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Abstract. The problem of state-observation is addressed for nonlinear systems that can be modelled by an ODE-PDE series association. The ODE subsystem assumes a triangular structure while the PDE element is of heat diffusion type. The aim is to accurately estimate online the state vector of the ODE subsystem and the distributed state of the PDE element. One major difficulty is that the state observation must only rely on the global system output i.e. the PDE state at the terminal boundary. In particular, the connection point between the ODE and the PDE blocs is not accessible to measurements. The observation problem is dealt with by designing a high-gain type observer. Sufficient conditions involving the PDE domain length are formally established that ensure the observer exponential convergence.

1. Introduction

In the last decades, the problems of nonlinear system observability and observer design has intensively been investigated for systems that can be described by ordinaries differential equations (ODEs). Several types of observers have been proposed, for several classes of nonlinear systems, including the high-gain observer e.g. (Gauthier et al., 1992; Deza et al., 1992; Khalil and Esfandiari, 1993; Shim et al., 2001), sliding-mode observers e.g. (Slotine, 1987; Edwards et al., 2000; Fridman et al., 2008), Luenberger-like observers e.g. (Andrieu and Praly, 2006). Additional references can be found in recent monographs e.g. (Besançon, 2007; Khalil, 2015).

The problem of infinite dimensional system (IDS) observability and observer design has also been given a great deal of interest, especially in recent years. The earliest works have focused on linear IDSs and a relatively complete theoretical framework exists since the nineties, including the infinite dimensional Luenberger observer, e.g. (Curtain and Zwart, 1995; Lasiecka and Triggiani, 2000) and reference list therein. Boundary observer design of bilinear IDSs have been studied in e.g. (Xu et al., 1995; Bounit and Hammouri, 1997, Vries et al., 2007). A unifying study of both interior and boundary observation for linear and bilinear systems is found in

(Amann, 1989). In (Smyshlyaev and Krstic, 2005), backstepping techniques have been used to design exponentially convergent boundary observers for a class of parabolic partial integro-differential equations. The problem of initial state recovery has also been given interest. In (Ramdani *et al.*, 2010), an iterative algorithm is proposed to recover the initial state of a linear infinite dimensional system. The proposed algorithm generalizes various algorithms, proposed earlier for specific classes of systems, and stands as an alternative to methods based on Gramian inversion (Tucsnak and Weiss, 2009). The ideas of Ramdani et al. (2010) have been extended to some nonlinear infinite dimensional systems, using LMI techniques (Fridman, 2013).

In this paper, we are interested in state observation of cascade systems including a ODE subsystem followed in series with PDE subsystem (Fig. 1). The aim is to recover the (finite-dimension) state of the ODE part and the (infinite-dimension) state of the PDE part. One major difficulty of this problem lies in the fact that the connecting point between the two parts is not accessible to measurements. In (Krstic and Smyshlyaev, 2008; Krstic, 2009) a boundary observer has been developed for a cascade involving a linear ODE and a (linear) heat PDE equation that may represent a distributed state sensor. In turn, the observer assumes a cascade structure with a finite- and infinite-dimensional parts. The observer design relies upon an infinite-dimensional transformation, inspired from the backstepping principle, and an exponentially stable target system. The observer thus obtained is shown to be exponentially convergent in the sense of a quadratic norm. Inspired by (Krstic and Smyshlyaev, 2008; Krstic, 2009), a new observer design is presently developed to address ODE-PDE systems that involve a triangular nonlinear ODE subsystem (the PDE part remains a heat equation).

The novelty of the present design approach is twofold: (i) it combines the backstepping infinite-dimensional transformation of (Krstic, 2009) and the high-gain observer design principles (Gauthier et al., 1992; Khalil and Esfandiari, 1993; Shim et al., 2001); (ii) it involves a quite different target system (as the ones used in (Krstic, 2009) are not usable for the present problem). The paper is organised as follows: first, the observation problem under study is formulated in Section 2; then, the observer design and analysis are dealt with in Section 3; a conclusion and reference list end the paper. To alleviate the presentation, some technical proofs are appended.

Notations. Throughout the paper, $\mathbf{R}^n$ denotes the n dimensional real space and the corresponding Euclidean norm is denoted $\|.\|$. $\mathbf{R}^{n\times n}$ denotes the set of all $n\times m$ real matrices and $\|.\|$ the induced Euclidian norm. Functions that are continuously differentiable with respect to all their arguments are denoted C^1. $L_2[0, D]$ is the Hilbert space of square integrable functions and the corresponding L_2 norm is denoted $\|.\|_2$. Accordingly, $\|\eta\|_2 = \left(\int_0^D \eta^2(\varsigma)d\varsigma\right)^{1/2}$ for all $\eta \in L_2[0, D]$. $H^1(0, D)$ is the Sobolev space of absolutely continuous functions $\eta : [0, D] \to \mathbf{R}$ with $d\eta / d\varsigma \in L_2[0, D]$. $H^2(0, D)$ is the Sobolev space of scalar functions $\eta : [0, D] \to \mathbf{R}$ with absolutely continuous $d\eta / d\varsigma \in L_2[0, D]$ and $d^2\eta / d\varsigma^2 \in L_2[0, D]$.

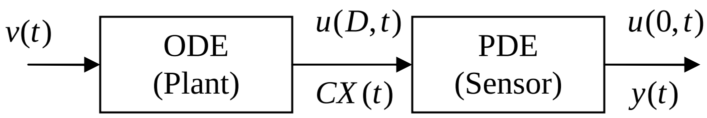

Fig. 1. System structure

2. Problem Formulation

Analytically, the system under study is modelled by a finite-order nonlinear ODE connected in series with a PDE (Fig. 1). The former could represents the plant dynamics which presently assume the following triangular-form state-space representation:

$$\dot{X}(t) = AX(t) + Bv(t) + f(X(t)), \quad t \geq 0 \tag{1a}$$

$$u(D,t) = CX(t) \tag{1b}$$

with:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & 0 & \dots & \dots & \dots & 0 \end{pmatrix} \in \mathbf{R}^{n\times n}, \quad B \in \mathbf{R}^n, \quad C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix} \in \mathbf{R}^{1\times n} \tag{1c}$$

where $v \in C([0, \infty) : \mathbf{R})$ denotes the system input, $X \in \mathbf{R}^n$ the state vector and $f : \mathbf{R}^n \to \mathbf{R}^n$ is a vector field with the triangular form:

$$f(X) = \begin{pmatrix} f_1(X_1) \\ f_2(X_1, X_2) \\ \vdots \\ f_n(X_1, \dots, X_n) \end{pmatrix}; \quad f_i : \mathbf{R}^i \to \mathbf{R} \tag{1d}$$

It is supposed that $f(0) = 0$ and f is class C^2 with bounded Jacobian matrix i.e.

$$\exists \beta > 0, \ \forall X \in \mathbf{R}^n : \ \|f_X(X)\| \leq \beta \tag{1e}$$

The system PDE part represents a diffusive sensing system modelled by the following heat equation and associated boundary condition:

$$u_t(x,t) = u_{xx}(x,t), \quad 0 \le x \le D \tag{2a}$$

$$u_x(0,t) = 0, \quad u(D,t) = CX(t) \tag{2b}$$

where D is a known scalar representing the length of the PDE domain. The whole system is observed through the output signal,

$$y(t) \stackrel{def}{=} u(0,t) \tag{2c}$$

The aim is to design an observer that provides accurate online estimates of both the finite-dimension state vector $X(t)$ and the distributed state variable $u(x,t)$, $0 \le x \le D$. The observer must only make use of the system input $v(t)$ and output $y(t)$. In particular, the connection signal $u(D,t)$ is not supposed to be accessible to measurements.

Note that, a similar state observation problem has been dealt with in (Krstic, 2009) for ODE-PDE systems where the ODE subsystem is linear i.e. the vector field $f(.)$ is identically null.

Before proceeding with the observer design and analysis, let us check that the system described by (1a-e)-(2a-c) is well posed. This is the subject of following statement proved in Appendix A.

Proposition 1. The system (1a-e)-(2a-c) has a unique classical solution

$$u(t) \in C([0,\infty):Y) \cap C^1((0,\infty):Y), \quad X(t) \in C^1([0,\infty):\mathbf{R}^n)$$

provided that $u(0) \in Y$, with $Y = \{\xi \in H^2(0,D) : \xi(D) = 0, \xi_x(0) = 0\}$ □

3. Observer Design and Analysis

3.1 Observer Design

Inspired by the high-gain observer design approach, the following observer structure is considered for the system (1a-d)-(2a-c):

$$\dot{\hat{X}} = A\hat{X} + Bv(t) + f(\hat{X}) - \theta\Delta^{-1}K(\hat{u}(0,t) - y(t)) \tag{3a}$$

$$\hat{u}(D,t) = C\hat{X}(t) \tag{3b}$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) - \theta k(x)(\hat{u}(0,t) - y(t)) \tag{3c}$$

$$\hat{u}_x(0,t) = 0 \tag{3d}$$

for all $t \ge 0$ and all $x \in [0,D]$, with

$$\Delta \stackrel{def}{=} diag\left\{1, \frac{1}{\theta}, \dots, \frac{1}{\theta^{n-1}}\right\} \in \mathbf{R}^{n\times n} \tag{3e}$$

where the scalar $\theta > 1$ is a design parameter. The vector and scalar gains, $K \in \mathbf{R}^n$ and $k(x) \in \mathbf{R}$, have yet to be defined. To this end, introduce the state estimation errors:

$$\tilde{X} = \hat{X} - X \,, \quad \tilde{u} = \hat{u} - u \tag{4}$$

Then, subtracting each of the system equations (1a-b)-(2a-c) from the corresponding equation in the observer (3a-d), one gets the following error system:

$$\dot{\tilde{X}} = A\tilde{X} + \left(f(\hat{X}) - f(X)\right) - \theta\Delta^{-1}K\,\tilde{u}(0,t) \tag{5a}$$

$$\tilde{u}(D,t) = C\tilde{X}(t) \tag{5b}$$

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) - \theta k(x)\,\tilde{u}(0,t) \tag{5c}$$

$$\tilde{u}_x(0,t) = 0 \tag{5d}$$

Inspired by (Krstic, 2009), the following backstepping transformation is considered:

$$\tilde{Z} = M^{-1}(D)\Delta\tilde{X} \,, \tag{6a}$$

$$\tilde{w}(x,t) = \tilde{u}(x,t) - CM(x)\tilde{Z}(t) \tag{6b}$$

where $M(x)$ is matrix function yet to be defined. Then, the error system (5a-d) rewrites, in terms of $\tilde{Z}$ and $\tilde{w}(x,t)$, as follows (see Appendix B):

$$\dot{\tilde{Z}} = \theta(M^{-1}(D)AM(D) - LCM(0))\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta L\tilde{w}(0,t) \tag{7a}$$

$$\tilde{w}(D,t) = 0 \tag{7b}$$

$$\begin{aligned}\tilde{w}_t(x,t) = {} & \tilde{w}_{xx}(x,t) - CM(x)M^{-1}(D)\Delta(f(\hat{X}) - f(X)) \\ & - \theta\left(k(x)\tilde{w}(0,t) + k(x)CM(0)\tilde{Z} - CM(x)L\tilde{w}(0,t)\right) \\ & + C\left(\frac{d^2M}{dx}(x) - \theta M(x)\left(M^{-1}(D)AM(D) - LCM(0)\right)\right)\tilde{Z}(t) \quad \text{(using (6b))}\end{aligned} \tag{7c}$$

$$\tilde{w}_x(0,t) = -C\frac{dM}{dx}(0)\tilde{Z} \tag{7d}$$

with:

$$L = M^{-1}(D)K \tag{7e}$$

We seek a gain $k(x)$ and a matrix function $M(x)$ that make the error system (7a-d) coincide with the following target system (which will be shown to be exponentially convergent in Subsection 3.2):

$$\dot{\tilde{Z}} = \theta(A - LCM(0))\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta L\tilde{w}(0,t) \tag{8a}$$

$$\tilde{w}(D,t) = 0 \tag{8b}$$

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - CM(x)M^{-1}(D)\Delta(f(\hat{X}) - f(X)) \quad (8c)$$

$$\tilde{w}_x(0,t) = 0 \quad (8d)$$

Comparing (7a-d) and (8a-d), it is checked that $k(x)$ and $M(x)$ must be defined as follows:

$$k(x) = CM(x)L \quad (9a)$$

$$\frac{d^2M}{dx^2}(x) = \theta M(x)A \quad (9b)$$

$$M(0) = I\,,\ \ \frac{dM}{dx}(0) = 0 \quad (9c)$$

Indeed, doing so equations (7d) reduces to (8b) while (7c) further develops as follows:

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) - CM(x)M^{-1}(D)\Delta(f(\hat{X}) - f(X))$$
$$+ C\left(\frac{d^2M}{dx}(x) - \theta M(x)M^{-1}(D)AM(D)\right)\tilde{Z}(t) \quad (10)$$

Additional properties of the matrix function $M(x)$ are given in Lemma 1 (see Appendix C).

Using Part 2 of that Lemma 1, one gets $M^{-1}(D)AM(D) = A$. Accordingly, equation (10) boils down to (8c).

Now, substituting to $k(x)$ and K their expressions given by (7e) and (9a), the observer (5a-d) rewrites in the following more suitable form:

$$\dot{\hat{X}} = A\hat{X} + Bv(t) + f(\hat{X}) - \theta\Delta^{-1}M(D)L(\hat{u}(0,t) - y(t)) \quad (11a)$$

$$\hat{u}(D,t) = C\hat{X}(t) \quad (11b)$$

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) - \theta CM(x)L(\hat{u}(0,t) - y(t)) \quad (11c)$$

$$\hat{u}_x(0,t) = 0 \quad (11d)$$

for all $t \geq 0$ and all $x \in [0,D]$. For convenience, (3e) is also rewritten:

$$\Delta \overset{def}{=} diag\left\{1, \frac{1}{\theta}, \ldots, \frac{1}{\theta^{n-1}}\right\} \in \mathbf{R}^{n\times n} \quad (11e)$$

The state observer thus designed is a high-gain type involving two design parameters, $L \in \mathbf{R}^n$ and $\theta > 1$. The analysis of Subsection 3.2 will provide insights on how to select these parameters.

Remark 1. a) The above observer design is also a generalization of the observer design proposed in (Krstic, 2009). Indeed, both observers address ODE-PDE cascaded systems and involve matrix gains $M(x)$ which play an instrumental role in the achievement of exponential convergence properties. The generalisation lies in the fact that the ODE part of the present class of systems (1a-e)-(2a-c) is nonlinear, whereas only linear systems are considered in (Krstic, 2009).

b) A major novelty of the present work is the definition of the new target system (8a-d) which quite different from the one used in (Krstic, 2009). The target system in (Krstic, 2009) is not usable here due to the nonlinearities in the error system (5a-d).

c) For convenience, the target system based upon in (Krstic, 2009) is rewritten here (see equations (97)-(100)):

$$\dot{\tilde{X}} = \left(A - M(D)LCM^{-1}(D)\right)\tilde{X} - M(D)L\tilde{w}(0,t) \tag{12a}$$

$$\tilde{w}(D,t) = 0 \tag{12b}$$

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) \tag{12c}$$

$$\tilde{w}_x(0,t) = 0 \tag{12d}$$

Clearly, the system (12a-d) is linear while (8a-d) is not. Furthermore(8a-d) involves a feedback interconnection between the finite dimensional and the infinite dimensional parts, whereas (12a-d) is a cascade structure. Consequently, the exponential stability analysis of the system (12a-d) is simpler than that of the system (8a-d). Indeed, the subsystems (12b-c) as well as the (autonomous part of) the subsystem (12a) are both well known to be exponentially stable. The proof of exponential stability is not that easy when it comes to the target system (8a-d).

d) Also, it is worth noticing that the presently designed observer (11a-e) is a High gain type while that in (Krstic, 2009) is not. However (11a-e) is not a standard high-gain observer due to the presence of the matrix gain $M(x)$

3.2 Observer Analysis

First, the well posedness of the observer (11a-e) is established in the following proposition the proof of which is placed in Appendix D.

Proposition 2. Let the gain L of the observer (11a-e) be selected so that $A - LC$ has all its eigenvalues with negative real parts and the input $v(t)$ be bounded and piecewise continuous. If $\hat{X}(0) \in \mathbf{R}^n$ and $\hat{u}(0) \in \left\{\xi \in H^1(0,D) : \xi(D) = 0, \xi_x(0) = 0\right\}$ then, the system (11a-e) admits a strong solution $\hat{u}(t) \in \left\{\xi \in H^2(0,D) : \xi(D) = 0, \xi_x(0) = 0\right\}$ and $\hat{X}(t)$ absolutely continuous □

Now, The exponential convergence of the observer (11a-e) is described in the following theorem which constitutes the main result:

Theorem 1. Letting the gain L of the observer (11a-e) be selected as in Proposition 2, there exists a scalar $\theta^* > 0$ such that, for all $\theta > \theta^*$ and $D \in (0, D^*(\theta))$ with $D^*(\theta) = 1/\sqrt{\theta}$, the observer (11a-e) a global exponential observer of the system (1a-d). Accordingly, the norm,

$$\left(\left\| X(t) - \hat{X}(t) \right\|^2 + \int_0^D (u(x,t) - \hat{u}(x,t))^2 dx \right)^{1/2},$$

is exponentially vanishing, as $t \to \infty$ □

Proof. The observer (11a-e) has been designed so that the corresponding error system (7a-d) coincides with the target system (8a-d), which expresses in terms of the variables $\tilde{Z}$ and $\tilde{w}(x,t)$ defined by (6a-b). To analyze the system (8a-b), the following Lyapunov function candidate is considered:

$$V = \tilde{Z}^T P \tilde{Z} + \frac{a}{2} \int_0^D \tilde{w}^2(x,t) dx \tag{13a}$$

with P any symmetric positive definite matrix satisfying the following algebraic equation:

$$P(A - LC) + (A - LC)^T P \le -\mu \mathbf{I} \tag{13b}$$

where $\mu > 0$ is arbitrarily chosen while $a > 0$ will be selected later in this proof. Time-derivation of (13a) yields, using (8a), (8c) and (13b):

$$\begin{aligned} \dot{V} &= \dot{\tilde{Z}}^T P \tilde{Z} + \tilde{Z}^T P \dot{\tilde{Z}} + a \int_0^D \tilde{w}(x,t) \tilde{w}_t(x,t) dx \\ &= -\mu\theta \left\| \tilde{Z} \right\|^2 + 2\tilde{Z}^T P M^{-1}(D) \Delta (f(\hat{X}) - f(X)) - 2\theta \tilde{Z}^T P L \tilde{w}(0,t) \\ &\quad + a \int_0^D \tilde{w}(x,t) \tilde{w}_{xx}(x,t) dx - a \int_0^D \tilde{w}(x,t) C M(x) M^{-1}(D) \Delta (f(\hat{X}) - f(X)) dx \end{aligned} \tag{14}$$

Let us analyse the different terms on the right side of (14), starting with the second term. One has:

$$\left| 2\tilde{Z}^T P M^{-1}(D) \Delta (f(\hat{X}) - f(X)) \right| \le \left| 2\tilde{Z}^T P M^{-1}(D) \Delta \left(\int_0^1 f_X(X(t) + s\tilde{X}(t)) \right) \tilde{X}(t) \right| \tag{15}$$

using (1e) and the mean value theorem, where f_X denotes the Jacobian matrix of f. By (6a-b) on has $\tilde{X} = \Delta^{-1} M(D) \tilde{Z}$. Then, (11) becomes:

$$\left| 2\tilde{Z}^T P M^{-1}(D) \Delta (f(\hat{X}) - f(X)) \right| \le \left| 2\tilde{Z}^T P M^{-1}(D) \Delta \left(\int_0^1 f_X(X(t) + s\tilde{X}(t)) \right) \Delta^{-1} M(D) \tilde{Z} \right| \tag{16}$$

Letting the x-domain length D be such that $D^2\theta < 1$, one gets using Parts 1 and 2 of Lemma 1 (Appendix C) that, $\forall x \in [0, D]$:

$$\left\| M(x) \right\| \le 1 + \sum_{k=1}^{n-1} \frac{1}{(2k)!} \overset{def}{=} c_1 < \infty \tag{17a}$$

$$\left\|M^{-1}(x)\right\| \leq 1+\sum_{k=1}^{n-1} \frac{|\alpha_k|}{(2k)!} \stackrel{def}{=} c_2 < \infty \tag{17b}$$

using the fact that $\left\|A^k\right\| \leq 1$, whatever k. Moreover, it is readily checked using (1e) and (3d) that, as long as $\theta > 1$, $\left\|\Delta\left(\int_0^1 f_X(X(t)+s\tilde{X}(t))\right)\Delta^{-1}\right\| \leq \beta$. Based on these observations, (16) yields:

$$\left|2\tilde{Z}^T PM^{-1}(D)\Delta(f(\hat{X})-f(X))\right| \leq 2c_1c_2\beta\|P\|\,\|\tilde{Z}\|^2 \tag{18}$$

where the right side of this inequality does not depend on θ (as long as $\theta > 1$). In turn, the third term on the right side of (14) develops as follows:

$$2\theta\left|\tilde{Z}^T PL\,\tilde{w}(0,t)\right| = 2\theta\left|\sqrt{\frac{\mu}{2}}\tilde{Z}^T\sqrt{\frac{2}{\mu}}PL\,\tilde{w}(0,t)\right| \leq \frac{\mu\theta}{2}\tilde{Z}^2 + \frac{2\theta\|PL\|^2}{\mu}|\tilde{w}(0,t)|^2$$

$$\leq \frac{\mu\theta}{2}\|\tilde{Z}\|^2 + \frac{2D\theta\|PL\|^2}{\mu}\int_0^D |\tilde{w}_x(x,t)|^2 dx \tag{19}$$

where the last inequality is obtained using the fact that:

$$|\tilde{w}(0,t)|^2 = \left|\int_0^D \tilde{w}_x(x,t)dx\right|^2 \quad \text{(using (7b))}$$

$$\leq D\int_0^D \tilde{w}_x^2(x,t)dx \quad \text{(using Schwartz inequality)}$$

Using an integration by parts , the fourth term on the right side of (14) develops as follows:

$$a\int_0^D \tilde{w}(x,t)\tilde{w}_{xx}(x,t)dx = -a\int_0^D \tilde{w}_x^2(x,t)dx \tag{20}$$

where the boundary conditions (7b) and (7d) have been used. Finally, the last term on the right side of (10) can be bounded as follows, whatever $\zeta > 0$:

$$a\left|\int_0^D \tilde{w}(x,t)CM(x)M^{-1}(D)\Delta(f(\hat{X})-f(X))dx\right|$$

$$\leq \frac{a\zeta}{2}\int_0^D |\tilde{w}(x,t)|^2 dx + \frac{a}{2\zeta}\int_0^D \left\|CM(x)M^{-1}(D)\Delta(f(\hat{X})-f(X))\right\|^2 dx$$

$$\leq \frac{a\zeta}{2}\int_0^D |\tilde{w}(x,t)|^2 dx + \frac{ac_1^2c_2^2}{2\zeta}\int_0^D \left\|\Delta(f(\hat{X})-f(X))\right\|^2 dx$$

where the last inequality is obtained using (17a-b). Following the same argument as the one used to get (18) from (16), the above inequality leads to:

$$a\left|\int_0^D \tilde{w}(x,t)CM(x)M^{-1}(D)\Delta(f(\hat{X})-f(X))dx\right|$$

$$\leq \frac{a\zeta}{2}\int_0^D |\tilde{w}(x,t)|^2 dx + \frac{aD\beta c_1^4c_2^2}{2\zeta}\|\tilde{Z}\|^2 \tag{21}$$

Using (18) to (21), it follows from (14) that:

$$\dot{V} \leq -\mu\theta\left\|\tilde{Z}\right\|^2 + 2c_1c_2\beta\|P\|\left\|\tilde{Z}\right\|^2 + \frac{\mu\theta}{2}\left\|\tilde{Z}\right\|^2 + \frac{2D\theta\|PL\|^2}{\mu}\int_0^D\left|\tilde{w}_x(x,t)\right|^2 dx$$

$$- a\int_0^D \tilde{w}_x^2(x,t)dx + \frac{a\zeta}{2}\int_0^D\left|\tilde{w}(x,t)\right|^2 dx + \frac{aD\beta c_1^4c_2^2}{2\zeta}\left\|\tilde{Z}\right\|^2$$

$$\leq -\left(\frac{\mu\theta}{2} - 2c_1c_2\beta\|P\| - \frac{aD\beta c_1^4c_2^2}{2\zeta}\right)\left\|\tilde{Z}\right\|^2 + \frac{2D\theta\|PL\|^2}{\mu}\int_0^D\left|\tilde{w}_x(x,t)\right|^2 dx - a\int_0^D \tilde{w}_x^2(x,t)dx$$

$$+ \frac{2aD^2\zeta}{\pi^2}\int_0^D\left|\tilde{w}_x(x,t)\right|^2 dx \tag{22}$$

where we have used Wirtinger's inequality (Hardy *et al.*, 1934):

$$\int_0^D\left|\tilde{w}(x,t)\right|^2 dx \leq \frac{4D^2}{\pi^2}\int_0^D\left|\tilde{w}_x(x,t)\right|^2 dx \tag{23}$$

This is presently possible because $\tilde{w}(D,t) = 0$ and $\tilde{w}(.,t) \in H^1(0,D)$. Rearranging terms on the right side of (22), one gets:

$$\dot{V} \leq -\left(\frac{\mu\theta}{2} - 2c_1c_2\beta\|P\| - \frac{aD\beta c_1^4c_2^2}{2\zeta}\right)\left\|\tilde{Z}\right\|^2 - \frac{a}{2}\int_0^D \tilde{w}_x^2(x,t)dx$$

$$-\left(\frac{a}{2} - \frac{2D\theta\|PL\|^2}{\mu} - \frac{2aD^2\zeta}{\pi^2}\right)\int_0^D \tilde{w}_x^2(x,t)dx$$

$$\leq -\frac{\mu\theta}{4}\left\|\tilde{Z}\right\|^2 - \underbrace{\left(\frac{\mu\theta}{4} - 2c_1c_2\beta\|P\| - \frac{aD\beta c_1^4c_2^2}{2\zeta}\right)}_{Term\,1}\left\|\tilde{Z}\right\|^2 - \frac{a\pi^2}{8D^2}\int_0^D \tilde{w}^2(x,t)dx$$

$$-\underbrace{\left(\frac{a}{2} - \frac{2D\theta\|PL\|^2}{\mu} - \frac{2aD^2\zeta}{\pi^2}\right)}_{Term\,2}\int_0^D \tilde{w}_x^2(x,t)dx \tag{24}$$

where the last inequality is obtained using again (23). At this stage, the free parameters $a > 0$ and $\zeta > 0$ and the design parameter $\theta > 1$ have yet to be chosen. An adequate choice is one that makes *Term* 1 and *Term* 2, on the right side of (24), nonnegative. As $0 < D < 1/\sqrt{\theta} < 1$, *Term* 2 is bounded from below as follows:

$$Term2 = \frac{a}{2} - \frac{2D\theta\|PL\|^2}{\mu} - \frac{2aD^2\zeta}{\pi^2} \geq a\left(\frac{1}{2} - \frac{2\zeta}{\pi^2}\right) - \frac{2\sqrt{\theta}\|PL\|^2}{\mu} \tag{25a}$$

Then, a sufficient condition for $Term2$ to be nonnegative is to let:

$$\frac{1}{2}-\frac{2\zeta}{\pi^2}>0 \text{ and } a=\frac{2\sqrt{\theta}\|PL\|^2}{\mu}\frac{2\pi^2}{\pi^2-4\zeta} \tag{25b}$$

The first inequality in (25b) is satisfied with e.g. $\zeta=\frac{\pi^2}{8}$. Then, *Term* 1 can be made nonnegative by letting:

$$\frac{\mu\theta}{4}\geq 2c_1c_2\beta\|P\|+\frac{4a\beta c_1^4c_2^2}{\pi^2\sqrt{\theta}} \qquad \text{(using } 0<D<1/\sqrt{\theta}\text{)}$$

$$\geq 2c_1c_2\beta\|P\|+\frac{4\beta c_1^4c_2^2}{\pi^2}\frac{2\|PL\|^2}{\mu}\frac{2\pi^2}{\pi^2-4\zeta} \qquad \text{(using (25b))} \tag{26}$$

This suggests the choice $\theta\geq\theta^*$ with $\theta^*=\frac{4}{\mu}\left(2c_1c_2\beta\|P\|+\frac{16\beta c_1^4c_2^2}{\pi^2}\frac{2\|PL\|^2}{\mu}\right)$. Doing so, inequality (26) yields:

$$\dot{V}\leq -c_3\left\|\tilde{Z}\right\|^2-c_4\int_0^D\tilde{w}^2(x,t)dx\leq -c_0V \tag{27}$$

with $c_3=\frac{\mu\theta^*}{4}$, $c_4=\frac{a\pi^2}{8D^2}$ and $c_0=\min(\frac{\mu\theta^*}{4\lambda_{\max}(P)},\frac{2\pi^2}{8D^2})$. This ends the proof of Theorem 1 ■

3.3. Extension

The result of Theorem 1 can be adapted to the case where the PDE subsystem is a delay/transport element. Then, the system (1a-e)-(2a-c) becomes:

$$\dot{X}(t)=AX(t)+Bv(t)+f(X(t)) \tag{28a}$$

$$u(D,t)=CX(t) \tag{28b}$$

$$u_t(x,t)=u_x(x,t), \quad 0\leq x\leq D \tag{28c}$$

$$y(t)=u(0,t) \tag{28d}$$

where the remaining notations are as in Section 2. Then, the observer (11a-e) adapts to this case as follows:

$$\dot{\hat{X}}=A\hat{X}+Bv(t)+f(\hat{X})-\theta\Delta^{-1}M(D)L(\hat{u}(0,t)-y(t)) \tag{29a}$$

$$\hat{u}(D,t)=C\hat{X}(t) \tag{29b}$$

$$\hat{u}_t(x,t)=\hat{u}_x(x,t)-\theta CM(x)L(\hat{u}(0,t)-y(t)) \tag{29c}$$

$$\hat{u}_x(0,t)=0 \tag{29d}$$

where Δ is defined by (11e) and $M(x) = e^{\theta Ax}$. Following mutatis-mutandis the proof of Theorem 1, the same result can be established with the observer (29a-d) being applied to the system (28a-d).

4. Conclusion

The problem of state observation is addressed for the class of nonlinear systems, represented by the ODE-PDE association of Fig. 1, analytically modelled by equations (1a-e). The aim is to get online estimates of both the finite-dimensional state $X(t)$ and the infinite-dimensional state $u(x,t)$ over the x-domain $(0, D)$, for some $D > 0$. A major difficulty is that the connexion point (between the ODE and the PDE subsystems), is not accessible to measurements making useless existing observers developed separately for ODE and PDE systems. The problem is dealt with using the high-gain type observer defined by equations (11a-e) which is a generalization of (Krstic, 2009) to the case where the ODE subsystem is nonlinear with triangular structure. The matrix function $M(x)$ emphasizes the difference with standard high-gain observers and plays an instrumental role in making (11a-e) an exponential convergence (Theorem 1). The present study can be pursued in several directions including: (i) re-designing the observer so that to make its convergence rate dependent on the the design parameters μ and θ; (ii) the design of an adaptive version of the observer and the generalisation to other ODE and PDE subsystems.

Appendices

Appendix A. Proof of Proposition 1.

First notice that, by the standard existence theorem, the solution $X(t)$ of the ODE subsystems (1a-e) exists, whatever $\|X(0)\| < \infty$, because f is continuous and Lipschitz (due to (1e)). Then, it remains to show that, in turn the solution $u(x,t)$ exists. To this end, introduce the following auxiliary signal:

$$\omega(x,t) = u(x,t) - \gamma(x,t)X(t) \tag{A1}$$

where $\gamma(x,t) \in \mathbf{R}^{1\times n}$ has yet to be defined. Clearly, $u(x,t)$ exists if $\omega(x,t)$ and $\gamma(x,t)$ do so. Presently, $\gamma(x,t)$ is selected so that $\omega(x,t)$ undergoes the following target system:

$$\omega_t(x,t) = \omega_{xx}(x,t) \tag{A2}$$

$$\omega_x(0,t) = 0 \tag{A3}$$

$$\omega(D,t) = 0 \tag{A4}$$

This parabolic system is analyzed in many places and its well posedness can be established in many ways. Applying e.g. Theorem 2.6.5 in (Zheng, 2004) it follows that (A2)-(A4) admits a unique local solution:

$$\omega \in C([0,\infty):Y) \cap C^1([0,\infty):Y) \qquad \text{(A5)}$$

whatever $\omega(0,x) \in Y$, where

$$Y = D(\mathbf{A}) = \left\{ \xi \in H^2(0,D) : \xi(D) = 0, \xi_x(0) = 0 \right\} \qquad \text{(A6)}$$

is the dense domain of the operator $\mathbf{A} = \dfrac{\partial^2}{\partial x^2}$. Note that (A5) is achieved making use of the fact that that $\mathbf{A}$ is a closed operator generating on Y a strongly continuous exponentially stable semigroup T satisfying the inequality $\|T(t)\| \le \kappa e^{-\delta t}$ $(t \ge 0)$ for some constant $\delta > 0$ (e.g. Curtain and Zwart, 1995).

Similarly, deriving both sides of (A1) with respect to time, one gets using (1a) and (2a):

$$\begin{aligned} \omega_t(x,t) &= u_{xx}(x,t) - \gamma(x,t)(AX(t) + f(X(t)) - \gamma_t(x,t)X(t) \\ &= \omega_{xx}(x,t) + \gamma_{xx}(x,t)X(t) - \gamma(x,t)\big(AX(t) + f(X(t))\big) - \gamma_t(x,t)X(t) \\ &= \omega_{xx}(x,t) + \gamma_{xx}(x,t)X(t) - \gamma(x,t)AX(t) \\ &\quad - \gamma(x,t)\left(\int_0^1 (f_X(sX(t))ds \right) X(t) - \gamma_t(x,t)X(t) \end{aligned} \qquad \text{(A7)}$$

where the last equality is obtained using the mean-value theorem. Equality (A4) suggests the following model for $\gamma(x,t)$:

$$\gamma_t(x,t) = \gamma_{xx}(x,t) - \gamma(x,t)A - \gamma(x,t)a(t) \qquad \text{(A8)}$$

where $a(t) = \int_0^1 (f_X(sX(t))ds$ is a bounded matrix function, due to (1e). Equation (A6) is completed with the corresponding boundary conditions. First, one gets from (A1), using (A4) and (2b):

$$0 = \omega(D,t) = u(D,t) - \gamma(D,t)X(t) = CX(t) - \gamma(D,t)X(t)$$

which suggest that one must let

$$\gamma(D,t) = C \qquad \text{(A9)}$$

Also, one gets from (A1), using (A3) and (2b):

$$0 = \omega_x(0,t) = u_x(0,t) - \gamma_x(0,t)X(t) = -\gamma_x(0,t)X(t)$$

which entails:

$$\gamma_x(0,t) = 0 \qquad \text{(A10)}$$

In the sequel, the notation $\bar{\gamma}(t)$ refers to the family of functions, parameterized by $t \in [0,\infty)$, defined by: $\bar{\gamma}(t):[0,D] \to \mathbf{R};\ x \to \gamma(x,t) - C$. This is a usual practice in the semigroups theory (e.g. Pazy, 1983). Then, equation (A8) rewrites in the form of a differential equation defined on $H = L_2[0,D]$ as follows:

$$\bar{\gamma}_t(t) = \mathbf{A}\bar{\gamma}(t) + F(t,\bar{\gamma}(t)) \tag{A11}$$

with $\mathbf{A}$ as above and $F(t,\bar{\gamma}) = \bar{\gamma}A + \bar{\gamma}a(t) + Ca(t) + CA$. Clearly, $F(t,\bar{\gamma})$ is affine w.r.t. $\bar{\gamma}$ and so it is Lipschitz. Then, again applying Theorem 2.6.5 in (Zheng, 2004) it follows that the system (A9)-(A11), with $\bar{\gamma}(0) \in Y$, admits a unique local solution:

$$\bar{\gamma} \in C([0,\infty):Y) \cap C^1((0,\infty):Y),$$

using again the fact that the function $F(t,\bar{\gamma})$ is Lipschitz and the operator $\mathbf{A}$ is generating on Y a strongly continuous exponentially stable semigroup.

Combining the above results on $\omega(x,t)$ and $\gamma(x,t)$, one gets that $u = \omega + \gamma X \in C([0,\infty):Y) \cap C^1((0,\infty):Y)$, whatever $u(0) \in Y$. This completes the proof of Proposition 1 ■

Appendix B. **Proof of (7a-b)**. Deriving $\tilde{Z} = M^{-1}(D)\Delta\tilde{X}$ with respect to time yields, using (5a):

$$\begin{aligned}\dot{\tilde{Z}} &= M^{-1}(D)\Delta\left(A\tilde{X} + (f(\hat{X}) - f(X)) - \theta\Delta^{-1}K\tilde{u}(0,t)\right)\\ &= M^{-1}(D)\Delta A\tilde{X} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta M^{-1}(D)K\tilde{u}(0,t)\\ &= M^{-1}(D)\Delta A\Delta^{-1}M(D)\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta L\tilde{u}(0,t)\end{aligned} \tag{B1}$$

with $L = M^{-1}(D)K$. It is readily checked using (1c) and (3e), that:

$$\Delta A\Delta^{-1} = \theta A \tag{B2}$$

Then, one gets:

$$M^{-1}(D)\Delta A\Delta^{-1}M(D) = \theta M^{-1}(D)AM(D) \tag{B3}$$

Then, equation (B1) reduces to:

$$\dot{\tilde{Z}} = \theta M^{-1}(D)AM(D)\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta L\tilde{u}(0,t) \tag{B4}$$

Furthermore, it is readily seen from (6b) that, $\tilde{w}(0,t) + CM(0)\tilde{Z}$ can be substituted to $\tilde{u}(0,t)$ in (B4) which then becomes:

$$\begin{aligned}\dot{\tilde{Z}} &= \theta M^{-1}(D)AM(D)\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta L\left(\tilde{w}(0,t) + CM(0)\tilde{Z}\right)\\ &= \theta M^{-1}(D)AM(D)\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta LCM(0)\tilde{Z} - \theta L\tilde{w}(0,t)\\ &= \theta(M^{-1}(D)AM(D) - LCM(0))\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta L\tilde{w}(0,t)\end{aligned} \tag{B5}$$

Equation (7a) is established.

To prove (7b), write the second equality in (6b) for $x = D$:

$$\tilde{w}(D,t) = \tilde{u}(D,t) - CM(D)\tilde{Z}$$

$$= C\tilde{X}(t) - C\Delta\tilde{X} \quad \text{(using (5b) and (6a))}$$

$$= C\tilde{X}(t) - C\tilde{X}(t) = 0$$

where the last equality is obtained using the fact that $C\Delta = C$, due to (1c) and (3e). This proves (7b).

To prove (7c), it follows deriving both sides of (6b) with respect to time:

$$\tilde{w}_t(x,t) = \tilde{u}_t(x,t) - CM(x)\dot{\tilde{Z}}$$

$$= \tilde{u}_{xx}(x,t) - \theta k(x)\,\tilde{u}(0,t) - CM(x)\dot{\tilde{Z}} \quad \text{(using (5c))}$$

$$= \tilde{w}_{xx}(x,t) + C\frac{d^2M}{dx}(x)\tilde{Z}(t) - \theta k(x)\,\tilde{u}(0,t) - CM(x)\dot{\tilde{Z}} \tag{B6}$$

where the last equality is obtained using (6b). Using (7a), equation (B6) further develops as follows:

$$\tilde{w}_t(x,t) = \tilde{w}_{xx}(x,t) + C\frac{d^2M}{dx}(x)\tilde{Z}(t) - \theta k(x)\,\tilde{u}(0,t)$$

$$- CM(x)\Big(\theta\big(M^{-1}(D)AM(D) - LCM(0)\big)\tilde{Z} + M^{-1}(D)\Delta(f(\hat{X}) - f(X)) - \theta L\tilde{w}(0,t)\Big)$$

$$= \tilde{w}_{xx}(x,t) - CM(x)M^{-1}(D)\Delta(f(\hat{X}) - f(X))$$

$$- \theta\big(k(x)\tilde{u}(0,t) - CM(x)L\tilde{w}(0,t)\big)$$

$$+ C\left(\frac{d^2M}{dx}(x) - \theta M(x)\big(M^{-1}(D)AM(D) - LCM(0)\big)\right)\tilde{Z}(t)$$

$$= \tilde{w}_{xx}(x,t) - CM(x)M^{-1}(D)\Delta(f(\hat{X}) - f(X))$$

$$- \theta\big(k(x)\tilde{w}(0,t) + k(x)CM(0)\tilde{Z} - CM(x)L\tilde{w}(0,t)\big)$$

$$+ C\left(\frac{d^2M}{dx}(x) - \theta M(x)\big(M^{-1}(D)AM(D) - LCM(0)\big)\right)\tilde{Z}(t) \tag{B7}$$

where we have used (6b). Equation (B7) is nothing other than (7c). Finally, equality (7d) is obtained by deriving both sides of (6b) with respect to x and then letting $x = 0$ in the obtained equality. Doing so, one gets:

$$\tilde{w}_x(0,t) = \tilde{u}_x(0,t) - C\frac{dM}{dx}(0)\tilde{Z} = -C\frac{dM}{dx}(0)\tilde{Z}$$

where the second equality is an immediate consequence of (5d). This completes the proof that the system (7a-d) holds ■

Appendix C. Properties of the matrix function $M(x)$.

Lemma 1. The function $M(x)$ defined by (9b-c), where A is as in (1c), has the following properties:

1) $M(x) = \mathbf{I} + \sum_{k=1}^{n-1} \frac{x^{2k}}{(2k)!}(\theta A)^k$

2) $M(x)A = AM(x)$

3) $M^{-1}(x) = \mathbf{I} + \sum_{k=1}^{n-1} \alpha_k x^{2k} (\theta A)^k$ with $\alpha_1 = -\frac{1}{2!}$, $\alpha_k = -\frac{\alpha_{k-1}}{2!} - \frac{\alpha_{k-2}}{4!} \ldots - \frac{1}{(2k)!}$

4) $M^{-1}(x)A = AM^{-1}(x)$

5) $M(x) \overset{def}{=} \begin{pmatrix} \mathbf{I} & 0 \end{pmatrix} e^{\begin{pmatrix} 0 & \theta A \\ \mathbf{I} & 0 \end{pmatrix} x} \begin{pmatrix} \mathbf{I} \\ 0 \end{pmatrix} \in \mathbf{R}^{n \times n}$, $\forall x$ □

Proof. **Part 1**. This is simply proved by checking that the presumed expression of $M(x)$ undergoes the differential equation and border condition (9b-c). It is readily checked that the first derivative is:

$$\frac{dM}{dx}(x) = \sum_{k=1}^{n-1} \frac{x^{2k-1}}{(2k-1)!}(\theta A)^k \tag{C1}$$

Deriving once again, one gets:

$$\begin{aligned}
\frac{d^2M}{dx^2}(x) &= \sum_{k=1}^{n-1} \frac{x^{2k-2}}{(2k-2)!}(\theta A)^k \\
&= \left(\sum_{k=1}^{n-1} \frac{x^{2k-2}}{(2k-2)!}(\theta A)^{k-1} \right)(\theta A) \\
&= \left(M(x) - \frac{x^{2(n-1)}}{(2(n-1))!}(\theta A)^{n-1} \right)(\theta A) \\
&= M(x)(\theta A)
\end{aligned}$$

where the last equality is obtained using the fact that the matrix A is nilpotent i.e. $A^n = 0$. This proves (9b). Furthermore, letting $x = 0$ on the right side of the equality in Part 1, yields $M(0) = I$. Also, letting $x = 0$ on the right side of (C1) gives $\frac{dM}{dx}(0) = 0$. Part 1 is established.

Part 2. This is an immediate consequence of part 1 using the fact that $A(\theta A)^k = (\theta A)^k A$.

Part 3. Let us develop the product $M(x)M^{-1}(x)$ replacing there $M^{-1}(x)$ by its presumed expression. Doing so one gets:

$$M(x)M^{-1}(x)=\left(\mathbf{I}+\sum_{k=1}^{n-1}\frac{x^{2k}}{(2k)!}(\theta A)^k\right)\left(\mathbf{I}+\sum_{k=1}^{n-1}\alpha_k x^{2k}(\theta A)^k\right)$$

$$=I+\sum_{k=1}^{2n-2}\beta_k x^{2k}(\theta A)^k \tag{C2}$$

Direct computations yield:

$$\beta_1=\alpha_1+\frac{1}{2!}$$

$$\beta_k=\alpha_k+\frac{\alpha_{k-1}}{2!}+\ldots+\frac{\alpha_1}{(2(k-1))!}+\frac{1}{(2k)!} \quad (k=2\ldots 2n-2)$$

By definition of the α_k's, it follows that all β_k's are equal to zero. Then (C2) implies that the equality $M(x)M^{-1}(x)=I$ does hold.

Part 4. This part is readily obtained from Part 3, pre-multiplying and post-multiplying the expression of $M^{-1}(x)$ by A.

Part 5. It readily follows from Part 1 that:

$$\frac{d^2M^T}{dx^2}(x)=\theta A^T M^T(x) \tag{C3}$$

Introduce the augmented matrix:

$$\vartheta(x)=\begin{pmatrix} M^T(x) \\ \dfrac{dM^T}{dx}(x)\end{pmatrix} \tag{C4}$$

It follows from (C3) that $\vartheta(x)$ undergoes the differential equation:

$$\frac{d\vartheta}{dx}(x)=\begin{pmatrix} 0 & I \\ \theta A^T & 0\end{pmatrix}\vartheta(x) \tag{C5}$$

with the initial condition $\vartheta(0)=\begin{pmatrix}\mathbf{I} & 0\end{pmatrix}$, using Part 4. The solution of (C5) is:

$$\vartheta(x)=e^{\begin{pmatrix} 0 & I \\ \theta A^T & 0\end{pmatrix}}\begin{pmatrix}\mathbf{I} \\ 0\end{pmatrix}$$

which, together with (C4) yields:

$$M^T(x)=\begin{pmatrix}\mathbf{I} & 0\end{pmatrix}e^{\begin{pmatrix} 0 & I \\ \theta A^T & 0\end{pmatrix}x}\begin{pmatrix}\mathbf{I} \\ 0\end{pmatrix}$$

which proves Part 5 and completes the proof of Lemma 1 ■

Appendix D. Proof of Proposition 2.

By Proposition 1, the true system state $(X(t), u(x,t))$ exists for all $t \geq 0, 0 \leq x \leq D$. Then, a sufficient condition for the observer (3a-d) to be well posed is that the error system (5a-d) is so. The well posedness of the latter will now be established. To this end, introduce the following variable change:

$$\tilde{Z} = M^{-1}(D)\Delta\tilde{X}\,, \qquad \tilde{X} = \Delta^{-1}M(D)\tilde{Z} \tag{D1}$$

$$\omega(x,t) = \tilde{u}(x,t) - CM(x)\tilde{Z} \tag{D2}$$

The last expression is referred to backstepping transformation (Krstic, 2009). Then, the error system (5a-d) rewrites, in terms of $\tilde{X}$ and $\omega(x,t)$, as follows:

$$\dot{\tilde{X}} = (A - \theta\Delta^{-1}M(D)LCM^{-1}(D)\Delta)\tilde{X} + (f(\hat{X}) - f(X)) - \theta\Delta^{-1}M(D)L\omega(0,t) \tag{D3}$$

$$= (A - \theta\Delta^{-1}M(D)LCM^{-1}(D)\Delta)\tilde{X}$$

$$+\left(\int_0^1 (f_X(X(t) + s\tilde{X}(t))ds\right)\tilde{X} - \theta\Delta^{-1}M(D)L\omega(0,t) \tag{D4}$$

$$\omega(D,t) = 0 \tag{D5}$$

$$\omega_t(x,t) = \omega_{xx}(x,t) - CM(x)M^{-1}(D)\Delta\left(\int_0^1 (f_X(X(t) + s\tilde{X}(t))ds\right)\tilde{X} \tag{D6}$$

$$\omega_x(0,t) = 0 \tag{D7}$$

where (D4) is obtained using the mean value theorem. Let us define a new function $\eta(x,t)$ as follows:

$$\eta_t(x,t) = (A - \theta\Delta^{-1}M(D)LCM^{-1}(D)\Delta)\eta(x,t)$$

$$+ M^{-1}(D)\Delta\left(\int_0^1 (f_X(X(t) + s\eta(x,t))ds\right)\eta(x,t) - \theta L\omega(0,t) \tag{D8}$$

$$\eta(x,0) = \tilde{X}(0) \tag{D9}$$

Comparing equations (D8-D9) and (D3), it is seen that:

$$\eta(x,t) = \tilde{X}(t), \quad \forall t \geq 0, \forall x \in [0,D] \tag{D10}$$

Therefore, analyzing the well-posedness of (D3-D7) amounts to analyzing the well posedness of the following system:

$$\eta_t(x,t) = (A - \theta\Delta^{-1}M(D)LCM^{-1}(D)\Delta)\eta(x,t)$$

$$+ M^{-1}(D)\Delta\left(\int_0^1 (f_X(X(t) + s\eta(x,t))ds\right)\eta(x,t) + \theta L\int_0^D \omega_x(x,t)dx \tag{D11}$$

$$\eta(x,0) = \tilde{X}(0) \tag{D12}$$

$$\omega(D,t)=0 \tag{D13}$$

$$\omega_t(x,t)=\tilde{w}_{xx}(x,t)-CM(x)M^{-1}(D)\Delta\left(\int_0^1 (f_X(X(t)+s\eta(x,t))ds\right)\eta(x,t) \tag{D14}$$

$$\omega_x(0,t)=0 \tag{D15}$$

where the last term on the right side of (D11) is obtained using the boundary condition $\omega(D,t)=0$. Following closely a similar analysis in (Fridman and Am, 2013, Appendix A), one defines the augmented state $W(x,t)=\begin{bmatrix}\eta(x,t)\\ \omega(x,t)\end{bmatrix}$. Then, the system (D11-D15) can be represented by the following differential equation, where

$$\dot{W}(t)=\Pi W(t)+F(t,W(t)), \tag{D16}$$

defined in $L_2(0,D)$ where

$$\Pi=\begin{bmatrix}A-\theta\Delta^{-1}M(D)LCM^{-1}(D)\Delta & 0\\ 0 & \dfrac{\partial^2}{\partial x^2}\end{bmatrix} \tag{D17}$$

and the nonlinear $F(.):\mathbf{R}^+\times H^1(0,D)\to L_2[0,D]$ is defined as follows:

$$F(t,W)=\phi(t,\eta)\eta+\begin{bmatrix}\theta L\int_0^D \omega_x(x,t)dx\\ 0\end{bmatrix} \tag{D18}$$

$$\phi(t,\eta)=\begin{bmatrix}M^{-1}(D)\Delta\left(\int_0^1 (f_X(X(t)+s\eta)ds\right)\\ -CM(x)M^{-1}(D)\Delta\left(\int_0^1 (f_X(X(t)+s\eta)ds\right)\end{bmatrix} \tag{D19}$$

The operator Π has the dense domain:

$$D(\Pi)=\left\{W\in H^2(0,D):W(D)=0,W_x(0)=0\right\}$$

At this point, recall that one has $M(D)A=AM(D)$, $AM^{-1}(D)=M^{-1}(D)A$ and $\Delta A\Delta^{-1}=\theta A$. Then, it is readily checked that $A-\theta\Delta^{-1}M(D)LCM^{-1}(D)\Delta=\theta\Delta^{-1}M(D)(A-LC)M^{-1}(D)\Delta$. This shows that the matrix $A-\theta\Delta^{-1}M(D)LCM^{-1}(D)\Delta$ is similar to $A-LC$ which we know it is Hurwitz. Also, it is well known that the operator $\dfrac{\partial^2}{\partial x^2}$ generates a strongly continuous exponentially stable semigroup. Then, it follows that in turn the operator Π generates a strongly continuous exponentially stable semigroup. Furthermore, it is easily checked that the operator $-\dfrac{\partial^2}{\partial x^2}$ is positive definite. On the other hand, since the matrix $A-LC$ has negative real-part eigenvalues, it has a square root. It turns out that, the operator $-\Pi$ has a square root $(-\Pi)^{1/2}$ in the domain:

$$H_{1/2} = D((-\Pi)^{1/2}) = \{W \in H^1(0,D) : W(D) = 0, W_x(0) = 0\}$$

which is a Hilbert space with the scalar product $< a,b >=< (-\Pi)^{1/2} a, (-\Pi)^{1/2} b >$.

On the other hand, since f is class C^2 satisfying (1e) and $X(t)$ exists (by Proposition 1), it follows from (D19) that $\phi(t,W)$ is bounded and belongs to C^1. Then, it follows from (D18) that $F(t,W)$ is class C^1 and so the following Lipschitz condition:

$$\|F(t_1,W_1) - F(t_2,W_2)\| \le \kappa_1 \left(|t_1 - t_2| + \|(-\Pi)^{1/2}(W_1 - W_2)\|\right), \ \forall (t_i, W_i) \in R \times H_{1/2} \qquad \text{(D20)}$$

with some constant $\kappa_1 > 0$, hold locally in $\mathbf{R} \times H_{1/2}$. Furthermore, since $\phi(t,W)$ is bounded there exist $\kappa_2 > 0$ such that:

$$\|F(t,W)\| \le \kappa_2 \|(-\Pi)^{1/2} W\|, \ \forall W \in H_{1/2} \qquad \text{(D21)}$$

Thus, Theorem 3.3.3 of Henry (1993) is applicable to (D16) ensuring that a strong solution $W(t) \in H_{1/2} \cap D(\Pi)$, for all $t > 0$, initialized with $W(0) \in H_{1/2}$. Then, one gets from (D1)-(D2) and (D10) that, $\tilde{u}(t) \in \{\xi \in H^1(0,D) : \xi(D) = 0, \xi_x(0) = 0\} \cap D(\Pi)$ and $\tilde{X}(t)$ is absolutely continuous. Then, it follows using Proposition 1 that, $\hat{X}$ is absolutely continuous and $\hat{u} = u + \tilde{u} \in \{\xi \in H^2(0,D) : \xi(D) = 0, \xi_x(0) = 0\}$ ■

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