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A CONSTRAINED HYBRID CRAMÉR-RAO BOUND FOR PARAMETER ESTIMATION

Chengfang Ren(1), Julien Le Kernec(2), Jerome Galy(3), Eric Chaumette(4), Pascal Larzabal(5), Alexandre Renaux(1)

1. INTRODUCTION

While Bayesian or non-Bayesian estimation algorithms are widely used in statistical signal processing, hybrid parameter estimation refers to the case where the parameters vector to estimate contains both non-random and random parameters. Numerous works have shown the versatility of deterministic constrained Cramér-Rao bound for estimation performance analysis and design of a system of measurement. However in many systems both random and non-random parameters may occur simultaneously. In this communication, we propose a constrained hybrid lower bound which take into account of equality constraint on deterministic parameters. The usefulness of the proposed bound is illustrated with an application to radar Doppler estimation.

Index Terms— Parameter estimation, hybrid Cramer-Rao bounds, equality constraints

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mation of random and non-random parameters with a set of equality constraints. The usefulness of the CHCRB is illustrated with an application to radar Doppler estimation.

2. RELATION TO PRIOR WORK

In deterministic parameter estimation, the CCRB has proven its usefulness for estimation performance analysis and design of a system of measurement by exploiting constraints between parameters to estimate. However, some systems of measurement cannot be adequately modelled by resorting to deterministic parameters only, since both random and non-random parameters may occur simultaneously. Therefore the purpose of the present paper is to extend the taking into account of equality constraint on deterministic parameters to the hybrid parameters context via the HCRB.

3. THE CONSTRAINED HYBRID CRAMÉR-RAO BOUND

3.1. Problem statement and notations

Let us first remind the estimation context in which the proposed bound can be useful. Consider Ω an observation space of points x and let θ = [θ₁ T θ₂ T] T denotes a (D + R) –dimensional hybrid real parameters vector to estimate, where θ₁ is a vector of unknown deterministic parameters belonging to Π₁ ⊆ ℝ^D and θ₂ is a vector of unknown random parameters belonging to Π₂ ⊆ ℝ^R with a known prior p.d.f. f (x; θ₁, θ₂). Let f (x; θ) = f (x, θ₁, θ₂) denotes the joint p.d.f. of x and θ parameterized by θ. Additionally, the deterministic parameters θ₁ are assumed to be constrained in a non empty subset C of Π₁ defined by K < D non redundant equality constraints:

\[ C = \{ \theta_1 \in \Pi_1 \mid c (\theta_1) = 0 \} \],

where c (θ₁) is a K –dimensional vector of derivable functions defined on Π₁. Let C (θ₁) denote the K × (D + R) matrix defined by

\[ C (\theta_1) = \frac{dc (\theta_1)}{d\theta_1} = \left[ \frac{dc (\theta_1)}{d\theta_1 \theta_1^T} \frac{dc (\theta_1)}{d\theta_1 \theta_2} \right] = [C_d (\theta_1) 0] \],

where C_d (θ₁) is a K × D matrix. Since the constraints are assumed to be non redundant, the rank of C_d (θ₁) is K for any θ₁ satisfying (1). Then there exists a D × (D – K) matrix U_d (θ₁) such that:

\[ C_d (\theta_1) U_d (\theta_1) = 0 \text{ and } U_d^T (\theta_1) U_d (\theta_1) = I_{D-K}, \]

where I_{D-K} denotes the identity matrix of size D – K. Moreover, if (3) holds, then the matrix U (θ₁) = \[ \begin{bmatrix} U_d (\theta_1) & 0 \\ 0 & I_R \end{bmatrix} \] satisfies

\[ C (\theta_1) U (\theta_1) = 0 \text{ and } U^T (\theta_1) C (\theta_1) U (\theta_1) = I_{D+R-K}. \]

Note that the column vectors of U_d (θ₁) is a basis of the kernel of C_d (θ₁) and the column vector of U (θ₁) is a basis of the kernel of C (θ₁). If the constraints are also applied over random parameters θ₂, then the matrix U will depend on θ₁, leading to a lower bound depending on the estimate of θ₁ (see section 3.3).

3.2. Estimator class requirement and preliminary results

Let \( \hat{\theta} (x) \) be an estimator of \( \theta \). The proposed bound is applicable for a class of estimator \( \hat{\theta} \) which are unbiased, as for the classical HCRB [1][17], i.e.:

\[ E_{\hat{\theta}} \left( (\hat{\theta} (x) - \theta) \right) = 0. \]

Any unbiased estimators satisfies the following relationship: for any integer \( i \in [1; D + R] \), one has:

\[ \int_{\mathbb{R}} \int_{\mathbb{C}_N} \left( \hat{\theta} (x) - \theta \right) \left[ \frac{\partial f (\theta, \theta_1)}{\partial \theta_1} \right]_{\theta_1 = \hat{\theta}_i} dx d\theta_1 = E_{\hat{\theta}} \left( (\hat{\theta} (x) - \theta) \right) = 0 + e_i, \]

where \( e_i \) is a vector such that \( e_i = 1 \) and \( e_i, i \neq i \) = 0 where \{e_i\} denotes the i-th element of the vector \( e_i \). Thus, one has:

\[ \int_{\mathbb{R}} \int_{\mathbb{C}_N} \left( \hat{\theta} (x) - \theta \right) \left[ \frac{\partial f (\theta, \theta_1)}{\partial \theta_1} \right]_{\theta_1 = \hat{\theta}_i} dx d\theta_1 = 0. \]

Additionally, let us set \( v = \frac{\partial f (\theta, \theta_1)}{\partial \theta_1} \) then:

\[ E_{\hat{\theta}} \left( (\hat{\theta} (x) - \theta) v^T \right) = \int_{\mathbb{R}} \int_{\mathbb{C}_N} \left( \hat{\theta} (x) - \theta \right) \left[ \frac{\partial f (\theta, \theta_1)}{\partial \theta_1} \right]_{\theta_1 = \hat{\theta}_i} dx d\theta_1. \]

Finally, by mixing (6) and (7), one obtains:

\[ E_{\hat{\theta}} \left( (\hat{\theta} (x) - \theta) v^T \right) = 0. \]

3.3. The proposed bound

In the following, for sake of legibility, let us set \( \hat{\theta} = \hat{\theta} (x) - \theta \) and \( U = U (\theta_1) \). For any square matrix M:

\[ E_{\hat{\theta}} \left( (\hat{\theta} - MU^T v) (\hat{\theta} - MU^T v)^T \right) = E_{\hat{\theta}} \left( \hat{\theta}^T \right) + MU^T E_{\hat{\theta}} \left( v v^T \right) MU^T - MU^T E_{\hat{\theta}} \left( v v^T \right) MU^T \]

Since \( E_{\hat{\theta}} \left( \hat{\theta}^T \right) \) is positive semidefinite and, from (8), one has:

\[ E_{\hat{\theta}} \left( \hat{\theta}^T \right) \succeq \left( \begin{bmatrix} MU^T + MU^T M^T - MU^T E_{\hat{\theta}} \left( v v^T \right) MU^T \end{bmatrix} \right) \]

Since this inequality holds for any matrix M, the tightest lower bound denoted CHCRB is obtained by maximizing the right hand side of (9) over M:

\[ \text{CHCRB} = \max_M \left( \begin{bmatrix} MU^T + MU^T M^T - MU^T E_{\hat{\theta}} \left( v v^T \right) MU^T \end{bmatrix} \right). \]

As \( U^T E_{\hat{\theta}} \left( v v^T \right) U \) is symmetric positive definite, there exists an invertible diagonal matrix D and an unitary matrix Q such that \( U^T E_{\hat{\theta}} \left( v v^T \right) U = QDQ^T \). Consequently, (10) can be rewritten as:

\[ \text{CHCRB} = \max_M \left( \begin{bmatrix} UQD^{-1} Q^T U - MU^T D (UQD^{-1} - MUQ) U \end{bmatrix} \right). \]

Since \( UQD^{-1} Q^T U \) is independent of M and since the CHCRB is formulated as the difference of two positive semidefinite matrix, the maximum is achieved if and only if \( MUQ = UQD^{-1} \), i.e.:

\[ MU = UQD^{-1} Q^T = U \left( U^T E_{\hat{\theta}} \left( v v^T \right) U \right)^{-1}. \]
Finally by substituting (12) in (10), one obtains:

$$\text{CHCRB} = U \left( U^T E_{x, \theta_1, \theta_2} [vv^T] U \right)^{-1} U^T. \quad (13)$$

Remarks:

• Another possible derivation of the CHCRB can be obtained by using the covariance inequality [18, p.124][4]:

$$E \left[ \theta \theta^T \right] \succeq E \left[ \theta v \theta^T \right] E^{-1} \left[ \psi \psi^T \right] E \left[ \psi \theta^T \right]$$

(14)

with $\psi = U^T v$.

• In general, the proposed bound does not need the invertibility of the Fisher matrix $E_{x, \theta_1, \theta_2} [vv^T]$ but of $U^T E_{x, \theta_1, \theta_2} [vv^T] U$ only. This condition is also required for the CCRB in the deterministic estimation context [16].

• If the matrix $U$ depends on $\theta$ and $E_{x, \theta_1, \theta_2} [\theta v \theta^T] \neq U$ then the lower bound will depend on $\theta$, what is pointless.

3.4. Comparison with existing Cramér-Rao Bounds

3.4.1. The CHCRB versus the HCRB

The unconstrained HCRB is given by [1][17]:

$$\text{HCRB} = E_{x, \theta_1, \theta_2} [vv^T]^{-1}$$

(15)

where $F = E_{x, \theta_1, \theta_2} [vv^T]$ is the so-called hybrid Fisher information matrix. The HCRB can be obtained from the CHCRB when $K = 0$ leading to $U = I_{d_1, d_2}$. In other cases, the HCRB and the CHCRB are different. However, a comparison between the CHCRB and the HCRB is possible when $F$ is non singular (otherwise the HCRB does not exist). Since $F$ is symmetric positive definite, there exists a symmetric invertible matrix $F^{+}$ such that $F = F^{+} F^{-}$. Thus the CHCRB can be rewritten as:

$$\text{CHCRB} = F^{-\frac{1}{2}} F^{+} U \left( U^T F^{+} F^+ U \right)^{-1} U^T F^{+} F^{-} F^{-\frac{1}{2}}$$

where $P_{F^{+} U} = F^{+} U \left( F^{+} U \right)^{-1} \left( F^{+} U \right)^T$ is the projection matrix onto the column space of $F^{+} U$. Let $P_{F^{+} U}^{\perp}$ denotes the projection matrix onto the vector space orthogonal to the previous one, then one has $P_{F^{+} U}^{\perp} + P_{F^{+} U} = I$ and:

$$\text{CHCRB} = F^{-\frac{1}{2}} \left( I - P_{F^{+} U} \right) F^{-\frac{1}{2}} = F^{-1} - F^{-\frac{1}{2}} P_{F^{+} U} F^{-\frac{1}{2}} F^{-\frac{1}{2}} \leq F^{-1}.$$ 

Therefore:

$$\text{CHCRB} \leq \text{HCRB}. \quad (16)$$

This result is expected since the constraints can be interpreted as additional informations in order to estimate more accurately the parameters of interest. It has been shown in [19] that estimation algorithms which include parameters constraints could be lower than the unconstrained lower bounds. This is why the CHCRB, even lower than HCRB, is helpful in the hybrid estimation context with parameter constraints.

3.4.2. The CHCRB versus the marginal CCRB

Another question that we can ask is what is the difference between the CHCRB and the marginal CCRB for the deterministic parameters with constraints where in the first case, we estimate simultaneously non random parameters $\theta_d$ and random parameters $\theta_r$, whereas in second case, we estimate non-random parameters $\theta_d$ only, $\theta_r$ being regarded as nuisance parameters? To answer this question, note that, first, the CHCRB can be split into four blocks:

$$\text{CHCRB} = \begin{bmatrix}
\text{CHCRB}_{d} & \text{CHCRB}_{dr} \\
\text{CHCRB}_{rd} & \text{CHCRB}_{r}
\end{bmatrix} \quad (17)$$

where the diagonal blocks $\text{CHCRB}_{d}$ and $\text{CHCRB}_{r}$ are respectively the lower bounds on the MSE of non-random parameters $\theta_d$ and random parameters $\theta_r$, i.e.:

$$E_{x, \theta_1, \theta_2} \left[ \left( \hat{\theta}_d (x) - \theta_d \right) \left( \hat{\theta}_d (x) - \theta_d \right)^T \right] \succeq \text{CHCRB}_{d}$$

$$E_{x, \theta_1, \theta_2} \left[ \left( \hat{\theta}_r (x) - \theta_r \right) \left( \hat{\theta}_r (x) - \theta_r \right)^T \right] \succeq \text{CHCRB}_{r}.$$ 

Second, let $v_d = \frac{\partial \ln f(x, \theta_1, \theta_2)}{\partial \theta_d}$ and $v_r = \frac{\partial \ln f(x, \theta_1, \theta_2)}{\partial \theta_r}$, Then the Fisher information matrix can be decomposed as:

$$F = E_{x, \theta_1, \theta_2} \left[ \begin{array}{c}
v_d v_d^T \\
v_r v_d^T \\
v_r v_r^T
\end{array} \right] \quad (18)$$

Similarly:

$$\text{CHCRB} = U E_{x, \theta_1, \theta_2} \left[ \begin{array}{ccc}
U_d v_d v_d^T & U_d v_d v_r^T \\
v_r v_d^T & v_r v_r^T
\end{array} \right] U^T \quad (18)$$

Let $S = E_{x, \theta_1, \theta_2} \left[ U_d v_d v_d^T U_d \right] - R$, where

$$R = U_d E_{x, \theta_1, \theta_2} \left[ v_d v_d^T \right] E_{x, \theta_1, \theta_2} \left[ v_r v_r^T \right] E_{x, \theta_1, \theta_2} \left[ v_r v_d^T \right] U_d,$$

then an inversion by block of (18) leads to the following expression of the CHCRB:

$$\text{CHCRB} = \begin{bmatrix}
U_d & 0 \\
E_{x, \theta_1, \theta_2} & v_r v_r^T
\end{bmatrix} \begin{bmatrix}
I \\
S^{-1}
\end{bmatrix} \begin{bmatrix}
U_d & 0 \\
E_{x, \theta_1, \theta_2} & v_r v_r^T
\end{bmatrix}$$

$$\text{CHCRB} \succeq U_d \left( U_d^T E_{x, \theta_1, \theta_2} \left[ v_d v_d^T \right] U_d \right)^{-1} U_d^T \quad (19)$$

Then, by identification between (17) and (19), one has:

$$\text{CHCRB}_{d} \leq U_d \left( U_d^T E_{x, \theta_1, \theta_2} \left[ v_d v_d^T \right] U_d \right)^{-1} U_d^T \quad (20)$$

The right hand side of (20) is the so-called marginal CCRB when $\theta_r$ is considered as nuisance parameters. Consequently, the CHCRB is lower than the marginal CCRB. This is an extension of the order relation existing between the unconstrained hybrid lower bound and the unconstrained marginal lower bound [4].
4. APPLICATION TO DOPPLER ESTIMATION

We consider a radar system consisting of a 1-element antenna array receiving scaled, time-delayed, and Doppler-shifted echoes of a known complex bandpass signal \( e^{j2\pi f_0 t} \), where \( f_0 \) is the carrier frequency and \( e^{j2\pi f_0 t} \) is the envelope of the emitted signal. The antenna receives a pulse train (burst) of \( L \) pulses of duration \( T_0 \) and bandwidth \( B \), with a pulse repetition interval (PRI) \( T \). backscattered by a “slow” moving target in comparison with \( e^{j2\pi f_0 t} \), i.e. \( |2v(L-1)T| << \frac{\pi}{f_0} \) (no range migration) and \( \frac{2v}{2T}T_0 f_0 << 1 \) (Doppler effect on \( e^{j2\pi f_0 t} \) is negligible), where \( c \) is the speed of light and \( v \) is the radial velocity of the target. Under the standard hypothesis of temporally white nuisance signal (thermal noise) of power \( \sigma_n^2 \) and a non fluctuating target during the burst duration, a simplified observation model for the \( 1^{th} \), \( 1 \leq l \leq L \), pulse is given by [20]:

\[
x_l(t) = e^{j2\pi f_0 t} r(t - \tau) + n_l(t), \quad r(t) = \alpha e^{j2\pi f(t-1)}
\]

where \( f = -2f_0 \frac{\tau}{T} \), \( -\frac{1}{2} \leq f \leq \frac{1}{2} \), is the normalized Doppler frequency and \( \alpha \) represents the complex amplitude of the target (including power budget equation). For the sake of simplicity, we assume that the target range is known. Therefore at the output of the delay/range matched filter at time \( t = \tau \), the observation model is:

\[
y_l = s e^{j2\pi f(t-1)} + n_l, \quad s = \sqrt{BT} \alpha = r + jq.
\]

and the vector of unknown parameters to estimate is \( \theta = (r, q, f)^T \) where \( (r, q) \) are assumed to be deterministic, \( f \) is assumed to be random with a known Gaussian prior distribution \( N(f_0, \sigma_f^2) \) and independent from the noise \( n_l \) assumed to be circular complex Gaussian distributed \( n_l \sim \mathcal{CN}(0, \sigma_n^2) \). This scenario corresponds to a multifunction radar entering a tracking mode after a target detection in a surveillance mode. The radar budget, i.e. \( |s|^2 \), and \( f_0 \) associated to the target have been previously assessed by the detection step of the surveillance mode. However, during the inherent delay associated to the mode switch, the radial velocity of the target may vary, what we model by a prior distribution. An interesting question is whether it is worth taking into account this radar budget knowledge for the estimation of the \( f \). Indeed, this amounts to introduce the following equality constraint:

\[
\tau^2 + q^2 = |s|^2 = c.
\]

Therefore, the answer can be provided by a comparison between the CHCRB and the HCRB. Using (15), the classical HCRB is:

\[
U = \begin{pmatrix}
\frac{\delta f}{\sigma_f} & 0 & 0 \\
0 & \frac{\delta q}{\sigma_q} & 0 \\
\frac{\delta c}{\sigma_c} & \frac{\delta c}{\sigma_c} & 1
\end{pmatrix}^{-1}
\]

The CHCRB is obtained using the following matrix \( U \) (13):

\[
U = \begin{pmatrix}
\frac{1}{\sigma_f} & 0 & 0 \\
0 & \frac{1}{\sigma_q} & 0 \\
0 & 0 & 1
\end{pmatrix}^T.
\]

In order to validate the proposed approach, we compute the MSE of the classical Maximum-A Posteriori MLE (MAPMLE) defined as:

\[
\hat{f}, \hat{q} = \arg\max_{(r, q) \in \mathbb{R}^2, f \in \mathbb{R}} f, q \in \{ r, q \}^2 \quad y_r, y_q, \hat{f}, \hat{q}, f, q.
\]

and the MSE of the Constrained MAPMLE (CMAPMLE) which restricts the \( (r, q) \) domain from \( \mathbb{R}^2 \) to \( S = \{ (r, q) \in \mathbb{R}^2 \mid |r|^2 + q^2 = |s|^2 \} \).

The simulation settings are: \( r = \frac{\delta f}{\sigma_f}, |s|^2 = 0.8, f = 0.25, \sigma_f = 0.05 \) and \( L = 32 \). The empirical MSE are assessed with 5000 Monte-Carlo trials and a frequency step \( \delta f = 2^{-18} \). In figure (1), the total empirical MSE of the MAPMLE and the CMAPMLE are compared with the trace of HCRB and CHCRB. One can note that the CMAPMLE total MSE is lower than the classical HCRB whereas the CHCRB adequately predicts the asymptotic behavior of the CMAPMLE total MSE. In figure (2), the empirical MSE of \( \hat{f} \) is compared with the HCRB and the CHCRB. Since the HCRB and the CHCRB are identical, therefore the estimation of \( \hat{f} \) is independent of the knowledge of the radar budget at least in the asymptotic region.

This theoretical result is confirmed by the same asymptotic performance of the MAPMLE and CMAPMLE. It is an extension of a well known property of the deterministic single tone estimation problem [21] to the random parameter case.

Fig. 1. Comparison of MAPMLE total MSE and HCRB versus SNR

Fig. 2. Comparison of MAPMLE MSE of \( f \) and HCRB versus SNR

5. CONCLUSION

In this paper, a constrained hybrid lower bound, called the CHCRB, has been developed in order to take into account equality constraints between deterministic parameters. The CHCRB is not only the relevant bound to predict the asymptotic behavior of constrained estimators but also a versatile tool for estimation performance analysis and design of a system of measurement involving hybrid parameters.
6. REFERENCES


