just the simple concatenation of the Bayesian and non-Bayesian techniques. Indeed, new estimator has to be derived as one can generally no longer use the maximum likelihood estimator (MLE) for the non-Bayesian part and the maximum *a posteriori* estimator (MAP) for the Bayesian part since the parameters are generally statistically linked [11, §1.1]. Similarly, performance analysis method of such estimators has to be modified accordingly, which is the aim of hybrid lower bounds on the mean square error (MSE).

The first hybrid lower bound, the so-called Hybrid Cramér-Rao bound (HCRB), has been introduced in the context of random parameters with prior probability density function (p.d.f.) independent of deterministic parameters [2]. This initial characterization of hybrid estimation has been generalized first in [4] where the so-called Hybrid Barankin Bound (HBB) is derived in order to handle the threshold phenomena. Additionally it is shown in [4] that one limiting form of the HBB yields the HCRB. An extension of the HCRB where the prior p.d.f. of the random parameters depends on deterministic parameters has been proposed in [8] and further analyzed in [12] providing a necessary and sufficient condition under which the HCRB of the nonrandom parameters is equal to the CRB and asymptotically tight. All these works have shown that in many estimation problems, like the deterministic CRB and Bayesian CRB (BCRB), the HCRB is tight in the asymptotic region only, i.e., when the signal-to-noise ratio (SNR) is high and/or the number of independent observations is large. Thus, since the knowledge of the particular value for which this threshold appears is fundamental in order to define estimators optimal operating area, others lower bounds of the class of Large Error bounds, i.e. able to reveal the threshold phenomena, have been studied to predict the threshold value [7].

As a contribution, we proposed in [1], for a single deterministic parameter, a single random parameter and two test points, a new hybrid lower bound which combines the McAulay-Seidman bound (MSB) [13] and the Weiss-Weinstein bound (WWB) [14]. This combination was motivated by the fact that, among the Bayesian bounds [15][16], the WWB is known to be one of the tightest, and, among the deterministic bounds, the MSB is usually used to approximate the Barankin Bound (BB) [13][17][18][19], the greatest lower bound on the MSE on deterministic parameters. In the present letter, we deal with the multivariate case, whatever the number of test points. Additionally, we introduce a more general family of Hybrid bounds (tighter that the existing ones [4][7]) which combines the generalization of the HBB introduced in [7] and the WWB in order to extend the family of Bayesian bounds introduced in [16] to hybrid estimation. Last, for use in applications, we give a closed-form expression of the proposed bound for a set of Gaussian observation models with parameterized mean, including tones estimation which exemplifies the threshold prediction capability of the proposed bound.

II. The hybrid McAulay-Seidman-Weiss-Weinstein bound (HMSWWB)

Throughout the present letter $\mathbf{x}$ denotes the random observation vector, Ω denotes the observations space. $\boldsymbol{\theta} = (\boldsymbol{\theta}_d; \boldsymbol{\theta}_r)$[1] denotes a K-dimensional hybrid real parameter vector to estimate $(K = D + R)$, where $\boldsymbol{\theta}_d$ is a vector of unknown deterministic parameters belonging to a subset Π_d of $\mathbb{R}^D$ and $\boldsymbol{\theta}_r$ is a vector of unknown random parameters belonging to $\mathbb{R}^R$. $\Pi_r \subseteq \mathbb{R}^R$ denotes the support of the prior p.d.f. of $\boldsymbol{\theta}_r$ denoted $f(\boldsymbol{\theta}_r|\boldsymbol{\theta}_d)$. Let $\Lambda_{\boldsymbol{\theta}_d} = \left\{\mathbf{h}_d \in \mathbb{R}^D \mid \boldsymbol{\theta}_d + \mathbf{h}_d \in \Pi_d\right\}$, $\Lambda_{\boldsymbol{\theta}_r} = \left\{\mathbf{h}_r \in \mathbb{R}^R \mid \boldsymbol{\theta}_r + \mathbf{h}_r \in \Pi_r\right\}$ and $\Lambda_{\boldsymbol{\theta}} = \Lambda_{\boldsymbol{\theta}_d} \times \Lambda_{\boldsymbol{\theta}_r} = \left\{\mathbf{h} \in \mathbb{R}^K \mid \boldsymbol{\theta} + \mathbf{h} \in \Pi_d \times \Pi_r\right\}$. $f(\mathbf{x}, \boldsymbol{\theta}) = f(\mathbf{x}, \boldsymbol{\theta}_r|\boldsymbol{\theta}_d) =$

[1]For L column vectors $\mathbf{a}_l$, $(\mathbf{a}_1; \mathbf{a}_2; \ldots; \mathbf{a}_L) \triangleq (\mathbf{a}_1^T, \mathbf{a}_2^T, \ldots, \mathbf{a}_L^T)^T$ denotes the vertical concatenation.

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$f(\mathbf{x}|\boldsymbol{\theta}_r,\boldsymbol{\theta}_d) f(\boldsymbol{\theta}_r|\boldsymbol{\theta}_d)$ denotes the joint p.d.f. of $\mathbf{x}$ and $\boldsymbol{\theta}_r$ parameterized by $\boldsymbol{\theta}_d$, and $\boldsymbol{\Theta} = \{\boldsymbol{\theta} : f(\mathbf{x},\boldsymbol{\theta}) > 0, \mathbf{x} \in \Omega\}$. $\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}[\mathbf{g}(\mathbf{x},\boldsymbol{\theta}_r)]$ denotes the statistical expectation of the vector of functions $\mathbf{g}(.)$ with respect to $\mathbf{x}$ and $\boldsymbol{\theta}_r$ parameterized by $\boldsymbol{\theta}_d$. We assume that for any non-empty couple set $(S,T) \subset \Omega \times \Pi_r$, $\int_S \int_T f(\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d) d\mathbf{x} d\boldsymbol{\theta}_r$ exists. Then, for any joint estimator $\widehat{\boldsymbol{\theta}} \triangleq \widehat{\boldsymbol{\theta}}(\mathbf{x}) = \left(\widehat{\boldsymbol{\theta}}_d; \widehat{\boldsymbol{\theta}}_r\right)$ defined on Ω, where $\widehat{\boldsymbol{\theta}}_d \triangleq \widehat{\boldsymbol{\theta}}_d(\mathbf{x})$ is an estimator of $\boldsymbol{\theta}_d$ and $\widehat{\boldsymbol{\theta}}_r \triangleq \widehat{\boldsymbol{\theta}}_r(\mathbf{x})$ is an estimator of $\boldsymbol{\theta}_r$, for any N-dimensional real-valued vector $\mathbf{v}(\mathbf{x},\boldsymbol{\theta})$ defined on $\Omega \times \Pi_d \times \Pi_r$ with a finite second order moment, *i.e.* $\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[v_n(\mathbf{x},\boldsymbol{\theta})^2\right] < \infty$, $1 \leq n \leq N$, the covariance inequality principle [20, p. 124] yields:

$$\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\mathbf{e}(\mathbf{x})\mathbf{e}(\mathbf{x})^T\right] \succeq \mathbf{C}\mathbf{V}^{-1}\mathbf{C}^T, \tag{1}$$

$$\mathbf{V} = \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\mathbf{v}(\mathbf{x},\boldsymbol{\theta})^T\right], \tag{2}$$

$$\mathbf{C} = \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\mathbf{e}(\mathbf{x})\mathbf{v}(\mathbf{x},\boldsymbol{\theta})^T\right], \tag{3}$$

where $\mathbf{e}(\mathbf{x}) = \widehat{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}$, and for two matrices, $\mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive semi-definite. Note that one must have $N \geq K$ in order to have $\mathbf{C}\mathbf{V}^{-1}\mathbf{C}^T$ positive definite. Note also that $\mathbf{C}$ depends on the estimation scheme $\widehat{\boldsymbol{\theta}}$ in general; however, some judicious choices of $\mathbf{v}(\mathbf{x},\boldsymbol{\theta})$ lead to lower bounds on the MSE as previously shown in [4][7] and exemplified hereinafter with the derivation a new hybrid lower bound which combines the MSB and the WWB. Note that the derivation is conducted in the multivariate context with multiple test points. The starting point is the result provided in [21, §III] showing that the class of estimators $\widehat{\boldsymbol{\theta}}$ satisfying:

$$\begin{aligned}&\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}[\mathbf{e}(\mathbf{x})] = (\mathbf{0};\boldsymbol{\lambda})\\ &\quad\Leftrightarrow \mathbb{E}_{\mathbf{x}|\boldsymbol{\theta}_d}\left[\widehat{\boldsymbol{\theta}}(\mathbf{x})\right] = \left(\boldsymbol{\theta}_d; \mathbb{E}_{\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}[\boldsymbol{\theta}_r] + \boldsymbol{\lambda}\right),\end{aligned} \tag{4}$$

where $\boldsymbol{\lambda}$ is an arbitrary vector independent of $\boldsymbol{\theta}$, which includes wide-sense unbiased estimates [8][12], verifies:

$$\begin{aligned}&\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}[\mathbf{e}(\mathbf{x})\nu(\mathbf{x},\boldsymbol{\theta},\mathbf{h})] = \mathbf{h},\\ &\nu(\mathbf{x},\boldsymbol{\theta},\mathbf{h}) = \left|\begin{array}{l}\frac{f(\mathbf{x},\boldsymbol{\theta}+\mathbf{h})}{f(\mathbf{x},\boldsymbol{\theta})} - 1,\ \boldsymbol{\theta} \in \boldsymbol{\Theta},\\ 0,\text{ otherwise.}\end{array}\right.\end{aligned} \tag{5}$$

where $\mathbf{h} = (\mathbf{h}_d;\mathbf{h}_r) \in \mathbb{R}^K$, whatever the statistical link between random and deterministic parameters provided that the support of random parameters prior p.d.f. is of the form:

$$1_{\Pi_r}(\boldsymbol{\theta}_r) = \left|\begin{array}{l}0 \text{ if } \sum\limits_{\mathbf{h}_r \in A}\left(\sum\limits_{l\in\mathbb{Z}} 1_{\Pi_r^0}(\boldsymbol{\theta}_r + l\mathbf{h}_r)\right) = 0,\\ 1,\text{ otherwise,}\end{array}\right. \tag{6}$$

where $1_S(\boldsymbol{\theta}_r)$ denotes the indicator function of subset S of $\mathbb{R}^R$, and A and Π_r^0 are subsets of $\mathbb{R}^R$. Thus Π_r may be a discrete subset of $\mathbb{R}^R$ or a subset of intervals of $\mathbb{R}^R$. However, if the existence of the HCRB is required, $\mathbf{h}_r$ must be free to be infinitesimally small [4, III.A], compelling (6) to reduce to: $\forall\boldsymbol{\theta}_r \in \mathbb{R}^R, 1_{\Pi_r}(\boldsymbol{\theta}_r) = 1$, that is $\Pi_r = \mathbb{R}^R$.

The key point allowing to obtain the HMSWWB is the additional identity (9) regarding the vector of random parameters $\boldsymbol{\theta}_r$ and which demonstration is given below. As $\boldsymbol{\theta}_r$ satisfies (6), the change of variables $\boldsymbol{\theta}_r' = \boldsymbol{\theta}_r - \mathbf{h}_r$ leads to:

$$\begin{aligned}&\int_{\Pi_r} f^m(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r|\boldsymbol{\theta}_d) f^{1-m}(\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d) d\boldsymbol{\theta}_r =\\ &\qquad\int_{\Pi_r} f^m(\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d) f^{1-m}(\mathbf{x},\boldsymbol{\theta}_r - \mathbf{h}_r|\boldsymbol{\theta}_d) d\boldsymbol{\theta}_r,\end{aligned}$$

and:

$$\begin{aligned}&\int_{\Pi_r} \boldsymbol{\theta}_r f^m(\mathbf{x},\boldsymbol{\theta}_r + \mathbf{h}_r|\boldsymbol{\theta}_d) f^{1-m}(\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d) d\boldsymbol{\theta}_r =\\ &\qquad\int_{\Pi_r} (\boldsymbol{\theta}_r - \mathbf{h}_r) f^m(\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d) f^{1-m}(\mathbf{x},\boldsymbol{\theta}_r - \mathbf{h}_r|\boldsymbol{\theta}_d) d\boldsymbol{\theta}_r.\end{aligned}$$

Thus, for any $\mathbf{h} \in \Delta_{\boldsymbol{\theta}}$, $\Delta_{\boldsymbol{\theta}} = \{\mathbf{h} = (\mathbf{0};\mathbf{h}_r)\,|\mathbf{h}_r \in \Lambda_{\boldsymbol{\theta}_r}\}$:

$$\left\{\begin{array}{l}\int_{\Pi_r} \upsilon(\mathbf{x},\boldsymbol{\theta},\mathbf{h},m) f(\mathbf{x},\boldsymbol{\theta}) d\boldsymbol{\theta}_r = 0,\\ \quad\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}[\boldsymbol{\theta}_r \upsilon(\mathbf{x},\boldsymbol{\theta},\mathbf{h},m)] = -\mathbf{h}_r \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\frac{f^{1-m}(\mathbf{x},\boldsymbol{\theta}-\mathbf{h})}{f^{1-m}(\mathbf{x},\boldsymbol{\theta})}\right],\end{array}\right.$$

where:

$$\upsilon(\mathbf{x},\boldsymbol{\theta},\mathbf{h},m) = \left|\begin{array}{l}\frac{f^m(\mathbf{x},\boldsymbol{\theta}+\mathbf{h})}{f^m(\mathbf{x},\boldsymbol{\theta})} - \frac{f^{1-m}(\mathbf{x},\boldsymbol{\theta}-\mathbf{h})}{f^{1-m}(\mathbf{x},\boldsymbol{\theta})},\ \boldsymbol{\theta} \in \boldsymbol{\Theta},\\ 0,\text{ otherwise,}\end{array}\right. \tag{7}$$

which are extensions to $f(\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d)$ of properties satisfied by functions of the Weiss-Weinsten family in Bayesian estimation [22, (1)(2)] where $f(\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d)$ reduces to $f(\mathbf{x},\boldsymbol{\theta}_r)$. As a consequence, for any real-valued function $\mathbf{g}(\mathbf{x},\boldsymbol{\theta}_d)$ defined on $\Omega \times \Pi_d$, $\forall\mathbf{h} \in \Delta_{\boldsymbol{\theta}}$, and $\forall\boldsymbol{\theta}_d \in \Pi_d$:

$$\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}[\mathbf{g}(\mathbf{x},\boldsymbol{\theta}_d)\upsilon(\mathbf{x},\boldsymbol{\theta},\mathbf{h},m)] = \mathbf{0}, \tag{8}$$

yielding:

$$\mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}[\mathbf{e}(\mathbf{x})\upsilon(\mathbf{x},\boldsymbol{\theta},\mathbf{h},m)] = (\mathbf{0};\mathbf{h}_r\mu(-\mathbf{h},m)), \tag{9}$$

where:

$$\mu(\mathbf{h},m) = \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\frac{f^m(\mathbf{x},\boldsymbol{\theta}+\mathbf{h})}{f^m(\mathbf{x},\boldsymbol{\theta})}\right]. \tag{10}$$

Note that if we choose $\mathbf{h} \notin \Delta_{\boldsymbol{\theta}}$, then (8) does not hold and the right-hand side of (9) depends on $\widehat{\boldsymbol{\theta}}$.

Finally, we can combine identities (5) and (9) to build the following N-dimensional real-valued vector $\mathbf{v}(\mathbf{x},\boldsymbol{\theta})$ partitioned into two subvectors with respectively I and J components, where $I + J = N$:

$$\begin{aligned}\{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_i &= \nu(\mathbf{x},\boldsymbol{\theta},\mathbf{h}_i), \qquad \mathbf{h}_i \in \Lambda_{\boldsymbol{\theta}}, \qquad 1 \leq i \leq I, &(11)\\ \{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_j &= \upsilon(\mathbf{x},\boldsymbol{\theta},\mathbf{h}_j,m_j), \mathbf{h}_j \in \Delta_{\boldsymbol{\theta}}, I+1 \leq j \leq N, &(12)\end{aligned}$$

where $0 < m_j < 1$. Indeed, by plugging (11) and (12) into (1), identities (5) and (9) allow to show, that subject to (6), the resulting matrix $\mathbf{C}$ in (3) is independent of any estimator $\widehat{\boldsymbol{\theta}}$ of class (4) yielding the HMSWWB defined as:

$$\mathbf{HMSWWB}(\boldsymbol{\theta}_d) = \sup_{\substack{\mathbf{h}_1,\cdots,\mathbf{h}_I \in \Lambda_{\boldsymbol{\theta}}\\ \mathbf{h}_{I+1},\cdots,\mathbf{h}_N \in \Delta_{\boldsymbol{\theta}}\\ 0<m_{I+1},\cdots,m_N<1}} \left\{\mathbf{C}\mathbf{V}^{-1}\mathbf{C}^T\right\}, \tag{13}$$

where matrices $\mathbf{C}$ (3) and $\mathbf{V}$ (2) are given by:

$$\mathbf{C} = [\mathbf{h}_1 \cdots \mathbf{h}_I\ \ \mu(\mathbf{h}_{I+1},m_{I+1})\mathbf{h}_{I+1} \cdots \mu(\mathbf{h}_N,m_N)\mathbf{h}_N], \tag{14}$$

$\circ$ $1 \leq i \leq I$ and $1 \leq i' \leq I$:

$$\begin{aligned} \{\mathbf{V}\}_{i,i'} &= \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_i \{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_{i'}\right] \\ &= \xi(\mathbf{h}_i, \mathbf{h}_{i'}, 1, 1) - 1, \end{aligned} \quad (15)$$

$\circ$ $1 \leq i \leq I$ and $I+1 \leq j \leq N$:

$$\begin{aligned} \{\mathbf{V}\}_{i,j} &= \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_i \{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_j\right] \\ &= \xi(\mathbf{h}_i, \mathbf{h}_j, 1, m_j) - \xi(\mathbf{h}_i, -\mathbf{h}_j, 1, 1-m_j) \\ &\quad -\mu(\mathbf{h}_j, m_j) + \mu(-\mathbf{h}_j, 1-m_j), \end{aligned} \quad (16)$$

$$\{\mathbf{V}\}_{j,i} = \{\mathbf{V}\}_{i,j}, \quad (17)$$

$\circ$ $I+1 \leq j \leq N$ and $I+1 \leq j' \leq N$:

$$\begin{aligned} \{\mathbf{V}\}_{j,j'} &= \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_j \{\mathbf{v}(\mathbf{x},\boldsymbol{\theta})\}_{j'}\right] \\ &= \xi(\mathbf{h}_j, \mathbf{h}_{j'}, m_j, m_{j'}) - \xi(-\mathbf{h}_j, \mathbf{h}_{j'}, 1-m_j, m_{j'}) \\ &\quad -\xi(\mathbf{h}_j, -\mathbf{h}_{j'}, m_j, 1-m_{j'}) \\ &\quad +\xi(-\mathbf{h}_j, -\mathbf{h}_{j'}, 1-m_j, 1-m_{j'}), \end{aligned} \quad (18)$$

and:

$$\xi(\mathbf{h}_a, \mathbf{h}_b, k, l) = \mathbb{E}_{\mathbf{x},\boldsymbol{\theta}_r|\boldsymbol{\theta}_d}\left[\frac{f^k(\mathbf{x},\boldsymbol{\theta}+\mathbf{h}_a) f^l(\mathbf{x},\boldsymbol{\theta}+\mathbf{h}_b)}{f^{k+l}(\mathbf{x},\boldsymbol{\theta})}\right]. \quad (19)$$

Note that since $\mu(\mathbf{h}, m) = \xi(\mathbf{h}, \mathbf{0}, m, 0)$, only the calculation of $\xi(\mathbf{h}_a, \mathbf{h}_b, k, l)$ is required to assess the HMSWWB for a particular hybrid estimation problem.

III. A NEW FAMILY OF HYBRID LARGE ERROR BOUNDS

As shown in [7] in the restricted case where $f(\boldsymbol{\theta}_r|\boldsymbol{\theta}_d) = f(\boldsymbol{\theta}_r)$, linear transformation on the centered likelihood-ratio (CLR) function is the cornerstone to generate a large class of hybrid bounds including any existing approximation of the Barankin bound [18][19]. Actually, these results can be extended to the class of estimators defined by (4) and the general case where the random parameters prior p.d.f. depends on the deterministic parameters, provided that the support Π_r of $f(\boldsymbol{\theta}_r|\boldsymbol{\theta}_d)$ is of the form (6) ($\mathbb{R}^R$ if the existence of the HCRB is required). As proposed below, a sketch of a proof is readily obtained from the simplest case: a single unknown deterministic parameter and a single unknown random parameter, i.e. $\boldsymbol{\theta} = (\theta_d; \theta_r)$, case for which (5) becomes:

$$\mathbf{h}_i = \mathbb{E}_{\mathbf{x},\theta_r|\theta_d}\left[\mathbf{e}(\mathbf{x})\, v(\mathbf{x},\boldsymbol{\theta},\mathbf{h}_i)\right], \quad \mathbf{h}_i = (h_{ri}; h_{di}).$$

Then, if $\{\ldots, \alpha(\tau, \mathbf{h}_i), \ldots\}$ is a set of samples of a parametric function $\alpha(\tau, \mathbf{h}), \tau \in \Phi \subset \mathbb{R}, \mathbf{h} = (h_r; h_d)$, then subject to (4)(6), $\widehat{\boldsymbol{\theta}}$ satisfies:

$$\sum_i \alpha(\tau, \mathbf{h}_i)\, \mathbf{h}_i = \mathbb{E}_{\mathbf{x},\theta_r|\theta_d}\left[\mathbf{e}(\mathbf{x}) \sum_i \alpha(\tau, \mathbf{h}_i)\, v(\mathbf{x},\boldsymbol{\theta},\mathbf{h}_i)\right]. \quad (20)$$

If $\alpha(\tau, \mathbf{h})$ is integrable over $\Lambda_{\boldsymbol{\theta}}, \forall \tau \in \Phi$, then (20) is, up to a scale factor (differential element $d\mathbf{h} = dh_r dh_d$), a numerical approximation (with rectangle rule) of the following identity:

$$\mathbb{E}_{\mathbf{x},\theta_r|\theta_d}\left[\mathbf{e}(\mathbf{x})\, \eta_\alpha(\mathbf{x},\boldsymbol{\theta},\tau)\right] = \boldsymbol{\Gamma}_\alpha(\tau) = \mathbf{C}, \quad (21)$$

where $\eta_\alpha(\mathbf{x},\boldsymbol{\theta},\tau) = \int_{\Lambda_{\boldsymbol{\theta}}} \alpha(\tau, \mathbf{h})\, v(\mathbf{x},\boldsymbol{\theta},\mathbf{h})\, d\mathbf{h}$ and $\boldsymbol{\Gamma}_\alpha(\tau) = \int_{\Lambda_{\boldsymbol{\theta}}} \alpha(\tau, \mathbf{h})\, \mathbf{h} d\mathbf{h}$. Moreover, subject to (21), (2) becomes:

$$\begin{aligned} V &= \mathbb{E}_{\mathbf{x},\theta_r|\theta_d}\left[\eta_\alpha(\mathbf{x},\boldsymbol{\theta},\tau)^2\right] \\ &= \iint_{\Lambda_{\boldsymbol{\theta}} \times \Lambda_{\boldsymbol{\theta}}} \alpha(\tau, \mathbf{h})\, K(\mathbf{h}, \mathbf{h}')\, \alpha(\tau, \mathbf{h}')\, d\mathbf{h} d\mathbf{h}' \end{aligned} \quad (22)$$

where $K(\mathbf{h}, \mathbf{h}') = \mathbb{E}_{\mathbf{x},\theta_r|\theta_d}\left[v(\mathbf{x},\boldsymbol{\theta},\mathbf{h})\, v(\mathbf{x},\boldsymbol{\theta},\mathbf{h}')\right]$, which corresponds to [7, (18)(19)] in the case of a single function $\alpha(\tau, \mathbf{h})$ (QED).

We can now extrapolate a bit further: similarly, by resorting to the numerical approximation with rectangle rule of integrals, we can state that proper linear transformations of $\mathbf{v}(\mathbf{x},\boldsymbol{\theta})$ defined by (11)(12) associated with an ad hoc number of test points (I and J) allows to generate:

- if $I \neq 0$, $J = 0$, $\boldsymbol{\theta} = \boldsymbol{\theta}_d$, any approximation of the deterministic BB [19][18][23],
- if $I \neq 0$, $J = 0$, $\boldsymbol{\theta} = (\boldsymbol{\theta}_d; \boldsymbol{\theta}_r)$, the CCRB introduced in [7],
- if $I \neq 0$, $J \neq 0$, $\boldsymbol{\theta} = \boldsymbol{\theta}_r$, the family of Bayesian bounds introduced in [16, (38)],
- if $I \neq 0$, $J \neq 0$, $\boldsymbol{\theta} = (\boldsymbol{\theta}_d; \boldsymbol{\theta}_r)$, the extension of the family of Bayesian bounds introduced in [16, (38)] to hybrid estimation, for the class of hybrid estimators defined by (4) whatever the statistical link between random and deterministic parameters, provided that the support of $f(\boldsymbol{\theta}_r|\boldsymbol{\theta}_d)$ is of the form (6) ($\mathbb{R}^R$ if the existence of the HCRB is required).

Logically, this family of bounds should be called the Hybrid Barankin-Weiss-Weinstein bounds (HBWWB) family.

Last, let us remind that there is an equivalence between the covariance inequality and the minimization of a Gram matrix under linear constraints [15][18], i.e. (3) in the current setting. As mentioned in [15][18], the effect of adding constraints, that is components in $\mathbf{v}(\mathbf{x},\boldsymbol{\theta})$ (1)(11)(12), is to increase (or leave unchanged) the associated lower bound $\mathbf{C}\mathbf{V}^{-1}\mathbf{C}^T$.

Therefore, as the HBWWB family derive from choices of $\mathbf{v}(\mathbf{x},\boldsymbol{\theta})$ including components (11), it allows to define bounds tighter that the existing ones [4][7].

IV. GAUSSIAN OBSERVATIONS WITH PARAMETERIZED MEAN

As previously mentioned, the only quantity to assess is $\xi(\mathbf{h}_a, \mathbf{h}_b, k, l)$ given by (19) which generally requires a numerical evaluation. However for some observation models, further analysis on $\xi(\mathbf{h}_a, \mathbf{h}_b, k, l)$ lead to analytical expressions with a reduced computational cost and even to closed-forms for some cases of interest. As an illustration, we consider the Gaussian observation model with parameterized mean:

$$\mathbf{x} = \boldsymbol{\psi}(\boldsymbol{\theta}) + \mathbf{n}, \ \mathbf{x} \in \mathbb{C}^P, \quad (23)$$

where $\boldsymbol{\psi}(.) \in \mathbb{C}^P$ is a known deterministic function and $\mathbf{n} \in \mathbb{C}^P$ is an additional noise which is assumed to be complex zero-mean Gaussian circular with a known covariance matrix $\sigma_n^2\mathbf{I}$, yielding: $\mathbf{x}|\boldsymbol{\theta} \sim \mathcal{CN}(\boldsymbol{\psi}(\boldsymbol{\theta}), \sigma_n^2\mathbf{I})$. This model is widely met in a plethora of signal processing problems such as: spectral analysis [24], array processing [3], digital communications [25], etc. Under (23):

$$\xi\left(\mathbf{h}_a,\mathbf{h}_b,k,l\right)=\mathbb{E}_{\boldsymbol{\theta}}\left[A\left(\boldsymbol{\theta},\mathbf{h}_a,\mathbf{h}_b,k,l\right)\frac{f^k(\boldsymbol{\theta}+\mathbf{h}_a)f^l(\boldsymbol{\theta}+\mathbf{h}_b)}{f^{k+l}(\boldsymbol{\theta})}\right],$$
$$A\left(\boldsymbol{\theta},\mathbf{h}_a,\mathbf{h}_b,k,l\right)=\mathbb{E}_{\mathbf{x}|\boldsymbol{\theta}}\left[\frac{f^k(\mathbf{x}|\boldsymbol{\theta}+\mathbf{h}_a)f^l(\mathbf{x}|\boldsymbol{\theta}+\mathbf{h}_b)}{f^{k+l}(\mathbf{x}|\boldsymbol{\theta})}\right], \tag{24}$$

where [26, (15)]:

$$\begin{aligned}&\sigma_n^2\ln A\left(\boldsymbol{\theta},\mathbf{h}_a,\mathbf{h}_b,k,l\right)=\\&\left\|k\boldsymbol{\psi}\left(\boldsymbol{\theta}+\mathbf{h}_a\right)+l\boldsymbol{\psi}\left(\boldsymbol{\theta}+\mathbf{h}_b\right)-(k+l-1)\,\boldsymbol{\psi}\left(\boldsymbol{\theta}\right)\right\|^2-\\&k\left\|\boldsymbol{\psi}\left(\boldsymbol{\theta}+\mathbf{h}_a\right)\right\|^2+l\left\|\boldsymbol{\psi}\left(\boldsymbol{\theta}+\mathbf{h}_b\right)\right\|^2-(k+l-1)\left\|\boldsymbol{\psi}\left(\boldsymbol{\theta}\right)\right\|^2\end{aligned}$$

As $A\left(\boldsymbol{\theta},\mathbf{h}_a,\mathbf{h}_b,k,l\right)$ depends on $\boldsymbol{\theta}$ generally, then $\xi\left(\mathbf{h}_a,\mathbf{h}_b,k,l\right)$ must be numerically assessed according to (24). However, if $A\left(\boldsymbol{\theta},\mathbf{h}_a,\mathbf{h}_b,k,l\right)$ is independent of $\boldsymbol{\theta}_r$, which is encountered in some cases of interest, e.g. [11] and [26] (and references herein), the following closed-form expression of $\xi\left(\mathbf{h}_a,\mathbf{h}_b,k,l\right)$ can be obtained when the prior p.d.f. $f\left(\boldsymbol{\theta}_r\right)$ is Gaussian $\mathcal{N}\left(\boldsymbol{\theta}_{r0},\sigma^2_{\theta_r}\mathbf{I}\right)$ and does not depend on the deterministic parameter [26, (25)]:

$$\xi\left(\mathbf{h}_a,\mathbf{h}_b,k,l\right)=A\left(\left(\boldsymbol{\theta}_d;\mathbf{0}\right),\mathbf{h}_a,\mathbf{h}_b,k,l\right)\times e^{-\frac{1}{2\sigma^2_{\theta_r}}\left(k\|\mathbf{h}_{ar}\|^2+l\|\mathbf{h}_{br}\|^2-\|k\mathbf{h}_{ar}+l\mathbf{h}_{br}\|^2\right)}. \tag{25}$$

Actually it is possible to formalize further this result by noticing that if $\forall\mathbf{h}$, $\exists\mathbf{W}\left(\boldsymbol{\theta}_r\right)$ a square matrix and $\mathbf{q}\left(\boldsymbol{\theta}_d,\mathbf{h}\right)$ a vector such that:

$$\boldsymbol{\psi}\left(\boldsymbol{\theta}+\mathbf{h}\right)=\mathbf{W}\left(\boldsymbol{\theta}_r\right)\mathbf{q}\left(\boldsymbol{\theta}_d,\mathbf{h}\right),\ \mathbf{W}^H\left(\boldsymbol{\theta}_r\right)\mathbf{W}\left(\boldsymbol{\theta}_r\right)=\mathbf{S}, \tag{26}$$

where $\mathbf{S}$ is a matrix independent of $\boldsymbol{\theta}_r$, then:

$$\begin{aligned}&\sigma_n^2\ln A\left(\boldsymbol{\theta},\mathbf{h}_a,\mathbf{h}_b,k,l\right)=\sigma_n^2\ln A\left(\left(\boldsymbol{\theta}_d;\mathbf{0}\right),\mathbf{h}_a,\mathbf{h}_b,k,l\right)=\\&\quad k\left(k-1\right)\mathbf{q}^H\left(\boldsymbol{\theta}_d,\mathbf{h}_a\right)\mathbf{S}\mathbf{q}\left(\boldsymbol{\theta}_d,\mathbf{h}_a\right)+\\&\quad l\left(l-1\right)\mathbf{q}^H\left(\boldsymbol{\theta}_d,\mathbf{h}_b\right)\mathbf{S}\mathbf{q}\left(\boldsymbol{\theta}_d,\mathbf{h}_b\right)+\\&\quad (k+l)\left(k+l-1\right)\mathbf{q}^H\left(\boldsymbol{\theta}_d,\mathbf{0}\right)\mathbf{S}\mathbf{q}\left(\boldsymbol{\theta}_d,\mathbf{0}\right)+\\&\quad 2kl\,\mathrm{Re}\left\{\mathbf{q}^H\left(\boldsymbol{\theta}_d,\mathbf{h}_b\right)\mathbf{S}\mathbf{q}\left(\boldsymbol{\theta}_d,\mathbf{h}_a\right)\right\}-\\&\quad 2\left(k+l-1\right)\mathrm{Re}\left\{\begin{pmatrix}k\mathbf{q}^H\left(\boldsymbol{\theta}_d,\mathbf{h}_a\right)+\\ l\mathbf{q}^H\left(\boldsymbol{\theta}_d,\mathbf{h}_b\right)\end{pmatrix}\mathbf{S}\mathbf{q}\left(\boldsymbol{\theta}_d,\mathbf{0}\right)\right\}.\end{aligned} \tag{27}$$

A. *Estimation of a single tone*

We consider a particular case of (23) which is a reference problem in threshold analysis [15][18][7][16][19]:

$$\mathbf{x}=se^{j\varphi}\mathbf{b}\left(\omega\right)+\mathbf{n},\ \ \mathbf{b}\left(\omega\right)=\left(1;e^{j\omega};\cdots;e^{j(P-1)\omega}\right). \tag{28}$$

We assume that the phase $\varphi\in\left]-\pi,\pi\right]$ and the amplitude $s\in\mathbb{R}^+$ are the deterministic unknown parameters and that the frequency ω is a random unknown parameter with a Gaussian centred prior p.d.f. with variance σ^2_ω. Therefore $\boldsymbol{\theta}=\left(\boldsymbol{\theta}_d;\omega\right)$, $\boldsymbol{\theta}_d=\left(s;\varphi\right)^T$. Since for all $\mathbf{h}=\left(h_s;h_\varphi;h_\omega\right)$:

$$\boldsymbol{\psi}\left(\boldsymbol{\theta}+\mathbf{h}\right)=Diag\left(\mathbf{b}\left(\omega\right)\right)\left(s+h_s\right)e^{j\left(\varphi+h_\varphi\right)}\mathbf{b}\left(h_\omega\right), \tag{29}$$

where $Diag\left(\mathbf{b}\left(\omega\right)\right)$ is a $P\times P$ diagonal matrix with the elements of vector $\mathbf{b}\left(\omega\right)$ on the main diagonal, the condition (26) is satisfied and $\xi\left(k,l,\mathbf{h}_a,\mathbf{h}_b\right)$ can computed via (25) and (27) given by:

$$\begin{aligned}&\sigma_n^2\ln A\left(\boldsymbol{\theta},\mathbf{h}_a,\mathbf{h}_b,k,l\right)=\\&\quad\begin{pmatrix}k\left(k-1\right)\left(s+h_{as}\right)^2+l\left(l-1\right)\left(s+h_{bs}\right)^2\\+\left(k+l\right)\left(k+l-1\right)s^2\end{pmatrix}P\\&\quad+2kl\left(s+h_{as}\right)\left(s+h_{bs}\right)T\left(h_{a\omega}-h_{b\omega},h_{a\varphi}-h_{b\varphi}\right)\\&\quad-2\left(k+l-1\right)s\begin{pmatrix}k\left(s+h_{as}\right)T\left(h_{a\omega},h_{a\varphi}\right)\\+l\left(s+h_{bs}\right)T\left(h_{b\omega},h_{b\varphi}\right)\end{pmatrix},\end{aligned} \tag{30}$$

where $T\left(h_\omega,h_\varphi\right)=\cos\left(\frac{(P-1)}{2}h_\omega+h_\varphi\right)\frac{\sin\left(\frac{P}{2}h_\omega\right)}{\sin\left(\frac{h_\omega}{2}\right)}$ and where $\mathbf{h}_a=\left(h_{as};h_{a\varphi};h_{a\omega}\right)$ and $\mathbf{h}_b=\left(h_{bs};h_{b\varphi};h_{b\omega}\right)$.

As an application example, we present a comparison of the HMSWWB (13) with the HCRB, the HBB and the CCLRB and the empirical MSE of the maximum a posteriori maximum likelihood estimator (MAPMLE) [27][11, p12]. The scenario is the following: $s=1$, $\varphi=\frac{\pi}{4}$, $\sigma^2_\omega=\frac{1}{2}$ and $P=2^5$. The expressions of the HCRB, the HBB and the CCLRB are given in [7] Section 5. The HBB and the proposed bound denoted HMSWWB are computed with $\mathbf{h}_1\in\left[-1;1\right]\times\left\{0\right\}\times\left\{0\right\}$ where the sampling interval for the first component is $\delta h_s=0.01$, $\mathbf{h}_2\in\left\{0\right\}\times\left]-\pi;\pi\right]\times\left\{0\right\}$ where the sampling interval for the second component is $\delta h_\varphi=\frac{2\pi}{2^8}$ and $\mathbf{h}_3\in\left\{0\right\}\times\left\{0\right\}\times\left]-\pi;\pi\right]$ where the sampling interval for the second component is $\delta h_\omega=\frac{\pi}{2^8}$. The CCLRB $(J=2)$ is computed with 2^{10} test points where the test points are selected following the configuration proposed in [7] and using 2 eigenvectors and the same sampling interval as for the HMSWWB. The MAPMLE is obtained by searching the value of $\boldsymbol{\theta}$ maximizing the joint p.d.f. [11, (176)]:

$$\begin{aligned}&\widehat{\boldsymbol{\theta}}_{MAPMLE}=\arg\max_{s,\varphi,\omega}\left\{\frac{e^{-\frac{1}{\sigma_n^2}\left\|\mathbf{x}-se^{j\varphi}\mathbf{b}(\omega)\right\|^2-\frac{\omega^2}{2\sigma^2_\omega}}}{\left(\pi\sigma_n^2\right)^P\sqrt{2\pi\sigma^2_\omega}}\right\}\Rightarrow\\&\quad\omega_{MAPMLE}=\arg\max_{\omega}\left\{\left|\mathbf{b}^H\left(\omega\right)\mathbf{x}\right|^2-\frac{P\sigma_n^2}{2\sigma^2_\omega}\omega^2\right\},\end{aligned}$$

and its empirical MSE is assessed with 10^3 Monte-Carlo trials.

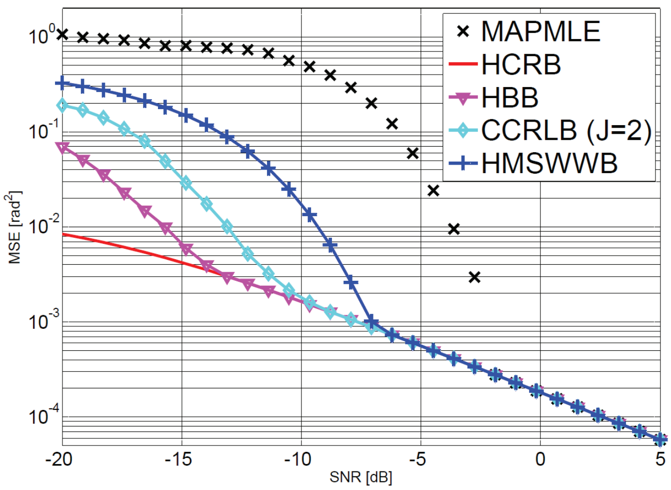

Fig. 1. Comparison of hybrid lower bounds on the MSE and the empirical MSE of the MAPMLE versus SNR.

We superimpose on the figure (1) the HCRB, the HBB, the CCRLB ($J = 2$), the HMSWWB, and the empirical MSE of the MAPMLE for the random parameter ω for which a SNR threshold phenomenon occurs [7]. As predicted by the theory, a significant better SNR threshold prediction is provided by the HMSWWB in comparison with other existing hybrid lower bound.

V. Conclusion

In this letter, a tighter family of hybrid lower bounds on the MSE has been proposed in the general context of multivariate parameters estimation. This family can be applied to a wide class of estimators including the case when random parameters prior p.d.f. depends on the deterministic parameters.

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