Adaptive Observer for a Class of Parabolic PDEs

T. Ahmed-Ali, F. Giri, M. Krstic, F. Lamnabhi-Lagarrigue, and L. Burlion

***Abstract*—The problem of state observation, based on spatially-sampled output measurements, is addressed for a class of infinite dimensional systems, modelled by a semi-linear heat equation augmented with a structured uncertain part involving a set of unknown parameters. An adaptive observer is designed that provides online estimates of the system (spatially distributed) state and unknown parameters based on sampled data (in space). Sufficient conditions for the observer to be exponentially convergent are established. These include an *ad-hoc* persistent excitation condition as well as a condition on how the observer gain must be selected in relation with the space sampling interval.**

***Index Terms*—Nonlinear system, observer design, PDE systems.**

I. Introduction

Adaptive state observers are resorted to deal with online state and parameter estimation. The first adaptive observers have been developed for finite-dimensional continuous-time linear systems and an extensive survey can be found in [1] and [2]. Then, research efforts have been devoted to designing nonlinear adaptive observers for (finite-dimensional) nonlinear systems, e.g., [3]–[7]. More recently, sampled-data (in time) observers have been developed for (finite-dimensional) nonlinear systems where output measurements are only available at sampling instants, e.g., [8]–[10].

The problem of observer design for infinite dimensional systems (IDSs) has also been given a great deal of interest, especially since the eighties. Several observer design techniques have been developed including the infinite dimensional Luenberger observer for linear IDSs (e.g., [11], [12]), the boundary observer design of bilinear IDSs (e.g., [13]–[15], [22]), backstepping-based boundary observers for parabolic partial integro-differential IDSs [16], initial state recovery algorithms for various linear and nonlinear IDSs [17]–[19], sampled-data (in time and space) observer for semilinear diffusion IDSs [24].

In the last few years, much interest has been paid to simultaneous parameter and state estimation for IDSs, within various contexts. In [20], simultaneous state and parameter estimation has been introduced to deal with output-feedback adaptive control design for parabolic PDEs. In this context, the unknown parameters are tuned by gradient-type laws while the (spatially distributed) state is estimated using open-loop filters. The convergence of the estimates to their true values is not established. But, this is not required for the achievement of control objectives. In [21], simultaneous state and parameter estimation has been performed to solve a parameter identification problem for reaction-advection type systems involving a single unknown parameter. Open- and closed-loop adaptive identifiers have been proposed where the unknown parameter is estimated using gradient-type estimators, while the (spatially distributed) state is estimated using open-loop filters. It is shown that the parameter estimate converges to its true values, by just using constant exciting inputs. In [22], simultaneous parameter and state estimation has been considered within the context of adaptive stabilization for a wave equation subject to a boundary harmonic disturbance linearly parameterized along a known set of functions. An adaptive observer estimating the system state and the (disturbance) unknown parameters is proposed and the estimation error system is shown to be asymptotically stable.

In this technical note, the problem of parameter and state estimation is addressed for IDSs that are described by a semilinear heat equation, based on sampled (in space) state measurements. The system includes a structured uncertain part, involving linearly a set of unknown parameters. Note that, in the absence of parameter uncertainty, the observer proposed in [24] applies to the present system. On the other hand, the class of systems considered here can be viewed as a generalization of the system considered in [21] as this involves a single unknown parameters. The corresponding observation problem is dealt with using an adaptive observer providing online estimates of the system (spatially distributed) state and unknown parameters. The observer design involves a backstepping state transformation, inspired from adaptive observers of nonlinear ODE systems [6], [26]. The observer state and parameter estimators are derived so that the transformed system coincides with an exponentially stable target system. The observer derivation entails an *ad-hoc* persistent excitation condition, under which all state and parameter estimation errors are guaranteed to be exponentially vanishing. A second sufficient condition is established showing that the observer gain depends on the space sampling period. The technical note is organized as follows: the observation problem statement, including the class of IDSs, is described in Section II; the adaptive observer design and analysis are dealt with in Section III; a conclusion and a reference list end the technical note.

II. Observation Problem Statement

The system under study is described by a parabolic type PDE of the form

$$u_t(x,t) = u_{xx}(x,t) + \theta^T \phi\left(y(t),t\right), \quad 0 < x < 1,\ t > 0 \tag{1a}$$

with the boundary condition

$$u_x(0,t) = 0, t \geq 0 \tag{1b}$$

and boundary input actuation:

$$u(1,t) = U(t), \quad t \geq 0,\ \text{(control input)} \tag{1c}$$

where $\phi : \mathbf{R}^p \times \mathbf{R} \to \mathbf{R}^n$ is a known C^1 function; $\theta \in \mathbf{R}^n$ is a fixed vector of unknown components but its dimension n is known. The quantity $\theta^T \phi(y(t),t)$ might represent a possible structured modelling error. The system is observed via an output vector $y(t) \in \mathbf{R}^p$ including all measurements acquired on the system at time t. Specifically, the spatial domain $0 \leq x \leq 1$ is divided in p known subintervals $[x_j, x_{j+1}]$, with $p \geq 1$ and $x_0 = 0 \leq x_j \leq x_{j+1} \leq x_p = 1$

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($j = 0 \cdots p-1$). A sensor is placed at the middle of each subinterval providing online state measurements at those positions

$$y_j(t) = u(\overline{x}_j, t); \quad \overline{x}_j = \frac{x_j + x_{j+1}}{2} \quad (j = 0, \ldots, p-1). \tag{2a}$$

Then, the system output vector $y(t)$ is defined as follows:

$$y(t) = [y_0(t), \ldots, y_{p-1}(t)]^T. \tag{2b}$$

Note that the case of single sensor $p = 1$, or equivalently $\Delta = 1$, is not ruled out, where Δ denotes the maximum space sampling interval, i.e.,

$$\Delta = \max_{j=0,\ldots,p-1} (x_{j+1} - x_j). \tag{2c}$$

The goal is to generate accurate online estimates $\hat{u}(x,t)$ and $\hat{\theta}$, respectively of the system state $u(x,t)$ ($0 \le x \le 1$; $t \ge 0$) and the parameter vector θ, based on the output measurements $y(t)$. To achieve this objective, the following assumption is considered:

Assumption 1: $u(x,t)$ is bounded and the function $\phi(.,t)$ is uniformly bounded with respect to the argument t. □

Remark 1:

a) Equation (1a) may capture several heat phenomena. For instance, the simpler case $\theta \in \mathbf{R}$ boils down to the heat system (1) in [28] (with the sensor being placed at the boundary $x = 0$). In the general case $\theta \in \mathbf{R}^n$, (1a) may be viewed as an approximation of a parabolic type equation of the form

$$u_t(x,t) = u_{xx}(x,t) + \theta^T \int_0^1 \psi\left(u(x,t), t\right) dx.$$

Then, the approximation of the integral term by the rectangle method leads to an equation similar to (1a).

b) The above boundedness assumption entails the existence of (not necessarily known) scalars $-\infty < u_m < u_M < \infty$ and $-\infty < \phi_m < \phi_M < \infty$ such that, $\forall x \in [0,1], \forall y \in [u_m, u_M]^p, \forall t \ge 0$

$$u_m \le u(x,t) \le u_M, \quad \phi_m \le \|\phi(y,t)\| \le \phi_M.$$

This assumption will prove to be crucial for the varying gain matrix $\lambda(x,t)$ of the observer (defined in the next section) to be bounded. □

III. Observer Design and Analysis

A. Observer Design

The system model (1a)–(1c) suggests the following observer structure:

$$\begin{aligned}\hat{u}_t(x,t) &= \hat{u}_{xx}(x,t) + \phi^T\left(y(t),t\right)\hat{\theta}(t) - K\left(\hat{u}(\overline{x}_j,t) - y_j(t)\right)\\ &\quad + v(x,t), \quad x_j \le x < x_{j+1}, \ (j = 0, \ldots, p-1) \quad (3a)\\ \hat{u}_x(0,t) &= 0 \quad (3b)\\ \hat{u}(1,t) &= U(t) \quad (3c)\end{aligned}$$

where $\hat{\theta}(t)$ is a parameter vector estimate, $v(x,t)$ is an additional correction term, and $K \ge 0$ is the observer gain. Suitable choices of these quantities will be made based on the subsequent analysis. First, introduce the following errors:

$$\begin{aligned}\widetilde{u}(x,t) &= \hat{u}(x,t) - u(x,t) \text{ (state estimation error)} \quad (4a)\\ \widetilde{\theta}(t) &= \hat{\theta}(t) - \theta \text{ (parameter estimation error)}. \quad (4b)\end{aligned}$$

Subtracting (1a) to (3a), it follows using (1a) and (4a), (4b) that $\widetilde{u}(x,t)$ undergoes the following equation:

$$\begin{aligned}\widetilde{u}_t(x,t) &= \widetilde{u}_{xx}(x,t) + \phi^T\left(y(t),t\right)\widetilde{\theta}(t) - K\widetilde{u}(\overline{x}_j,t) + v(x,t),\\ &\qquad x_j \le x < x_{j+1}, \ (j = 0, \ldots, p-1) \quad (5a)\end{aligned}$$

with the following boundary conditions:

$$\widetilde{u}_x(0,t) = \widetilde{u}(1,t) = 0 \text{ (using (1b–c) and (4b–c))}. \tag{5b}$$

Now, introduce the backstepping transformation, inspired by [6]

$$z(x,t) = \widetilde{u}(x,t) - \lambda(x,t)\widetilde{\theta}(t) \tag{6}$$

where $\lambda(x,t) \in \mathbf{R}^{1\times n}$ is an auxiliary vector function to be defined later. It is worth noting that, the above transformation has originally been introduced in [6] for finite dimensional systems described by nonlinear ODEs. It has proved in many places to be useful for the design of adaptive observers [7]–[10].

It follows from (6) that $z(x,t)$ undergoes the following equation:

$$\begin{aligned}z_t(x,t) &= \widetilde{u}_{xx}(x,t) + \phi^T\left(y(t),t\right)\widetilde{\theta}(t) - K\widetilde{u}(\overline{x}_j,t) + v(x,t)\\ &\quad - \lambda_t(x,t)\widetilde{\theta}(t) - \lambda(x,t)\dot{\widetilde{\theta}}(t)\end{aligned} \tag{7}$$

for all $t \ge 0$, $x_j \le x < x_{j+1}$ and $j = 0, \ldots, p-1$. Equation (7) suggests the following choice of $v(x,t)$:

$$v(x,t) = \lambda(x,t)\dot{\widetilde{\theta}}(t) \tag{8}$$

Doing so, (7) simplifies to

$$z_t(x,t) = \widetilde{u}_{xx}(x,t) + \phi^T\left(y(t),t\right)\widetilde{\theta}(t) - K\widetilde{u}(\overline{x}_j,t) - \lambda_t(x,t)\widetilde{\theta}(t) \tag{9}$$

for all $t \ge 0$, $x_j \le x < x_{j+1}$ and $j = 0, \ldots, p-1$. In view of (6), $z(x,t) + \lambda(x,t)\widetilde{\theta}(t)$ can be substituted to $\widetilde{u}(x,t)$ on the right side of (9). Doing so, one gets

$$\begin{aligned}z_t(x,t) &= z_{xx}(x,t) + \lambda_{xx}(x,t)\widetilde{\theta}(t) + \phi^T\left(y(t),t\right)\widetilde{\theta}(t)\\ &\quad - K\left(z(\overline{x}_j,t) + \lambda(\overline{x}_j,t)\widetilde{\theta}(t)\right) - \lambda_t(x,t)\widetilde{\theta}(t)\\ &= z_{xx}(x,t) - Kz(\overline{x}_j,t)\\ &\quad + \left(\lambda_{xx}(x,t) + \phi^T\left(y(t),t\right) - K\lambda(\overline{x}_j,t) - \lambda_t(x,t)\right)\widetilde{\theta}(t)\end{aligned} \tag{10}$$

for all $t \ge 0$, $x_j \le x < x_{j+1}$ and $j = 0, \ldots, p-1$. Equation (10) suggests the following trajectory for the auxiliary state vector $\lambda(x,t)$:

$$\begin{aligned}\lambda_t(x,t) &= \lambda_{xx}(x,t) - K\lambda(\overline{x}_j,t) + \phi^T\left(y(t),t\right);\\ &\qquad t \ge 0, \ x_j \le x < x_{j+1}, \ j = 0, \ldots, p-1\end{aligned} \tag{11a}$$

with the following boundary and initial conditions:

$$\lambda_x(0,t) = \lambda(1,t) = 0, \text{ and } \lambda(x,0) = 0. \tag{11b}$$

Doing so, (10) boils down to

$$z_t(x,t) = z_{xx}(x,t) - Kz(\overline{x}_j,t); t \ge 0, x_j \le x < x_{j+1}, (j = 0, \ldots, p-1). \tag{12a}$$

In view of (11b) and (5b), one gets from (6) the following boundary conditions:

$$z_x(0,t) = z(1,t) = 0 \tag{12b}$$

Owing to the unknown parameter vector, the following adaptive law is used:

$$\begin{aligned}\dot{\hat{\theta}}(t) &= -R(t)\Lambda^T(t)\widetilde{y}(t) \text{ and } \hat{\theta}(t) \in \mathbf{R}^n \quad (13a)\\ \dot{R}(t) &= R(t) - R(t)\Lambda^T(t)\Lambda(t)R(t) \text{ with } R(t) \in \mathbf{R}^{n\times n} \quad (13b)\\ \Lambda(t) &= \begin{bmatrix}\lambda(\overline{x}_0,t)\\ \vdots\\ \lambda(\overline{x}_{p-1},t)\end{bmatrix} \in \mathbf{R}^{p\times n} \quad (13c)\\ \widetilde{y}(t) &= \left[\hat{u}(\overline{x}_0,t) - y_0(t), \ldots, \hat{u}(\overline{x}_{p-1},t) - y_{p-1}(t)\right]^T \quad (13d)\end{aligned}$$

where the initial conditions $\hat{\theta}(0) = \theta_0$ and $R(0) = R_0$ are arbitrarily chosen but $R_0 = R_0^T > 0$. The structure of the parameter adaptive law (13a)–(13d) is similar to that used in (finite-dimensional) system observers, e.g., [5]–[7]. The main difference is that the adaptive gain $\Lambda(t)$ is presently generated by a PDE equation [namely, (11a), (11b)], while such a gain was generated by an ODE equation in the case of finite-dimension system observers. Except for this difference, the adaptive law (13a), (13b) is expected to perform as in the finite-dimensional case. In particular, the matrix $R(t)$ [generated by (13b)] will be required to stay all the time symmetric and positive definite. The observer thus designed is constituted of (3a)–(3c), (11a), (11b) and (13a)–(13c). For convenience, these equations are rewritten together as follows:

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \phi^T(y(t),t)\,\hat{\theta}(t) - K(\hat{u}(\overline{x}_j,t) - y_j(t)) + \lambda(x,t)\dot{\hat{\theta}}(t), \quad \text{for } x_j \le x < x_{j+1} \quad (j = 0,\ldots,p-1) \tag{14a}$$

$$\hat{u}_x(0,t) = 0, \quad \hat{u}(1,t) = U(t) \tag{14b}$$

$$\lambda_t(x,t) = \lambda_{xx}(x,t) - K\lambda(\overline{x}_j,t) + \phi^T(y(t),t); \quad t \ge 0, x_j \le x < x_{j+1}, j = 0,\ldots,p-1 \tag{14c}$$

$$\lambda_x(0,t) = \lambda(1,t) = 0, \quad \lambda(x,0) = 0 \text{ (by definition)} \tag{(4d)}$$

$$\dot{\hat{\theta}}(t) = -R(t)\Lambda^T(t)\widetilde{y}(t) \tag{14e}$$

$$\dot{R}(t) = R(t) - R(t)\Lambda^T(t)\Lambda(t)R(t). \tag{14f}$$

B. Observer Analysis

The observer analysis amounts to analyzing the stability of the following error system:

$$z_t(x,t) = z_{xx}(x,t) - Kz(\overline{x}_j,t); \quad \text{for all } t \ge 0 \text{ and } x_j \le x < x_{j+1} (j = 0,\ldots,p-1) \tag{15a}$$

$$\dot{\widetilde{\theta}}(t) = -R(t)\Lambda^T(t)\left(Z(t) + \Lambda(t)\widetilde{\theta}(t)\right) \text{ with } \widetilde{\theta} = \hat{\theta} - \theta \tag{15b}$$

$$Z(t) = [z(\overline{x}_0,t),\ldots,z(\overline{x}_{p-1},t)]^T \in \mathbf{R}^p \tag{15c}$$

$$z_x(0,t) = z(1,t) = 0 \tag{15d}$$

where (15a) and (15c) are, respectively, copies of (12a) and (14c), while (15b) is obtained from (14e) replacing there $\widetilde{y}(t)$ by $Z(t) + \Lambda(t)\widetilde{\theta}(t)$, due to (2a), (2b), (6), and (15d). The error equations (15a), (15b) are completed by the λ-equation (11a) and the associated boundary and initial conditions (11b).

It is worth noting that the solution $(z = 0, \lambda = 0, \widetilde{\theta} = 0)$ is an equilibrium of the system (15a), (15b). Indeed, it is readily checked that, if $(z(x,t_0) = 0, \lambda(x,t_0) = 0, \widetilde{\theta}(t_0) = 0)$, for all $0 < x < 1$ and some $t_0 \ge 0$, then one has $(z(x,t) = 0, \lambda(x,t) = 0, \widetilde{\theta}(t) = 0)$, for all $t \ge t_0$.

For space limitation, the well posedness of the problem at hand is concisely discussed in the following remark.

Remark 2:

a) It is readily checked that, the system of (1a), (15a), and (14c) are simpler forms of equation (21) in [24]. Following mutatis mutandis the well-posedness analysis developed there, one concludes that a strong solution $(u(x,t), z(x,t), \lambda(x,t))$ exists in the Hilbert space

$$H_{\frac{1}{2}} = D\left((-A)^{\frac{1}{2}}\right) = \left\{w \in H^1(0,1) : w_x(0) = w(1) = 0\right\}$$

where $A = \partial^2/\partial x^2$, $(-A)^{1/2}$ the square root of $(-A)$, and $H^1(0,1)$ is the Sobolev space of absolutely continuous scalar functions $w : [0,1] \to \mathbf{R}$ or $\mathbf{R}^n$ with $dw/dx \in L_2[0,1]$. Then, the existence and uniqueness of $R(t)$ and $\widetilde{\theta}(t)$ is immediately obtained from (14f) and (15b), applying the usual existence theorem of ODEs. Then, the existence and uniqueness of $\widetilde{u}(x,t) = z(x,t) + \lambda(x,t)\widetilde{\theta}(t)$ follows from (6). This also implies the existence of $\hat{u}(x,t) = \widetilde{u}(x,t) + u(x,t)$.

b) Also, the following Wirtinger's inequalities will be repeatedly used in the forthcoming analysis [25]

$$\int_a^b w^2(x,t)dx \le \frac{4(b-a)^2}{\pi^2}\int_a^b w_x^2(x,t)dx \tag{16a}$$

$$\max_{a \le x \le b} w^2(x,t) \le \int_a^b w_x^2(x,t)dx \tag{16b}$$

whatever the function $w \in H^1(a,b)$ such that $w(a) = 0$ or $w(b) = 0$. □

The next result is on the boundedness of the auxiliary vector $\lambda(x,t)$.

Proposition 1: The auxiliary state vector $\lambda(x,t)$, generated by (14c)–(14d), is uniformly bounded, provided that the gain observer K in (14a) and the space sampling interval Δ satisfy the condition: $0 \le K\Delta^2 < 4\pi^2$. □

Proof: See Appendix A.

Using Proposition 1, it is readily follows from (13c) that:

$$\|\Lambda(t)\| \le \Lambda_m \text{ with } \Lambda_m = \lambda_m\sqrt{p} \text{ and } \lambda_m = \sup_{0 \le x \le 1, t \ge 0} \|\lambda(x,t)\|. \tag{17}$$

To ensure the exponential stability of the system (15a)–(15d), it is required that the time-varying matrix $R(t)$ [solution of (14f)] exists and is symmetric positive definite. Now, it is shown in many places (e.g., [6], [7]) that $R(t)$ enjoys the required properties if the following persistent excitation (PE) condition holds:

$$\exists \delta, \varepsilon_0, \varepsilon_1 > 0,\ \forall t > 0 : \varepsilon_0 I_{(m+p)\times(m+p)} < \int_t^{t+\delta} \Lambda^T(s)\Lambda(s)ds < \varepsilon_1 I_{(m+p)\times(m+p)}. \tag{18}$$

Similar PE conditions are needed in system identification and adaptive observation. Note that, the right inequality simply means that $\Lambda(s)$ is bounded which actually is the case due to (17). The left inequality means that the column vectors of $\Lambda(s)$ span the vector space $\mathbf{R}^{n+p}$ on any finite time interval $[t, t+\delta]$, for all t. Under condition (18), it turns out that the inverse R^{-1} is also bounded and symmetric positive definite, i.e., there are positive scalars $(r, \overline{r})$, such that

$$rI_{(m+p)\times(m+p)} \le R^{-1}(t) = \left(R^{-1}(t)\right)^T \le \overline{r}I_{(m+p)\times(m+p)}, \forall t \ge 0. \tag{19}$$

In the sequel, condition (18) is supposed to be true, so that one can make use of (19). Then, one has the following main result:

Theorem 1: Let the adaptive observer described by (4a)–(4c), (9), (12a), (12b), and (14a), (14b) be applied to the system (1a), (1b), (2a), (2b) subject to Assumption 1 and the PE assumption (18). Let the observer gain K and the space sampling interval Δ be selected such that $K\Delta < 2\pi^2$. Then, the estimation errors $\widetilde{\theta}(t)$ and the quantity $\max_{0 \le x \le 1} |\widetilde{u}(x,t)|$ are globally exponentially vanishing. □

Proof: Consider the following Lyapunov function candidate:

$$V(z, \widetilde{\theta}) = V_0(\widetilde{\theta}) + \frac{\mu_0}{2}\int_0^1 z^2(x,t)dx + \frac{1}{2}\int_0^1 z_x^2(x,t)dx \tag{20a}$$

with $V_0(\widetilde{\theta}) = \widetilde{\theta}^T R^{-1} \widetilde{\theta}$ and μ_0 any positive scalar such that

$$\mu_0 > p \left(1 - \frac{K\Delta}{2\pi^2}\right)^{-1}. \tag{20b}$$

Note that μ_0 exists because $K\Delta < 2\pi^2$ by assumption (Theorem 1). From (20a), one gets the following time-derivative:

$$\dot{V}(z,\widetilde{\theta}) = \dot{V}_0(\widetilde{\theta}) + \mu_0 \int_0^1 z(x,t) z_t(x,t)dx + \int_0^1 z_x z_{xt}(x,t)dx. \tag{21}$$

Using (15b) and (14f), one immediately gets

$$\begin{aligned}
\dot{V}_0(\widetilde{\theta}) &= \widetilde{\theta}^T(t)\dot{R}^{-1}(t)\widetilde{\theta}(t) + 2\widetilde{\theta}^T(t)R^{-1}(t)\dot{\widetilde{\theta}}(t) \\
&= \widetilde{\theta}^T(t)\left(-R^{-1} + \Lambda^T(t)\Lambda(t)\right)\widetilde{\theta}(t) \\
&\quad - 2\widetilde{\theta}^T(t)\left(\Lambda^T(t)\Lambda(t)\widetilde{\theta}(t) + \Lambda^T(t)Z(t)\right) \\
&\leq -\widetilde{\theta}^T(t)R^{-1}(t)\widetilde{\theta}(t) + \|Z(t)\|^2 \text{ (using Young's inequality)} \\
&\leq -\widetilde{\theta}^T R^{-1}\widetilde{\theta} + p \max_{0<x<1} |z(x,t)|^2 \text{ (using (15c))} \\
&\leq -V_0(\widetilde{\theta}) + p\int_0^1 z_x^2(x,t)dx
\end{aligned} \tag{22}$$

using Wirtinger's inequality (16b). On the other hand, using (15a), the third term on the right side of (21) develops as follows:

$$\begin{aligned}
\int_0^1 z(x,t)z_t(x,t)dx &= \sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}} z(x,t)\left(z_{xx}(x,t) - Kz(\overline{x}_j,t)\right)dx \\
&= \sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}} z(x,t)z_{xx}(x,t)dx - \sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}} Kz(\overline{x}_j,t)z(x,t)dx \\
&= \int_0^1 z(x,t)z_{xx}(x,t)dx - \sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}} K\left(z(\overline{x}_j,t) - z(x,t)\right)z(x,t)dx \\
&\quad - \sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}} Kz^2(x,t)dx \\
&= -\int_0^1 z_x^2(x,t)dx - K\left(\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}\left(z(\overline{x}_j,t) - z(x,t)\right)z(x,t)dx - \int_0^1 z^2(x,t)dx\right)
\end{aligned} \tag{23}$$

where the last equality is obtained using an integration by parts. By Young's inequality, one has

$$\begin{aligned}
-\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}} K\left(z(\overline{x}_j,t) - z(x,t)\right)z(x,t)dx &\leq \frac{K\xi}{2}\int_0^1 z^2(x,t)dx \\
&\quad + \frac{1}{2\xi}\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}} K\left(z(\overline{x}_j,t) - z(x,t)\right)^2 dx
\end{aligned} \tag{24}$$

whatever $\xi > 0$. Also, the application of (16a) gives

$$\begin{aligned}
\frac{K}{2\xi}\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}\left(z(\overline{x}_j,t) - z(x,t)\right)^2 dx &= \frac{K}{2\xi}\sum_{j=0}^{p-1}\int_{x_j}^{\overline{x}_j}\left(z(\overline{x}_j,t) - z(x,t)\right)^2 dx \\
&\quad + \frac{K}{2\xi}\sum_{j=0}^{p-1}\int_{\overline{x}_j}^{x_{j+1}}\left(z(\overline{x}_j,t) - z(x,t)\right)^2 dx \leq \frac{K\Delta^2}{2\xi\pi^2}\int_0^1 z_x^2(x,t)dx.
\end{aligned}$$

This together with (23) and (24) yields

$$\begin{aligned}
&\int_0^1 z(x,t)z_t(x,t)dx \\
&\leq -\int_0^1 z_x^2(x,t)dx + \frac{K\xi}{2}\int_0^1 z^2(x,t)dx \\
&\quad + \frac{K\Delta^2}{2\xi\pi^2}\int_0^1 z_x^2(x,t)dx - K\int_0^1 z^2(x,t)dx \\
&\leq -\left(1 - \frac{K\Delta^2}{2\xi\pi^2}\right)\int_0^1 z_x^2(x,t)dx - K\left(1 - \frac{\xi}{2}\right)\int_0^1 z^2(x,t)dx.
\end{aligned} \tag{25}$$

Using (22) and (25), one gets

$$\begin{aligned}
&\widetilde{\theta}^T(t)\dot{R}^{-1}(t)\widetilde{\theta}(t) + 2\widetilde{\theta}^T(t)R^{-1}(t)\dot{\widetilde{\theta}}(t) + \mu_0\int_0^1 z(x,t)z_t(x,t)dx \\
&\leq -\widetilde{\theta}^T R^{-1}\widetilde{\theta} + p\int_0^1 z_x^2(x,t)dx \\
&\quad - \mu_0\left(\left(1 - \frac{K\Delta^2}{2\xi\pi^2}\right)\int_0^1 z_x^2(x,t)dx + K\left(1 - \frac{\xi}{2}\right)\int_0^1 z^2(x,t)dx\right) \\
&\leq -\widetilde{\theta}^T R^{-1}\widetilde{\theta} - \mu_0\left(1 - \frac{K\Delta^2}{2\xi\pi^2} - \frac{p}{\mu_0}\right)\int_0^1 z_x^2(x,t)dx \\
&\quad - K\mu_0\left(1 - \frac{\xi}{2}\right)\int_0^1 z^2(x,t)dx.
\end{aligned} \tag{26}$$

Focusing on the last term on the right side of (21), one has

$$\int_0^1 z_x(x,t)z_{xt}(x,t)dx = \int_0^1 z_x(x,t)z_{tx}(x,t)dx \tag{27}$$

using a similar argument as Remark A1 in Appendix A of [27]. Then, one has, using successively an integration by part, (12a) and the boundary conditions $z_x(0,t) = z(1,t) = 0$

$$\begin{aligned}
\int_0^1 z_x(x,t)z_{xt}(x,t)dx &= -\int_0^1 z_{xx}(x,t)z_t(x,t)dx \\
&= -\int_0^1 z_{xx}^2(x,t)dx + \sum_{j=1}^{p-1}\int_{x_j}^{x_{j+1}} Kz_{xx}(x,t)z(\overline{x}_j,t)dx \\
&= -\int_0^1 z_{xx}^2(x,t)dx + \sum_{j=1}^{p-1}\int_{x_j}^{x_{j+1}} Kz_{xx}(x,t)z(x,t)dx \\
&\quad + \sum_{j=1}^{p-1}\int_{x_j}^{x_{j+1}} Kz_{xx}(x,t)\left(z(\overline{x}_j,t) - z(x,t)\right)dx \\
&\leq -\int_0^1 z_{xx}^2(x,t)dx + \int_0^1 Kz_{xx}(x,t)z(x,t)dx \\
&\quad + \sum_{j=1}^{p-1}\int_{x_j}^{x_{j+1}}\left(K\frac{\eta}{2}z_{xx}^2(x,t) + K\frac{1}{2\eta}\left(z(\overline{x}_j,t) - z(x,t)\right)^2\right)dx
\end{aligned} \tag{28}$$

where Young's inequality has been used in the last inequality and $\eta > 0$ is arbitrary. Integrating by part the second integral on the right side of (28), this yields

$$\begin{aligned}
&\int_0^1 z_x(x,t)z_{xt}(x,t)dx \\
&\leq -\int_0^1 z_{xx}^2(x,t)dx + K\left(z_x(1,t)z(1,t) - z_x(0,t)z(0,t)\right) \\
&\quad - K\int_0^1 z_x^2(x,t)dx + \frac{K\eta}{2}\int_0^1 z_{xx}^2(x,t)dx \\
&\quad + \frac{K}{2\eta}\sum_{j=1}^{p-1}\int_{x_j}^{x_{j+1}}\left(z(\overline{x}_j,t)-z(x,t)\right)^2 dx \\
&\leq \left(\frac{K\eta}{2}-1\right)\int_0^1 z_{xx}^2(x,t)dx - K\int_0^1 z_x^2(x,t)dx \\
&\quad + \frac{K}{2\eta}\sum_{j=1}^{p-1}\int_{x_j}^{\overline{x}_j}\left(z(\overline{x}_j,t)-z(x,t)\right)^2 dx \\
&\quad + \frac{K}{2\eta}\sum_{j=1}^{p-1}\int_{\overline{x}_j}^{x_{j+1}}\left(z(\overline{x}_j,t)-z(x,t)\right)^2 dx \qquad (29)
\end{aligned}$$

where the conditions $z_x(0,t) = z(1,t) = 0$ have again been used. Applying (16a) to the last two terms on the right side of (29), one gets

$$\begin{aligned}
&\int_0^1 z_x(x,t)z_{xt}(x,t)dx \\
&\leq -\left(1-\frac{K\eta}{2}\right)\int_0^1 z_{xx}^2(x,t)dx - K\int_0^1 z_x^2(x,t)dx \\
&\quad + \frac{\Delta^2}{2\eta\pi^2}\int_0^1 z_x^2(x,t)dx \\
&\leq -\left(1-\frac{K\eta}{2}\right)\int_0^1 z_{xx}^2(x,t)dx \\
&\quad - K\left(1-\frac{\Delta^2}{2\eta\pi^2}\right)\int_0^1 z_x^2(x,t)dx). \qquad (30)
\end{aligned}$$

Let the free parameters η in (30) and ξ in (26) be such that:

$$\xi = \Delta \quad \text{and} \quad \eta = \frac{\Delta}{\pi^2}. \qquad (31)$$

This ensures that

$$1-\frac{\Delta^2}{2\eta\pi^2} = 1-\frac{\Delta}{2} > 0 \quad \text{and} \quad 1-\frac{\xi}{2} = 1-\frac{\Delta}{2} > 0 \qquad (32)$$

because $0 < \Delta \leq 1$. Then, the first term on the right side of (30) entails the following condition on K:

$$\frac{K\eta}{2} < 1 \text{ or, equivalently, } K\Delta < 2\pi^2 \qquad (33)$$

which is nothing else than the condition in Theorem 1. Combining (21), (26), and (30) gives, using (31)–(33)

$$\begin{aligned}
\dot{V} \leq & -\widetilde{\theta}^T R^{-1}\widetilde{\theta} - K\mu_0\left(1-\frac{\Delta}{2}\right)\int_0^1 z^2(x,t)dx \\
& -\mu_0\left(1-\frac{K\Delta}{2\pi^2}-\frac{p}{\mu_0}\right)\int_0^1 z_x^2(x,t)dx \\
& -K\left(1-\frac{\Delta}{2}\right)\int_0^1 z_x^2(x,t)dx. \qquad (34)
\end{aligned}$$

Note that by (20b), one has

$$1-\frac{K\Delta}{2\pi^2}-\frac{p}{\mu_0} > 0. \qquad (35)$$

Then, (34) yields

$$\begin{aligned}
\dot{V} \leq & -V_0(\widetilde{\theta}) - \mu_0\left(K\left(1-\frac{\Delta}{2}\right)+\frac{\pi^2}{8}\left(1-\frac{K\Delta}{2\pi^2}-\frac{p}{\mu_0}\right)\right) \\
& \times \int_0^1 z^2(x,t)dx \\
& -\left(K\left(1-\frac{\Delta}{2}\right)+\frac{\mu_0}{2}\left(1-\frac{K\Delta}{2\pi^2}-\frac{p}{\mu_0}\right)\right) \\
& \times \int_0^1 z_x^2(x,t)dx \\
\leq & -V_0(\widetilde{\theta}) - \sigma_0\frac{\mu_0}{2}\int_0^1 z^2(x,t)dx - \frac{\sigma_1}{2}\int_0^1 z_x^2(x,t)dx \qquad (36) \\
\leq & -\sigma V(z,\widetilde{\theta}) \qquad (37)
\end{aligned}$$

with $\sigma = \min(1, \sigma_0, \sigma_1)$, where

$$\sigma_0 = 2K\left(1-\frac{\Delta}{2}\right)+\frac{\pi^2}{4}\left(1-\frac{K\Delta}{2\pi^2}-\frac{p}{\mu_0}\right) > 0 \qquad (38)$$

$$\sigma_1 = K\left(1-\frac{\Delta}{2}\right)+\frac{\mu_0}{2}\left(1-\frac{K\Delta}{2\pi^2}-\frac{p}{\mu_0}\right) > 0. \qquad (39)$$

Clearly, inequality (37) implies that V is exponentially vanishing (as $t \to \infty$) and from (20a) so are $\widetilde{\theta}$, $\int_0^1 z^2(x,t)dx$ and $\int_0^1 z_x^2(x,t)dx$. Then, using Wirtinger's inequality (16b), one has

$$\max_{0\leq x\leq 1} z^2(x,t) \leq \int_0^1 z_x^2(x,t)dx. \qquad (40)$$

This holds because $z(1,t) = 0$ (due to (15d)). Then, in turn $\max_{0\leq x\leq 1}|z(x,t)|$ is also exponentially vanishing. On the other hand, one has from (6) that $\widetilde{u}(x,t) = z(x,t) + \lambda(x,t)\widetilde{\theta}(t)$. As $\lambda(x,t)$ is bounded (by Proposition 1), it follows that $\max_{0\leq x\leq 1}|\widetilde{u}(x,t)|$ is also exponentially vanishing. This completes the proof of Theorem 1. ■

Remark 3:

a) As $0 < \Delta \leq 1$, the condition $K\Delta/2\pi^2 < 1$, introduced in Theorem 1, is more restrictive than the condition $K\Delta^2/4\pi^2 < 1$ required in Proposition 1. Therefore, only the former is retained. Under that condition, the gain $K \geq 0$ can be arbitrarily chosen, but the spatial sampling interval Δ must be selected accordingly. The larger K, the smaller Δ.
b) Using (35), it follows from (38), (39) that σ_0 and σ_1 are increasing functions of the gain K. Then, it follows from (36) that the convergence rate of $\int_0^1 z^2(x,t)dx$ and $\int_0^1 z_x^2(x,t)dx$ can be made arbitrarily high by letting K be sufficiently large.

Then, it follows from (22) that the convergence rate of $V_0(\widetilde{\theta})$ can be made arbitrarily close to that of $V_0(\widetilde{\theta}(0))e^{-t}$ by letting K be sufficiently large. Consequently, it follows from (6) and (40) that, the convergence rate of $\max_{0\leq x\leq 1}|\widetilde{u}(x,t)|$ is also made higher with larger values of K. On the other hand, it has been pointed out in Part a that, large values of K entail small values of the sampling interval Δ or, equivalently, large number p of required sensors. □

IV. Conclusion

We have addressed the problem of estimating the state and parameters of the class of IDSs described by the model (1a)–(1c), (2a), (2b). The latter is basically a parabolic PDE augmented by the structured quantity $\theta^T\phi(y(t),t)$. The adaptive observer (14a)–(14f) is designed and shown to enjoy exponential convergence, under the persistent excitation property (18) and the condition (20b). The latter shows that the observer gain K must be selected taking into account the space sampling interval Δ. The smaller Δ the larger may be the gain K.

Appendix A
Proof of Proposition 1

Consider the Lyapunov functional candidate:

$$W(t)=\frac{1}{2}\int_0^1\lambda(x,t)\lambda^T(x,t)dx+\frac{1}{2}\int_0^1\lambda_x(x,t)\lambda_x^T(x,t)dx \quad \text{(A1)}$$

Using (14c), it follows from (A1):

$$\dot{W}(t)=\int_0^1\lambda_t(x,t)\lambda^T(x,t)dx+\int_0^1\lambda_x(x,t)\lambda_{xt}^T(x,t)dx. \quad \text{(A2)}$$

Following the same approach as in the proof of Theorem 1, the two terms on the right side of (A2) will be successively upper bounded. The first term develops as follows:

$$\begin{aligned}
&\int_0^1\lambda_t(x,t)\lambda^T(x,t)dx\\
&=\int_0^1\lambda_{xx}(x,t)\lambda^T(x,t)dx-\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}K\lambda(\overline{x}_j,t)\lambda^T(x,t)dx\\
&\quad+\int_0^1\phi^T(y(t),t)\lambda^T(x,t)dx\\
&=-\int_0^1\|\lambda_x(x,t)\|^2dx-\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}K\lambda(\overline{x}_j,t)\lambda^T(x,t)dx\\
&\quad+\int_0^1\lambda(x,t)\phi(y(t),t)\,dx\\
&=-\int_0^1\|\lambda_x(x,t)\|^2dx\\
&\quad-\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}K\left(\lambda(\overline{x}_j,t)-\lambda(x,t)\right)\lambda^T(x,t)dx\\
&\quad+\int_0^1\lambda(x,t)\phi(y(t),t)\,dx-\int_0^1K\|\lambda(x,t)\|^2dx
\end{aligned} \quad \text{(A3)}$$

where the penultimate last equality is obtained using an integration by parts of the first term on the right side and the boundary conditions in (14d). Using Young inequality, one has

$$\begin{aligned}
&\int_0^1\lambda(x,t)\phi(y(t),t)\,dx\\
&\leq\frac{\|\phi(y(t),t)\|^2}{2\zeta}+\frac{\zeta}{2}\int_0^1\|\lambda(x,t)\|^2dx &\text{(A4)}\\
&-\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}K\left(\lambda(\overline{x}_j,t)-\lambda(x,t)\right)\lambda^T(x,t)dx\\
&\leq\frac{1}{2\omega}\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}K\|\lambda(\overline{x}_j,t)-\lambda(x,t)\|^2dx\\
&\quad+\frac{K\omega}{2}\int_0^1\|\lambda(x,t)\|^2dx\& &\text{(A5)}
\end{aligned}$$

whatever $\zeta>0$ and $\omega>0$. Using Wirtinger's inequality (16a), the first term on the right side of (A5) is bounded as follows:

$$\begin{aligned}
&\frac{1}{2\omega}\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}K\|\lambda(\overline{x}_j,t)-\lambda(x,t)\|^2dx\\
&\leq\frac{1}{2\omega}\sum_{j=0}^{p-1}\int_{x_j}^{\overline{x}_j}K\|\lambda(\overline{x}_j,t)-\lambda(x,t)\|^2dx\\
&\quad+\frac{1}{2\omega}\sum_{j=0}^{p-1}\int_{\overline{x}_j}^{x_{j+1}}K\|\lambda(\overline{x}_j,t)-\lambda(x,t)\|^2dx\\
&\leq\frac{K}{2\omega}\frac{\Delta^2}{\pi^2}\sum_{j=0}^{p-1}\int_{x_j}^{x_{j+1}}\|\lambda_x(x,t)\|^2dx &\text{(A6)}
\end{aligned}$$

Using (A4)–(A6), it follows from (A3) that:

$$\begin{aligned}
\int_0^1\lambda_t(x,t)\lambda^T(x,t)dx&\leq-\left(1-\frac{K}{2\omega}\frac{\Delta^2}{\pi^2}\right)\int_0^1\|\lambda_x(x,t)\|^2dx\\
&-\left(K-\frac{K\omega}{2}-\frac{\zeta}{2}\right)\int_0^1\|\lambda(x,t)\|^2dx+\frac{\|\phi(y(t),t)\|^2}{2\zeta}. \quad \text{(A7)}
\end{aligned}$$

Letting $\omega=K\Delta^2/2\vartheta\pi^2$ for some $0<\vartheta<1$, inequality (A7) yields

$$\begin{aligned}
\int_0^1\lambda_t(x,t)\lambda^T(x,t)dx&\leq-(1-\vartheta)\int_0^1\|\lambda_x(x,t)\|^2dx\\
&-\left(K-\frac{K^2\Delta^2}{4\vartheta\pi^2}-\frac{\zeta}{2}\right)\int_0^1\|\lambda(x,t)\|^2dx+\frac{\|\phi(y(t),t)\|^2}{2\zeta}. \quad \text{(A8)}
\end{aligned}$$

Now, let us focus on the second term on the right side of (A2). First, notice that equality (27) still holds replacing there $z(x,t)$ by $\lambda(x,t)$. Then, one immediately has

$$\begin{aligned}
\int_0^1\lambda_x(x,t)\lambda_{xt}^T(x,t)dx&=\int_0^1\lambda_x(x,t)\lambda_{tx}^T(x,t)dx\\
&=-\int_0^1\lambda_{xx}(x,t)\lambda_t^T(x,t)dx \quad \text{(A9)}
\end{aligned}$$

using an integration by part and the boundary conditions (14d). The above equality develops further as follows, using (14c):

$$\begin{aligned}
&\int_0^1 \lambda_x(x,t)\lambda_{xt}^T(x,t)dx \\
&= -\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \lambda_{xx}(x,t)\lambda_t^T(x,t)dx \\
&= -\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \|\lambda_{xx}(x,t)\|^2 dx \\
&\quad + K\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \lambda_{xx}(x,t)\lambda^T(\overline{x}_j,t)dx \\
&\quad - \int_0^1 \lambda_{xx}(x,t)\phi\left(y(t),t\right)) \\
&= -\int_0^1 \lambda_{xx}^2(x,t)dx + K\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \lambda_{xx}(x,t)\lambda^T(x,t)dx \\
&\quad + K\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \lambda_{xx}(x,t)\left(\lambda^T(\overline{x}_j,t)-\lambda^T(x,t)\right)dx \\
&\quad - \int_0^1 \lambda_{xx}(x,t)\phi^T\left(y(t),t\right)) dx \\
&= -\int_0^1 \lambda_{xx}^2(x,t)dx + K\int_0^1 \lambda_{xx}(x,t)\lambda^T(x,t)dx \\
&\quad - \int_0^1 \lambda_{xx}(x,t)\phi^T\left(y(t),t\right)) dx \\
&\quad + K\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \lambda_{xx}(x,t)\left(\lambda^T(\overline{x}_j,t)-\lambda^T(x,t)\right)dx. \qquad \text{(A10)}
\end{aligned}$$

Applying Young's inequality to the third and fourth quantities on the right side of (A10) and integrating by part the second integral, equality (A10) yields, using the boundary conditions (14d)

$$\begin{aligned}
&\int_0^1 \lambda_x(x,t)\lambda_{xt}^T(x,t)dx \\
&\le -\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx - K\int_0^1 \|\lambda_x(x,t)\|^2 dx \\
&\quad + K\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \frac{\upsilon}{2}\|\lambda_{xx}(x,t)\|^2 \\
&\quad + \frac{1}{2\upsilon}\|\lambda(\overline{x}_j,t)-\lambda(x,t)\|^2 dx \\
&\quad + \frac{\varpi}{2}\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx + \frac{1}{2\varpi}\|\phi\left(y(t),t)\right)\|^2 \\
&\le -\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx - K\int_0^1 \|\lambda_x(x,t)\|^2 dx \\
&\quad + \frac{K\upsilon}{2}\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx \\
&\quad + \frac{K}{2\upsilon}\sum_{p=0}^{p-1}\int_{x_j}^{\overline{x}_j} \|\lambda(\overline{x}_j,t)-\lambda(x,t)\|^2 dx \\
&\quad + \frac{K}{2\upsilon}\sum_{p=0}^{p-1}\int_{\overline{x}_j}^{x_{j+1}} \|\lambda(\overline{x}_j,t)-\lambda(x,t)\|^2 dx \\
&\quad + \frac{\varpi}{2}\int_0^1 \lambda_{xx}^2(x,t)dx + \frac{1}{2\varpi}\|\phi\left(y(t),t)\right)\|^2 \qquad \text{(A11)}
\end{aligned}$$

whatever the scalars $\upsilon > 0$, $\varpi > 0$. Applying the Wirtinger's inequality (16a) to the two terms in the penultimate line in (A11), one gets

$$\begin{aligned}
&\int_0^1 \lambda_x(x,t)\lambda_{xt}^T(x,t)dx \\
&\le -\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx - K\int_0^1 \|\lambda_x(x,t)\|^2 dx \\
&\quad + \frac{K\upsilon}{2}\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx + \frac{K\Delta^2}{2\upsilon\pi^2}\sum_{p=0}^{p-1}\int_{x_j}^{x_{j+1}} \|\lambda_x(x,t)\|^2 dx \\
&\quad + \frac{\varpi}{2}\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx + \frac{1}{2\varpi}\|\phi\left(y(t),t)\right)\|^2 \\
&\le -\left(1-\frac{K\upsilon}{2}-\frac{\varpi}{2}\right)\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx \\
&\quad - K\left(1-\frac{\Delta^2}{2\upsilon\pi^2}\right)\int_0^1 \|\lambda_x(x,t)\|^2 dx + \frac{1}{2\varpi}\|\phi\left(y(t),t)\right)\|^2 . \qquad \text{(A12)}
\end{aligned}$$

Combining (A12) and (A8), one obtains

$$\begin{aligned}
\dot{W}(t) \le &-\left(K - \frac{K^2\Delta^2}{4\vartheta\pi^2} - \frac{\zeta}{2}\right)\int_0^1 \|\lambda(x,t)\|^2 dx \\
&- (1-\vartheta)\int_0^1 \|\lambda_x(x,t)\|^2 dx \\
&- K\left(1-\frac{\Delta^2}{2\upsilon\pi^2}\right)\int_0^1 \|\lambda_x(x,t)\|^2 dx \\
&- \left(1-\frac{K\upsilon}{2}-\frac{\varpi}{2}\right)\int_0^1 \|\lambda_{xx}(x,t)\|^2 dx \\
&+ \left(\frac{1}{2\zeta}+\frac{1}{2\varpi}\right)\|\phi\left(y(t),t\right)\|^2 . \qquad \text{(A13)}
\end{aligned}$$

The above inequality shows that, one first proceeds with the selection of the design parameters K, Δ according to the following condition:

$$\frac{K\Delta^2}{4\pi^2} < 1 \qquad \text{(A14)}$$

This is nothing other than the assumption (made on K and Δ) in Proposition 1. Then, let the free parameters $(\upsilon, \vartheta, \varpi)$ be such that

$$\frac{K\Delta^2}{4\pi^2} < \vartheta < 1,\ 0 < \frac{\zeta}{2} \le K - \frac{K^2\Delta^2}{4\vartheta\pi^2}$$
$$\frac{\Delta^2}{2\pi^2} < \upsilon < \frac{2}{K},\ \frac{\varpi}{2} < 1 - \frac{K\upsilon}{2}. \tag{A15}$$

It is readily checked that, this choice ensures that

$$K - \frac{K^2\Delta^2}{4\vartheta\pi^2} - \frac{\zeta}{2} \ge 0,\ 1 - \vartheta > 0$$
$$1 - \frac{\Delta^2}{2\upsilon\pi^2} > 0,\ 1 - \frac{K\upsilon}{2} - \frac{\varpi}{2} > 0.$$

Then, applying Wirtinger's inequality (16a) to the second and fourth terms on the right side of inequality (A13), this develops as follows:

$$\begin{aligned}\dot{W}(t) \le &- \left(\left(K - \frac{K^2\Delta^2}{4\vartheta\pi^2} - \frac{\zeta}{2}\right) + \frac{\pi^2(1-\vartheta)}{4}\right) \\ &\times \int_0^1 \|\lambda(x,t)\|^2\, dx \\ &- \left(K\left(1 - \frac{\Delta^2}{2\upsilon\pi^2}\right) + \frac{\pi^2}{4}\left(1 - \frac{K\upsilon}{2} - \frac{\varpi}{2}\right)\right) \\ &\times \int_0^1 \|\lambda_x(x,t)\|^2\, dx \\ &+ \left(\frac{1}{2\zeta} + \frac{1}{2\varpi}\right)\|\phi\left(y(t),t\right)\|^2 \\ \le &-\gamma W(t) + \left(\frac{1}{2\zeta} + \frac{1}{2\varpi}\right)\|\phi\left(y(t),t\right)\|^2 \end{aligned} \tag{A16}$$

with

$$\begin{aligned}\gamma = 2\min\Bigg(&\left(K - \frac{K^2\Delta^2}{4\vartheta\pi^2} - \frac{\zeta}{2}\right) + \frac{\pi^2(1-\vartheta)}{4}, \\ &K\left(1 - \frac{\Delta^2}{2\upsilon\pi^2}\right) + \frac{\pi^2}{4}\left(1 - \frac{K\upsilon}{2} - \frac{\varpi}{2}\right)\Bigg).\end{aligned}$$

As $\|\phi(y(t),t)\|$ is bounded (by Assumption 1), it follows from (A16) that so is $W(t)$. Then, it follows from (A1) that $\int_0^1 \|\lambda_x(x,t)\|^2 dx$ is bounded. Applying the inequality (16b), one gets that $\max_{0\le x\le 1} \|\lambda(x,t)\|^2$ is bounded. Proposition 1 is proved. ■

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