Robust Stabilization of Nonlinear Globally Lipschitz Delay Systems

Tarek Ahmed-Ali, I. Karafyllis, M. Krstic, Françoise Lamnabhi-Lagarrigue

Robust Stabilization of Nonlinear Globally Lipschitz Delay Systems

Tarek Ahmed-Ali, Iasson Karafyllis, Miroslav Krstic and Francoise Lamnabhi-Lagarrigue

Abstract This paper studies the application of a recently proposed control scheme to globally Lipschitz nonlinear systems for which the input is delayed and applied with zero order hold, the measurements are sampled and delayed, and only an output is measured (i.e., the state vector is not available). The control scheme consists of an observer for the delayed state vector, an inter-sample predictor for the output signal, an approximate predictor for the future value of the state vector, and the nominal feedback law applied with zero order hold and computed for the predicted value of the future state vector. The resulting closed-loop system is robust with respect to modeling and measurement errors and robust to perturbations of the sampling schedule.

1 Introduction

Predictor feedback is used frequently in the literature for systems with large input delays. The literature on predictor feedback under non-constant input delays is reviewed in [1, 2] (where time-varying delays and state-dependent input delays

T. Ahmed-Ali (✉)
Laboratoire GREYC CNRS-ENSICAEN, 06 Boulevard du Marechal Juin,
14050 Caen Cedex, France
e-mail: tarek.ahmed-ali@ensicaen.fr

I. Karafyllis
Department of Mathematics, National Technical University of Athens,
Heroon Polytechneiou 9, 15780 Athens, Greece
e-mail: iasonkar@central.ntua.gr

M. Krstic
Department of Mechanical and Aerospace Engineering, University of California,
San Diego, La Jolla, CA 92093-0411, USA
e-mail: krstic@ucsd.edu

F. Lamnabhi-Lagarrigue
Centre National de la Recherche Scientifique, CNRS-EECI SUPELEC,
3 Rue Joliot Curie, 91192 Gif-Sur-Yvette, France
e-mail: lamnabhi@lss.supelec.fr

are studied). For nonlinear systems with constant input delays, various forms of predictors have been used:

- Exact predictors based on the knowledge of the solution mapping [3–5].
- Approximate predictors based on successive approximations [6, 7].
- Approximate predictors based on numerical schemes [8, 9].
- Approximate predictors for which the prediction is provided by the output of a properly constructed control system [10–12].

Moreover, the literature on predictor feedback for nonlinear systems with constant input delays has considered cases where the input is applied continuously (as in [1–3, 6, 8, 12], or [9]) or is applied with zero order hold or ZOH (as in [3] and [7]). There is also a wide literature of predictor feedback design and implementation for linear systems with constant input delays; see the references in [4] and [5].

This chapter considers the application of a recently proposed control scheme to globally Lipschitz nonlinear systems for which the input is applied with ZOH, the measurements are sampled and delayed, and only an output is measured (since the state vector is not available). Moreover, we also consider the effect of possible modeling errors and measurement noise. The control scheme consists of an observer for the delayed state vector, an inter-sample predictor for the output signal, an approximate predictor for the future value of the state vector, and the nominal feedback law applied with ZOH and computed for the predicted value of the future state vector. The control scheme has been applied to globally Lipschitz nonlinear systems previously (as in [7]) but in this work we have used a different prediction action, namely, we are using approximate predictors for which the prediction is provided by the output of a properly constructed control system (namely, dynamic approximate predictors) instead of predictors that are based on successive approximations. The chapter generalizes the results provided in [7] to various directions:

- We show that the convergence is independent of the lower diameter of the sampling schedule (in contrast with [7], where the estimates depended on the lower diameter of the sampling schedule).
- We provide assumptions which can be applied to general nonlinear globally Lipschitz systems (in contrast with [7], where only triangular single input systems were considered).
- We provide explicit formulae for the asymptotic gains of various inputs (in contrast with [7], where only qualitative estimates were provided).
- We provide explicit inequalities for the upper diameter of the sampling partition and the holding period, which can be used in straightforward way by the potential control practitioner.

The application of the proposed control scheme guarantees robustness with respect to modeling errors, measurement noise and perturbations of the sampling schedule.

Prior to the submission of the present chapter, we were informed of the work [13]. The results in [13] also deal with globally Lipschitz systems using the control scheme proposed in [7]. The results [13] cover various cases of transmission protocols and generalized the results of [7] to non-triangular globally Lipschitz systems.

The results of the present paper are less conservative than the results in [13] and the main difference between the present work and the results of [13] is the use of dynamic approximate predictors instead of predictors which are based on successive approximations.

2 Notation

Throughout this chapter, we adopt the following notation:

- For a vector $x \in \mathbb{R}^n$ we denote by $|x|$ its usual Euclidean norm, by x' its transpose. For a real matrix $A \in \mathbb{R}^{n\times m}$, $A' \in \mathbb{R}^{m\times n}$ denotes its transpose and $|A| = \sup\{\, |Ax| \,;\ x \in \mathbb{R}^n \ ,\ |x| = 1 \,\}$ is its induced norm. $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix.
- $\mathbb{R}_+$ denotes the set of non-negative real numbers. For every $t \geq 0$, $[t]$ denotes the integer part of $t \geq 0$, i.e., the largest integer being less or equal to $t \geq 0$. A partition $\pi = \{T_i\}_{i=0}^{\infty}$ of $\mathbb{R}_+$ is an increasing sequence with $T_0 = 0$ and $T_i \to +\infty$.
- Let $x : [a-r, b) \to \mathbb{R}^n$ with $b > a \geq 0$ and $r \geq 0$. By x_t we denote the "history" of x from $t-r$ to t, i.e., $(x_t)\,(\theta) = x(t+\theta)\ ;\ \theta \in [-r, 0]$, for $t \in [a, b)$. By $\breve{x}_t$ we denote the "open history" of x from $t-r$ to t, i.e., $(\breve{x}_t)\,(\theta) = x(t+\theta)\ ;\ \theta \in [-r, 0)$, for $t \in [a, b)$.
- Let $I \subseteq \mathbb{R}_+$ be an interval. By $L^\infty(I; U)$ ($L^\infty_{loc}(I; U)$) we denote the space of measurable and (locally) bounded functions $u(\cdot)$ defined on I and taking values in $U \subseteq \mathbb{R}^m$. Notice that we do not identify functions in L^∞ *(I;U)* which differ on a measure zero set. For $L^\infty([-r, 0]; \mathbb{R}^n)$ or $x \in L^\infty([-r, 0); \mathbb{R}^n)$ we define $\|x\| = \sup_{\theta\in[-r,0]} |x(\theta)|$ or $\|x\| = \sup_{\theta\in[-r,0)} |x(\theta)|$. Notice that $\sup_{\theta\in[-r,0]} |x(\theta)|$ is not the essential supremum but the actual supremum. By $PC(I; \mathbb{R}^m)$ we denote the space of piecewise continuous functions $u(\cdot)$ defined on I and taking values in $\mathbb{R}^m$.
- By $C^0(A; \Omega)$, where $A \subseteq \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^m$, we denote the class of continuous functions taking values in $\Omega \subseteq \mathbb{R}^m$. A continuous mapping $F : C^0([-r, 0]; \mathbb{R}^l) \times \mathbb{R}^m \to \mathbb{R}^n$ is said to be Lipschitz on bounded sets if there exists a non-decreasing function $Q : \mathbb{R}_+ \to \mathbb{R}_+$ such that $|F(x, u) - F(y, u)| \leq Q\,(\|x\| + \|y\| + |u|)\,\|x - y\|$ for all $x, y \in C^0([-r, 0]; \mathbb{R}^l)$ and for all $u \in \mathbb{R}^m$.

3 Statement of Main Results

We consider a time-invariant control system of the form

$$\begin{array}{l} \dot{x}(t) = f(x(t), u(t-\tau)) + Gv(t) \\ x \in \mathbb{R}^n\ ,\ u \in \mathbb{R}^m\ ,\ v \in \mathbb{R}^q \end{array}\ , t \geq 0 \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous vector field with $f(0,0) = 0$, $\tau > 0$ is a constant, and $G \in \mathbb{R}^{n \times q}$ is a real constant matrix. The input $v(t) \in \mathbb{R}^q$ quantifies the effect of possible modeling errors. We assume that the following assumptions hold for system (1).

(H1) *There exist a continuous mapping $k : \mathbb{R}^n \to \mathbb{R}^m$ with $k(0) = 0$, a constant $\mu > 0$ and a symmetric, positive definite matrix $P \in \mathbb{R}^{n \times n}$ such that the following inequality holds for all $x \in \mathbb{R}^n$:*

$$x' P f(x, k(x)) \leq -4\mu |x|^2 \tag{2}$$

(H2) *There exist constants $L_1, L_2, K \geq 0$ such that the following inequalities hold for all $x, y \in \mathbb{R}^n$, $u, v \in \mathbb{R}^m$:*

$$|k(x) - k(y)| \leq K |x - y| \tag{3}$$

$$|f(x, u) - f(y, u)| \leq L_1 |x - y| \tag{4}$$

$$|f(x, u) - f(x, v)| \leq L_2 |u - v| \tag{5}$$

(H3) *There exist matrices $L \in \mathbb{R}^{n \times p}$, $H \in \mathbb{R}^{p \times n}$, a constant $\omega > 0$ and a symmetric, positive definite matrix $Q \in \mathbb{R}^{n \times n}$ such that the following inequality holds for all $x, e \in \mathbb{R}^n$, $u \in \mathbb{R}^m$:*

$$e' Q \left(f(x + e, u) - f(x, u) + LHe \right) \leq -2\omega |e|^2 \tag{6}$$

Discussion of the assumptions: Assumption (H1) guarantees that the "continuously applied" feedback law $u(t) = k(x(t))$ would globally exponentially stabilize the equilibrium point $0 \in \mathbb{R}^n$ of system (1) if the input delay τ were absent, i.e., if $\tau = 0$. Assumption (H2) guarantees that both the "nominal" feedback law $k : \mathbb{R}^n \to \mathbb{R}^m$ and the mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are globally Lipschitz mappings. Assumption (H3) guarantees that the system $\dot{z}(t) = f(z(t), u(t - \tau)) + L(Hz(t) - y(t))$ would be a global exponential observer for system (1) provided that the output $y(t) = Hx(t)$ were available for all $t \geq 0$ and that no modeling errors were present.

System (1) under assumptions (H1), (H2), and (H3) would be globally exponentially stabilized by the dynamic output feedback law $\dot{z}(t) = f(z(t), u(t)) + L(Hz(t) - y(t))$ with $u(t) = k(z(t))$ if (a) the input delay τ were absent, (b) the input $u(t)$ were allowed to be continuously adjusted, (c) no modeling errors were present, and (d) the output $y(t) = Hx(t)$ were available for all $t \geq 0$. In this work, we will assume that none of the previous requirements hold. More specifically, we assume that:

- The output measurement is sampled, corrupted and delayed, i.e., there is a partition $\{\tau_i\}_{i=0}^{\infty}$ of $\mathbb{R}_+$ with $\sup_{i \geq 0} (\tau_{i+1} - \tau_i) \leq T_s$ where $T_s > 0$ is a constant, an input $\xi \in L_{loc}^{\infty}(\mathbb{R}_+; \mathbb{R}^p)$ and a constant $r \geq 0$ so that $y(t) = Hx(\tau_i) + \xi(\tau_i)$, for all

$t \in [\tau_i, \tau_{i+1})$, $i = 0, 1, 2, ...$ The number $T_s > 0$ is called the upper diameter of the partition and is known, while the sampling times $\{\tau_i\}_{i=0}^{\infty}$ are not known.

- The input delay τ is present and modeling errors are present as well.
- The input cannot be continuously adjusted and can only be applied with ZOH, i.e., there exists a constant $T_H > 0$ (the holding period) such that $u(t) = u_l \in \mathbb{R}^m$, for all $t \in [lT_H, (l+1)T_H)$, $l = 0, 1, 2, ...$

One (or all) of the above complications are present when the control system is networked and there are communication or computation delays along the operation of the network; see the discussion in [3]. Furthermore, the existence of sampled, corrupted and delayed measurements is common for (bio)chemical processes and the inability of continuous adjustment of the input is also common for many systems. Our main result is given next.

Theorem 1 *Consider system* (1) *under assumptions (H1), (H2), and (H3). Let* $T_s > 0$, $T_H > 0$, *and* $r \geq 0$ *be real constants and* $N > 0$ *be an integer that satisfy the inequalities*

$$\frac{\exp(L_1 T_H)(L_1+L_2K)T_H}{1-\exp(L_1 T_H)(L_1+L_2K)T_H} |P| \, L_2 K \leq 2\mu \text{ and}$$
$$\sqrt{\frac{K_4}{2K_3}} |QL| \, |H| \, L_1 T_s < \omega, \; L_1(r+\tau) < N \tag{7}$$

where K_3 *and* $K_4 > 0$ *are constants that satisfy* $K_3 |x|^2 \leq x'Qx \leq K_4 |x|^2$ *for all* $x \in \mathbb{R}^n$. *Then for every* $c > 0$, *there exist constants* $\theta > 0$ *and* $\Theta > 0$ *such that for every partition* $\{\tau_i\}_{i=0}^{\infty}$ *of* $\mathbb{R}_+$ *satisfying*

$$\sup_{i \geq 0} (\tau_{i+1} - \tau_i) \leq T_s \tag{8}$$

and for all choices of $x_0 \in C^0([-r, 0]; \mathbb{R}^n)$, $z_{j,0} \in C^0([-\delta, 0]; \mathbb{R}^n)$ $(j = 0, ..., N)$, $\breve{u}_0 \in L^\infty([-r-\tau, 0); \mathbb{R}^m)$, $v \in L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^q)$, *and* $\xi \in L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^p)$, *the solution*

$$(x(t), z_0(t), ..., z_N(t), u(t), w(t)) \in \mathbb{R}^n \times \mathbb{R}^{Nn} \times \mathbb{R}^m \times \mathbb{R}^p \tag{9}$$

of the system (1) *with the choices*

$$\dot{z}_0(t) = f(z_0(t), u(t-\tau-r)) + L\left(Hz_0(t) - w(t)\right), \; t \geq 0 \tag{10}$$

$$\begin{aligned} \dot{z}_j(t) = \; & \dot{z}_{j-1}(t) \\ & + f(z_j(t), u(t + j\delta - r - \tau)) - f(z_j(t-\delta), u(t + (j-1)\delta - r - \tau)) \\ & - c\left(z_j(t) - z_{j-1}(t) - \int_{t-\delta}^{t} f(z_j(s), u(s + j\delta - r - \tau))ds\right), \\ & t \geq 0, \; j = 1, ..., N \end{aligned} \tag{11}$$

$$\dot{w}(t) = Hf(z(t), u(t-\tau-r)), t \in [\tau_i, \tau_{i+1}), i = 0, 1, 2, ... \tag{12}$$

$$w(\tau_i) = Hx(\tau_i - r) + \xi(\tau_i), i = 0, 1, 2, ... \tag{13}$$

$$u(t) = k\left(z_N(lT_H)\right), t \in [lT_H, (l+1)T_H), l = 0, 1, 2, ... \tag{14}$$

and $\delta = (r+\tau)/N$, *and with the initial conditions* $x(s) = (x_0)(s)$ *for* $s \in [-r, 0]$, $z_j(s) = (z_{j,0})(s)$ *for* $s \in [-\delta, 0]$, *and* $u(s) = (\breve{u}_0)(s)$ *for* $s \in [-r-\tau, 0)$ *corresponding to any inputs* $v \in L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^q)$ *and* $\xi \in L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^p)$, *is unique, exists for all* $t \geq 0$ *and satisfies the following estimates*:

$$\begin{aligned}
&|x(t)| \leq \sqrt{\tfrac{K_2}{K_1}} \exp\left(-\theta(t-\tau)\right) \sup_{0\leq s\leq \tau}\left(|x(s)|\right) \\
&+\Xi_1 \Xi_2 \Omega \Lambda^N \tfrac{|QL|\exp(\theta\,(\tau+T_H))}{1-g} \sup_{0\leq s\leq t}\left(|\xi\,(s)|\right) \\
&+\Xi_1\left(|PG| + |G|\,|P|\,C + \Omega\exp\left(\theta\, T_H\right)\tfrac{\Lambda^{N+1}-\Lambda}{\Lambda-1}\,|G|\right)\exp\left(\theta\,\tau\right)\sup_{0\leq s\leq t}\left(|v(s)|\right) \\
&+\Xi_1\Xi_2\Omega\Lambda^N \tfrac{\exp(\theta\,(\tau+T_H))}{1-g}\left(|QG|\,\times\exp(-\theta\, r) + |QL|\,|HG|\,T_s\right)\sup_{0\leq s\leq t}\left(|v\,(s)|\right) \\
&+\Xi_1\Omega\exp\left(-\theta\,(t-\tau-T_H)\right)\tfrac{\Lambda^{N+1}-\Lambda}{\Lambda-1}\left(L_2\delta\,\|\breve{u}_0\| + 3\max_{l=1,\ldots,N}\left(\|z_{l,0}\|\right)\right) \\
&+\Xi_1\Omega\exp\left(-\theta\,(t-r-\tau-T_H)\right)\sup_{0\leq s\leq r}\left(|x\,(s+\tau) - z_N(s)|\right) \\
&+\Xi_1\Omega\exp\left(-\theta\,(t-r-\tau-T_s-T_H)\right)\tfrac{\Lambda^N}{1-g}\sqrt{\tfrac{K_4}{K_3}}\sup_{r\leq s\leq r+T_s}\left(|x\,(s-r) - z_0\,(s)|\right) \\
&+\Xi_1\Omega\exp\left(-\theta\,(t-r-\tau-T_H)\right)\tfrac{\Lambda^N-1}{\Lambda-1} \\
&\times\max_{j=1,\ldots,N}\left(\sup_{-\delta\leq s\leq r}\left(|x\,(s-r+j\delta) - z_j\,(s)|\right)\right)
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\|x_t\| + \textstyle\sum_{j=0}^N \|z_{j,t}\| + \|\breve{u}_t\| \leq \Theta\Big(&\exp\left(-\theta\, t\right)\left(\|x_0\| + \textstyle\sum_{j=0}^N \|z_{j,0}\| + \|\breve{u}_0\|\right) + \\
&\sup\nolimits_{0\leq s\leq t}\left(|v(s)|\right) + \sup\nolimits_{0\leq s\leq t}\left(|\xi\,(s)|\right)\Big),
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
&\Omega = |P|\,L_2 K(1+C),\;\; \Xi_1 = \sqrt{\tfrac{K_2}{2\mu K_1(\mu-\theta K_2)}}, \\
&\Xi_2 = \sqrt{\tfrac{K_4}{2\omega K_3(\omega-\theta K_4)}},\;\; \Lambda = \tfrac{\theta}{\theta - L_1(\exp(\theta\,\delta)-1)}, \\
&g = \sqrt{\tfrac{K_4}{2\omega K_3(\omega-\theta K_4)}}\,|QL|\,|H|\,L_1\tfrac{\exp(\theta\,T_s)-1}{\theta},\;\text{ and } C = \tfrac{\exp(L_1T_H)L_2KT_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H}.
\end{aligned} \tag{17}$$

Inequality (16) guarantees the input-to-state stability property (as defined in [14]) for the closed-loop system given by (1), (10)–(14) with respect to modeling errors and measurement noise. More specifically, estimate (16) shows that the gain function for the external inputs v and ξ is linear. On the other hand, inequality (15) allow us to estimate explicitly the asymptotic gains of the external inputs v and ξ to the output $Y(t) = x(t)$: the asymptotic gain of the modeling error v is guaranteed to be any number greater than

$$\Xi_1 \left(|PG| + |G| |P| C + \Omega \frac{\Lambda^{N+1} - \Lambda}{\Lambda - 1} |G| + \frac{\Xi_2 \Omega \Lambda^N}{1 - g} |QG| + \frac{\Xi_2 \Omega \Lambda^N}{1 - g} |QL| |HG| T_s \right)$$

and the asymptotic gain of the measurement error ξ is guaranteed to be any number greater than

$$\Xi_1 \Xi_2 \Omega \Lambda^N \frac{|QL|}{1 - g}, \quad \text{where}$$

$$\begin{aligned} &\Omega = |P| L_2 K (1 + C), \ \Xi_1 = \tfrac{1}{\mu} \sqrt{\tfrac{K_2}{2K_1}}, \ \Xi_2 = \tfrac{1}{\omega} \sqrt{\tfrac{K_4}{2K_3}}, \ \Lambda = \tfrac{1}{1 - L_1 \delta}, \\ &g = \Xi_2 |QL| |H| L_1 T_s, \ \text{and } C = \tfrac{\exp(L_1 T_H) L_2 K T_H}{1 - \exp(L_1 T_H)(L_1 + L_2 K) T_H}. \end{aligned} \tag{18}$$

Robustness to perturbations of the sampling schedule is also guaranteed.

Estimates (15) and (16) are independent of the lower diameter of the sampling partition (i.e., of $\inf_{i \geq 0}(\tau_{i+1} - \tau_i)$). This feature is in sharp contrast with the result in [7]. This difference is explained by a different methodology in the proof; if the same methodology were followed in [7] then a similar result would be proved. A few words are needed for the explanation of the hybrid dynamic feedback given by (10)–(14).

- (10) is an observer for the delayed state vector $x(t - r)$. However, (10) does not use the continuous signal $y(t) = Hx(t - r)$, which is not available. The signal $w(t)$ replaces the output signal $y(t) = Hx(t - r)$.
- (12), (13) is an inter-sample predictor for the non-available output signal; it uses the output values at the sampling times and "tries" to predict the output signal between two consecutive sampling times.
- System (11) is an approximate predictor of the future value of the state vector $x(t + \tau)$. The approximate predictor uses the estimated value $z_0(t)$ of the delayed state vector $x(t - r)$, which is provided by the observer, and provides $z_N(t)$, which is an approximation of $x(t + \tau)$.
- Finally, (14) is the "nominal" feedback law computed at the predicted value of the future state vector $x(t + \tau)$ applied with ZOH (emulation).

Remarks:

- Contrary to the approach in [15], no tradeoff between the delays (in the input and the output) and the upper diameter of the sampling partition and the holding period is present for our control scheme.
- From (7), we see that for long delays, we can use sufficiently many predictors that ensure the robustness properties (15) and (16). The counterpart of this is that the gains corresponding to measurement and modeling errors will increase.

4 Key Lemmas

The proof of Theorem 1 is in the following section and is demanding because even the existence/uniqueness of the solution of the overall closed-loop system is not trivial, since the closed-loop system is a hybrid system with delays. For the proof of Theorem 1, we need the following two lemmas, which are stated below. Their proofs are simple and are omitted.

Lemma 1 *Let $r \geq 0$ be a constant and $F : C^0([-r, 0]; \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuous mapping that is Lipschitz on bounded sets and satisfies the inequality $|F(x, u)| \leq L \|x\| + M |u|$ for all $(x, u) \in C^0([-r, 0]; \mathbb{R}^n) \times \mathbb{R}^m$ for certain constants $L > 0$ and $M \geq 0$. Then for every $t_0 \geq 0$, $b \in (t_0, +\infty)$, $x_0 \in C^0([-r, 0]; \mathbb{R}^n)$, and $u \in L^\infty([t_0 - r, b); \mathbb{R}^m)$, the unique solution of $\dot{x}(t) = F(x_t, u(t))$ with initial condition $x(t_0 + s) = (x_0)(s)$ for all $s \in [-r, 0]$ exists for all $t \in [t_0, b]$ and satisfies the estimate*

$$\|x_t\| \leq \exp(L(t - t_0)) \left(\|x_{t_0}\| + M \int_{t_0}^{t} |u(s)|\, ds \right) \tag{19}$$

for all $t \in [t_0, b]$.

Lemma 2 *Let $P \in \mathbb{R}^{n \times n}$ be a symmetric, positive definite matrix and $K_1 > 0$ and $K_2 > 0$ be constants such that the inequality $K_1 |x|^2 \leq x' P x \leq K_2 |x|^2$ holds for all $x \in \mathbb{R}^n$. Let $x : [t_0, b) \to \mathbb{R}^n$ be an absolutely continuous mapping that satisfies the inequality*

$$x'(t) P \dot{x}(t) \leq -c |x(t)|^2 + \sum_{j=1}^{m} a_j |v_j(t)|^2 \tag{20}$$

for $t \in [t_0, b)$ a.e., where $t_0 \geq 0$, $b \in (t_0, +\infty]$, $c > 0$, and $a_j \geq 0$ ($j = 1, ..., m$) are constants, $p_j \geq 1$ are integers and $v_j \in L^\infty_{loc}(\mathbb{R}_+; \mathbb{R}^{p_j})$ ($j = 1, ..., m$) are measurable and locally essentially bounded functions. Then for every $\mu \in (0, c/K_2)$ and for every $t \in [t_0, b)$, the estimate

$$\begin{aligned} &\sup_{t_0 \leq s \leq t} (|x(s)| \exp(\mu s)) \leq \\ &\sqrt{\tfrac{K_2}{K_1}}\, |x(t_0)| \exp(\mu t_0) + \sqrt{\tfrac{K_2}{K_1(c - \mu K_2)}} \sum_{j=1}^{m} \sqrt{a_j} \sup_{t_0 \leq s \leq t} \left(|v_j(s)| \exp(\mu s) \right) \end{aligned} \tag{21}$$

holds.

5 Proof of Theorem 1

We divide the proof into the following three parts:

Part I: Existence and Uniqueness of Solutions

In this part of the proof, we show that for every $x_0 \in C^0([-r,0];\mathbb{R}^n)$, $z_{j,0} \in C^0([-\delta,0];\mathbb{R}^n)$ $(j = 0,...,N)$, $\breve{u}_0 \in L^\infty([-r-\tau,0);\mathbb{R}^m)$, $v \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^q)$, and $\xi \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^p)$, the closed-loop system given by (1), (10)-(14) has a unique solution with initial condition $x(s) = (x_0)(s)$ for $s \in [-r,0]$, $z_j(s) = (z_{j,0})(s)$ for $s \in [-\delta,0]$, and $u(s) = (\breve{u}_0)(s)$ for $s \in [-r-\tau,0)$ corresponding to inputs $v \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^q)$ and $\xi \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^p)$, and defined for all $t \geq 0$.

Part II: Proof of (15)

In this part of the proof, we show that estimate (15) holds for an appropriate constant $\theta > 0$.

Part III: Proof of (16)

In this part of the proof, we show that estimate (16) holds.

Part I: Existence and Uniqueness of Solutions

First we prove that for every $x_0 \in C^0([-r,0];\mathbb{R}^n)$, $z_{j,0} \in C^0([-\delta,0];\mathbb{R}^n)$ $(j = 0,...,N)$, $\breve{u}_0 \in L^\infty([-r-\tau,0);\mathbb{R}^m)$, $v \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^q)$, and $\xi \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^p)$, the closed-loop system given by (1)–(14) has a unique solution with initial condition $x(s) = (x_0)(s)$ for $s \in [-r,0]$, $z_j(s) = (z_{j,0})(s)$ for $s \in [-\delta,0]$, and $u(s) = (\breve{u}_0)(s)$ for $s \in [-r-\tau,0)$ corresponding to inputs $v \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^q)$ and $\xi \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}^p)$. The solution is defined for all $t \geq 0$ and is constructed step-by-step using the following claim:

Claim 1 Assume that there exist an integer $a \geq 0$ and $x \in C^0([-r,aT_H];\mathbb{R}^n)$ and $z_j \in C^0([-\delta,aT_H];\mathbb{R}^n)$ for $j = 0,...,N$ that are absolutely continuous on $[0,aT_H]$, and $w \in PC([0,aT_H];\mathbb{R}^p)$ and $u \in L^\infty([-r-\tau,aT_H);\mathbb{R}^m)$ that satisfy $x(s) = (x_0)(s)$ for $s \in [-r,0]$ and $z_j(s) = (z_{j,0})(s)$ for all $s \in [-\delta,0]$ and $u(s) = (\breve{u}_0)(s)$ for all $s \in [-r-\tau,0)$, as well as Eqs. (1), (10), (11), and (12) for $t \in [0,aT_H)$ a.e., Eq. (13) for all integers $i \geq 0$ with $\tau_i \leq aT_H$, and Eq. (14) for $l = 0,...,a-1$ (only when $a > 0$). Then there exist $x \in C^0([-r,(a+1)T_H];\mathbb{R}^n)$ and $z_j \in C^0([-\delta,(a+1)T_H];\mathbb{R}^n)$ $(j = 0,...,N)$ that are absolutely continuous on $[0,(a+1)T_H]$, $w \in PC([0,(a+1)T_H];\mathbb{R}^p)$, and $u \in L^\infty([-r-\tau,(a+1)T_H);\mathbb{R}^m)$ that satisfy $x(s) = (x_0)(s)$ for $s \in [-r,0]$, $z_j(s) = (z_{j,0})(s)$ for $s \in [-\delta,0]$, $u(s) = (\breve{u}_0)(s)$ for $s \in [-r-\tau,0)$, Eqs. (1), (10)–(12) for $t \in [0,(a+1)T_H)$ a.e., Eq. (13) for all integers $i \geq 0$ with $\tau_i \leq (a+1)T_H$, and Eq. (14) for $l = 0,...,a$.

Proof (Claim 1) Using (14) for $l = a$, we can (uniquely) define u on $[aT_H, (a+1)T_H)$. Since u is constant on $[aT_H,(a+1)T_H)$, we know that $u \in L^\infty([-r-\tau,(a+1)T_H);\mathbb{R}^m)$.

Since the right-hand side of (1) satisfies a linear growth condition and since u is defined on $[-r-\tau,(a+1)T_H)$, it follows from Lemma 1 that we can uniquely define x on $[aT_H,(a+1)T_H]$. The extended mapping $x : [-r,(a+1)T_H] \to \mathbb{R}^n$

satisfies $x \in C^0([-r, (a+1)T_H]; \mathbb{R}^n)$, is absolutely continuous on $[0, (a+1)T_H]$, and satisfies (1) for $t \in [0, (a+1)T_H)$ a.e.

Since $\lim_{i\to\infty} \tau_i = +\infty$, there are only a finite number of sampling times τ_i in the interval $[aT_H, (a+1)T_H]$ (and possibly none). The right-hand sides of (10) and (12) satisfy a linear growth condition and since u is defined on $[-r-\tau, (a+1)T_H)$ and x is defined on $[-r, (a+1)T_H]$, it follows from Lemma 1 that we can uniquely define (z_0, w) on $[aT_H, (a+1)T_H]$. The extended mapping $z_0 : [-\delta, (a+1)T_H] \to \mathbb{R}^n$ satisfies $z_0 \in C^0([-\delta, (a+1)T_H]; \mathbb{R}^n)$, is absolutely continuous on $[0, (a+1)T_H]$ and satisfies (10) and (12) for $t \in [0, (a+1)T_H)$ a.e.. Moreover, the extended mapping $w : [0, (a+1)T_H] \to \mathbb{R}^p$ satisfies $w \in PC([0, (a+1)T_H]; \mathbb{R}^p)$.

Finally, using (11) and Lemma 1, we can define z_1 and next $z_2, ..., z_N$ on $[aT_H, (a+1)T_H]$. The extended mappings $z_j : [-\delta, (a+1)T_H] \to \mathbb{R}^n$ (for $j = 1, ..., N$) satisfy $z_j \in C^0([-\delta, (a+1)T_H]; \mathbb{R}^n)$, are absolutely continuous on $[0, (a+1)T_H]$ and satisfy (11) for $t \in [0, (a+1)T_H)$ a.e.
Therefore, the claim holds. $\triangleleft$

Part II: Proof of (15)

We next present three inequalities, which are direct consequences of (1), (2), (6) and (10):

$$\begin{aligned}&(z_0(t) - x(t-r))' \, Q \tfrac{d}{dt} (z_0(t) - x(t-r)) \leq -2\omega \, |z_0(t) - x(t-r)|^2 \\ &\quad - (z_0(t) - x(t-r))' \, QGv(t-r) - (z_0(t) - x(t-r))' \, QL \, (w(t) - Hx(t-r)) \\ &\text{for a.e. } t \geq T,\end{aligned} \tag{22}$$

$$\begin{aligned}&x'(t)P\dot{x}(t) \leq \\ &-4\mu \, |x(t)|^2 + x'(t)PGv(t) - x'(t)P \left(f(x(t), k(x(t))) - f(x(t), u(t-\tau)) \right), \\ &\text{for a.e. } t \geq 0, \text{ and}\end{aligned} \tag{23}$$

$$\begin{aligned}&z_0'(t)Q\dot{z}_0(t) \leq -2\omega \, |z_0(t)|^2 + z_0'(t)Qf(0, u(t-r-\tau)) - z_0'(t)QLw(t), \\ &\text{for a.e. } t \geq 0,\end{aligned} \tag{24}$$

where $T = \min\{\tau_i \, : \, \tau_i \geq r \, , \, i = 1, 2, ...\}$ is the smallest sampling time for which $\tau_i \geq r$ holds.

The following equations hold for all $j = 1, ..., N$, $t \geq 0$ and are direct consequences of (11):

$$\begin{aligned}&z_j(t) = z_{j-1}(t) + \int_{t-\delta}^{t} f(z_j(s), u(s + j\delta - r - \tau))ds + \\ &exp\,(-ct) \left(z_j(0) - z_{j-1}(0) - \int_{-\delta}^{0} f(z_j(s), u(s + j\delta - r - \tau))ds \right) \\ &\text{for all } t \geq 0\end{aligned} \tag{25}$$

$$
\begin{aligned}
&x(t-r+j\delta) = \\
&x(t-r+(j-1)\delta) + \int_{t-\delta}^{t} f(x(s-r+j\delta), u(s+j\delta-r-\tau))ds \\
&+ \int_{t-r+(j-1)\delta}^{t-r+j\delta} Gv(s)ds \text{ for all } t \geq r
\end{aligned} \tag{26}
$$

Completing the squares in (22) and (24), and using (23) and (5), we get:

$$
\begin{aligned}
&(z_0(t)-x(t-r))' Q \tfrac{d}{dt} (z_0(t)-x(t-r)) \leq \\
&-\omega |z_0(t)-x(t-r)|^2 + \tfrac{1}{2\omega} |QG|^2 |v(t-r)|^2 \\
&+\tfrac{1}{2\omega} |QL|^2 |w(t)-Hx(t-r)|^2 \text{ for a.e. } t \geq T
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&x'(t+\tau)P\dot{x}(t+\tau) \leq \\
&-4\mu |x(t+\tau)|^2 + |PG| |v(t+\tau)| |x(t+\tau)| \\
&+|P| L_2 |x(t+\tau)| |k(x(t+\tau))-u(t)| \text{ for a.e. } t \geq 0.
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
z_0'(t)Q\dot{z}_0(t) \leq -\omega |z_0(t)|^2 + \tfrac{1}{2\omega} |Q|^2 L_2^2 |u(t-r-\tau)|^2 \\
+\tfrac{1}{2\omega} |QL|^2 |w(t)|^2, \text{ for a.e. } t \geq 0.
\end{aligned} \tag{29}
$$

For all $t \geq 0$, we can use (14) to get $u(t) = k(z_N(lT_H))$, where $l = [t/T_H]$. It follows from Eq. (1) that

$$
\begin{aligned}
&|x(t+\tau)-x(lT_H+\tau)| \leq \\
&\int_{lT_H+\tau}^{t+\tau} |f(x(s),u(s-\tau))| ds + |G| \int_{lT_H+\tau}^{t+\tau} |v(s)| ds.
\end{aligned} \tag{30}
$$

Therefore, we get the following for all $t \geq 0$, where $l = [t/T_H]$:

$$
\begin{aligned}
&|x(t+\tau)-x(lT_H+\tau)| \leq L_1 \int_{lT_H}^{t} |x(s+\tau)-x(lT_H+\tau)| ds \\
&+ (L_1+L_2K) T_H |x(lT_H+\tau)| \\
&+L_2KT_H |z_N(lT_H)-x(lT_H+\tau)| + |G| T_H \sup_{lT_H \leq s \leq t} (|v(s+\tau)|)
\end{aligned}
$$

(using the triangle inequality, (4) and (5) and the fact that $t \in [lT_H, (l+1)T_H)$) and

$$
\begin{aligned}
&|x(t+\tau)-x(lT_H+\tau)| \leq \exp(L_1T_H)(L_1+L_2K) T_H |x(lT_H+\tau)| \\
&+\exp(L_1T_H) L_2KT_H |z_N(lT_H)-x(lT_H+\tau)| \\
&+\exp(L_1T_H) |G| T_H \sup_{lT_H \leq s \leq t} (|v(s+\tau)|)
\end{aligned}
$$

(using the Gronwall-Bellman lemma and the fact that $t \in [lT_H, (l+1)T_H)$), and

$$
\begin{aligned}
&|x(t+\tau)-x(lT_H+\tau)| \leq \tfrac{\exp(L_1T_H)(L_1+L_2K)T_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H} |x(t+\tau)| \\
&+\tfrac{\exp(L_1T_H)L_2KT_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H} |z_N(lT_H)-x(lT_H+\tau)| \\
&+\tfrac{\exp(L_1T_H)|G|T_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H} \sup_{lT_H \leq s \leq t} (|v(s+\tau)|)
\end{aligned}
$$

(using the triangle inequality and the fact that $\exp(L_1T_H)(L_1+L_2K) T_H < 1$).

The above inequality in conjunction with the triangle inequality, (3), (28) and (14) (which implies that $u(t) = k\left(z_N(lT_H)\right)$, where $l = [t/T_H]$) and the inequality

$$\tfrac{\exp(L_1T_H)(L_1+L_2K)T_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\,|P|\,L_2K \le 2\mu \tag{31}$$

give

$$\begin{aligned}
&x'(t+\tau)P\dot{x}(t+\tau) \le -2\mu\,|x(t+\tau)|^2\\
&+\tfrac{|P|L_2K(1-L_1T_H\exp(L_1T_H))}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\,|x(t+\tau)|\,|x(lT_H+\tau)-z_N(lT_H)|\\
&+\left(\tfrac{\exp(L_1T_H)|G||P|L_2KT_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H}+|PG|\right)|x(t+\tau)|\sup_{lT_H\le s\le t}\left(|v(s+\tau)|\right)
\end{aligned} \tag{32}$$

for $t \ge 0$ a.e. and $l = [t/T_H]$. Completing the squares in (32), we get

$$\begin{aligned}
&x'(t+\tau)P\dot{x}(t+\tau) \le\\
&-\mu\,|x(t+\tau)|^2 + \tfrac{1}{2\mu}\left(\tfrac{|P|L_2K(1-L_1T_H\exp(L_1T_H))}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\right)^2|x(lT_H+\tau)-z_N(lT_H)|^2\\
&= +\tfrac{1}{2\mu}\left(\tfrac{\exp(L_1T_H)|G||P|L_2KT_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H}+|PG|\right)^2\sup_{lT_H\le s\le t}\left(|v(s+\tau)|^2\right)
\end{aligned} \tag{33}$$

for $t \ge 0$ a.e. and $l = [t/T_H]$. Combining (25) and (26), we obtain the following for all $j = 1, ..., N$ and $t \ge r$:

$$\begin{aligned}
&\left|z_j(t)-x(t-r+j\delta)\right| \le |G|\sup_{t-r+(j-1)\delta\le s\le t-r+j\delta}\left(|v(s)|\right)+\\
&\left|z_{j-1}(t)-x(t-r+(j-1)\delta)\right| + L_1\int_{t-\delta}^{t}\left|z_j(s)-x(s-r+j\delta)\right|ds\\
&+\exp(-ct)\left|z_j(0)-z_{j-1}(0)-\int_{-\delta}^{0}f(z_j(s),u(s+j\delta-r-\tau))ds\right|
\end{aligned} \tag{34}$$

It follows from (34) that the following inequalities hold for all $\theta \in (0, c]$, $j = 1, ..., N$, and $t \ge r$:

$$\begin{aligned}
\sup_{r\le s\le t}&\left(\left|z_j(s)-x(s-r+j\delta)\right|\exp(\theta s)\right) \le\\
&\sup_{r\le s\le t}\left(\left|z_{j-1}(s)-x(s-r+(j-1)\delta)\right|\exp(\theta s)\right)\\
&+L_1\tfrac{\exp(\theta\delta)-1}{\theta}\sup_{r-\delta\le s\le t}\left(\left|z_j(s)-x(s-r+j\delta)\right|\exp(\theta s)\right)\\
&+\left|z_j(0)-z_{j-1}(0)-\int_{-\delta}^{0}f(z_j(s),u(s+j\delta-r-\tau))ds\right|\\
&+|G|\exp(\theta t)\sup_{0\le s\le t-r+j\delta}\left(|v(s)|\right)
\end{aligned} \tag{35}$$

Since $L_1\delta < 1$, there is a small enough $\theta \in (0, c]$ such that $L_1\left(\exp(\theta\delta)-1\right) < \theta$. It follows from (35) that the following inequalities hold for all $\theta \in (0, c]$ sufficiently small, $j = 1, ..., N$ and $t \ge r$:

$$
\begin{aligned}
&\sup_{r \le s \le t} \left(\left| z_j(s) - x(s - r + j\delta) \right| \exp(\theta\, s) \right) \le \\
&\quad \Lambda \sup_{r \le s \le t} \left(\left| z_{j-1}(s) - x(s - r + (j-1)\delta) \right| \exp(\theta\, s) \right) \\
&\quad + \sup_{r-\delta \le s \le r} \left(\left| z_j(s) - x(s - r + j\delta) \right| \exp(\theta\, s) \right) \\
&\quad + \Lambda \, |G| \exp(\theta\, t) \sup_{0 \le s \le t - r + j\delta} (|v(s)|) \\
&\quad + \Lambda \left| z_j(0) - z_{j-1}(0) - \int_{-\delta}^{0} f(z_j(s), u(s + j\delta - r - \tau)) ds \right|
\end{aligned} \tag{36}
$$

where

$$
\Lambda = \frac{\theta}{\theta - L_1 \left(\exp(\theta\, \delta) - 1 \right)}. \tag{37}
$$

Using (4), (5), (36), the fact that $L_1 \delta < 1$, and induction, we conclude that the following inequalities hold for all sufficiently small $\theta \in (0, c]$ and all $j = 1, ..., N$ and $t \ge r$:

$$
\begin{aligned}
&\sup_{r \le s \le t} \left(\left| z_j(s) - x(s - r + j\delta) \right| \exp(\theta\, s) \right) \le \\
&\quad \Lambda^j \sup_{r \le s \le t} \left(|z_0(s) - x(s - r)| \exp(\theta\, s) \right) \\
&\quad + \tfrac{\Lambda^j - 1}{\Lambda - 1} \exp(\theta\, r) \max_{l = 1, \dots, j} \left(\sup_{r - \delta \le s \le r} \left(|z_l(s) - x(s - r + l\delta)| \right) \right) \\
&\quad + \Lambda \tfrac{\Lambda^j - 1}{\Lambda - 1} \left(L_2 \delta \, \|u_0\| + 3 \max_{l = 1, \dots, j} \left(\| z_{l,0} \| \right) \right) \\
&\quad + \Lambda \tfrac{\Lambda^j - 1}{\Lambda - 1} |G| \exp(\theta\, t) \sup_{0 \le s \le t - r + j\delta} (|v(s)|)
\end{aligned} \tag{38}
$$

Using Lemma 2 and inequalities (27) and (33) we obtain:

$$
\begin{aligned}
&\sup_{0 \le s \le t} \left(|x(s + \tau)| \exp(\theta\, s) \right) \le \sqrt{\tfrac{K_2}{K_1}} \, |x(\tau)| \\
&+ \sqrt{\tfrac{K_2}{2\mu K_1 (\mu - \theta K_2)}} \left(|PG| + \tfrac{\exp(L_1 T_H) |G| |P| L_2 K T_H}{1 - \exp(L_1 T_H)(L_1 + L_2 K) T_H} \right) \exp(\theta\, t) \sup_{\tau \le s \le t + \tau} (|v(s)|) \\
&+ \sqrt{\tfrac{K_2}{2\mu K_1 (\mu - \theta K_2)}} \left(\tfrac{|P| L_2 K (1 - L_1 T_H \exp(L_1 T_H))}{1 - \exp(L_1 T_H)(L_1 + L_2 K) T_H} \right) \\
&\quad \times \exp(\theta\, T_H) \sup_{0 \le s \le t} \left(|x\,(s + \tau) - z_N\,(s)| \exp(\theta\, s) \right)
\end{aligned} \tag{39}
$$

for all $t \ge 0$ and $\theta \in (0, \mu / K_2)$, where K_1 and $K_2 > 0$ are constants such that the inequality $K_1 \, |x|^2 \le x' P x \le K_2 \, |x|^2$ holds for all $x \in \mathbb{R}^n$ and

$$
\begin{aligned}
&\sup_{T \le s \le t} \left(|z_0(s) - x(s - r)| \exp(\theta\, s) \right) \le \sqrt{\tfrac{K_4}{K_3}} \, |z_0(T) - x(T - r)| \exp(\theta\, T) \\
&\quad + \sqrt{\tfrac{K_4}{2\omega K_3 (\omega - \theta K_4)}} \, |QG| \sup_{0 \le s \le t - r} \left(|v(s)| \exp(\theta\, s) \right) \\
&\quad + \sqrt{\tfrac{K_4}{2\omega K_3 (\omega - \theta K_4)}} \, |QL| \sup_{T \le s \le t} \left(|w(s) - H x(s - r)| \exp(\theta\, s) \right)
\end{aligned} \tag{40}
$$

for all $t \geq T$ and $\theta \in (0, \omega/K_4)$, where $T = \min\{\tau_i : \tau_i \geq r, i = 1, 2, ...\}$. Combining (39) and (38) for $j = N$, we obtain the following for all $t \geq 0$ and sufficiently small $\theta > 0$:

$$\begin{aligned}
&\sup_{0\leq s\leq t}\left(|x(s+\tau)|\exp(\theta s)\right) \leq \sqrt{\tfrac{K_2}{K_1}}\,|x(\tau)| \\
&+\sqrt{\tfrac{K_2}{2\mu K_1(\mu-\theta K_2)}}\left(|PG| + \tfrac{\exp(L_1T_H)|G||P|L_2KT_H}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\right) \\
&\quad\times \exp(\theta t)\sup_{\tau\leq s\leq t+\tau}(|v(s)|) \\
&+\sqrt{\tfrac{K_2}{2\mu K_1(\mu-\theta K_2)}}\left(\tfrac{|P|L_2K(1-L_1T_H\exp(L_1T_H))}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\right) \\
&\quad\times \exp(\theta(r+T_H))\sup_{0\leq s\leq r}(|x(s+\tau)-z_N(s)|) \\
&+\sqrt{\tfrac{K_2}{2\mu K_1(\mu-\theta K_2)}}\left(\tfrac{|P|L_2K(1-L_1T_H\exp(L_1T_H))}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\right)\exp(\theta T_H) \\
&\quad\times \Lambda\tfrac{\Lambda^N-1}{\Lambda-1}\left(L_2\delta\,\|u_0\| + 3\max_{l=1,\dots,N}\left(\|z_{l,0}\|\right)\right) \\
&+\sqrt{\tfrac{K_2}{2\mu K_1(\mu-\theta K_2)}}\left(\tfrac{|P|L_2K(1-L_1T_H\exp(L_1T_H))}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\right)\exp(\theta T_H) \\
&\quad\times \Lambda^N\sup_{r\leq s\leq t}(|x(s-r)-z_0(s)|\exp(\theta s)) \\
&+\sqrt{\tfrac{K_2}{2\mu K_1(\mu-\theta K_2)}}\left(\tfrac{|P|L_2K(1-L_1T_H\exp(L_1T_H))}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\right)\exp(\theta(r+T_H)) \\
&\quad\times \tfrac{\Lambda^N-1}{\Lambda-1}\max_{j=1,\dots,N}\left(\sup_{-\delta\leq s\leq r}\left(|x(s-r+j\delta)-z_j(s)|\right)\right) \\
&+\sqrt{\tfrac{K_2}{2\mu K_1(\mu-\theta K_2)}}\left(\tfrac{|P|L_2K(1-L_1T_H\exp(L_1T_H))}{1-\exp(L_1T_H)(L_1+L_2K)T_H}\right)\exp(\theta(t+T_H)) \\
&\quad\times \Lambda\tfrac{\Lambda^N-1}{\Lambda-1}|G|\sup_{0\leq s\leq t+\tau}(|v(s)|)
\end{aligned} \tag{41}$$

Combining (1) and (12), we obtain the following for all $t \in [\tau_i, \tau_{i+1})$ with $\tau_i \geq T$:

$$\begin{aligned}
|w(t) - Hx(t-r)| &\leq |w(\tau_i) - Hx(\tau_i - r)| + |H|\,L_1\int_{\tau_i}^{t}|z_0(s) - x(s-r)|\,ds \\
&\quad + |HG|\int_{\tau_i}^{t}|v(s-r)|\,ds
\end{aligned} \tag{42}$$

Using (13), (42) and the facts that $t \in [\tau_i, \tau_{i+1})$, $\tau_i \geq T$, and $\sup_{i\geq 0}(\tau_{i+1} - \tau_i) \leq T_s$, we get the following for all $\theta > 0$:

$$\begin{aligned}
&|w(t) - Hx(t-r)|\exp(\theta t) \leq \\
&|\xi(\tau_i)|\exp(\theta t) + |HG|\,T_s\exp(\theta t)\sup_{\tau_i\leq s\leq t}(|v(s-r)|) \\
&+ |H|\,L_1\sup_{\tau_i\leq s\leq t}(|z_0(s) - x(s-r)|\exp(\theta s))\,\tfrac{\exp(\theta T_s)-1}{\theta}
\end{aligned} \tag{43}$$

Estimate (43) implies the following estimate for all $t \geq T$ and $\theta > 0$:

$$\begin{aligned}
&\sup_{T\leq s\leq t}(|w(s) - Hx(s-r)|\exp(\theta s)) \leq \exp(\theta t)\sup_{T\leq s\leq t}(|\xi(s)|) \\
&\quad + |H|\,L_1\sup_{T\leq s\leq t}(|z_0(s) - x(s-r)|\exp(\theta s))\,\tfrac{\exp(\theta T_s)-1}{\theta} \\
&\quad + |HG|\,T_s\exp(\theta t)\sup_{0\leq s\leq t-r}(|v(s)|)
\end{aligned} \tag{44}$$

Since $\sqrt{\frac{K_4}{2K_3}}\,|QL|\,|H|\,L_1 T_s < \omega$, it follows that there exists $\theta > 0$ such that

$$g = \sqrt{\frac{K_4}{2\omega K_3(\omega-\theta K_4)}}\,|QL|\,|H|\,L_1 \frac{\exp(\theta\, T_s)-1}{\theta} < 1. \tag{45}$$

Combining (40) and (44), we get this for all $t \geq T$ and for all sufficiently small $\theta > 0$:

$$\begin{aligned}&\sup_{T\leq s\leq t}\left(|z_0(s)-x(s-r)|\exp\left(\theta\, s\right)\right) \leq \\ &\quad \frac{1}{1-g}\sqrt{\frac{K_4}{K_3}}\,|z_0(T)-x(T-r)|\exp\left(\theta\, T\right) \\ &\quad +\frac{\exp(\theta\, t)}{1-g}\sqrt{\frac{K_4}{2\omega K_3(\omega-\theta K_4)}}\left(|QG|\exp\left(-\theta\, r\right)+|QL|\,|HG|\,T_s\right)\sup_{0\leq s\leq t}\left(|v(s)|\right) \\ &\quad +\frac{1}{1-g}\sqrt{\frac{K_4}{2\omega K_3(\omega-\theta K_4)}}\,|QL|\exp\left(\theta\, t\right)\sup_{T\leq s\leq t}\left(|\xi(s)|\right)\end{aligned} \tag{46}$$

Combining (41) and (46) and using the fact that $\sup_{i\geq 0}(\tau_{i+1}-\tau_i) \leq T_s$ (which implies that $T = \min\{\tau_i \,:\, \tau_i \geq r\,,\, i=1,2,...\}$ satisfies $T \leq r + T_s$), we obtain estimate (15) for all $t \geq 0$ and sufficiently small $\theta > 0$.

Part III: Proof of (16)

Our strategy for the proof of estimate (16) is described next.

Using (38), (46) and (15) in conjunction with a standard causality argument, we conclude that there exists a constant $\kappa > 0$ such that the following inequality holds for all $t \geq 0$:

$$\begin{aligned}\|x_t\| + \textstyle\sum_{j=0}^{N}\|z_t\| \leq \kappa\exp\left(-\theta\, t\right)\sup_{0\leq s\leq \tau+r+T_s}\left(\|x_s\| + \|\breve{u}_s\| + \sum_{j=0}^{N}\|z_{j,s}\|\right) \\ +\kappa\left(\sup_{0\leq s\leq t}\left(|\xi\,(s)|\right) + \sup_{0\leq s\leq t}\left(|v\,(s)|\right)\right)\end{aligned} \tag{47}$$

Using (3), (14) and the fact that $k(0) = 0$, we obtain the following for all $t \geq 0$:

$$\begin{aligned}\|\breve{u}_t\| \leq \|\breve{u}_0\|\exp\left(-\theta(t-r-\tau)\right) \\ +K\exp\left(-\theta\,(t-r-\tau-T_H)\right)\sup_{0\leq s\leq t}\left(|z_N(s)|\exp(\theta s)\right)\end{aligned} \tag{48}$$

Therefore, inequalities (47) and (48) allow us to conclude the existence of a constant $\bar{\kappa} > 0$ such that the following inequality holds for all $t \geq 0$:

$$\begin{aligned}&\|x_t\| + \|\breve{u}_t\| + \textstyle\sum_{j=0}^{N}\|z_t\| \leq \\ &\bar{\kappa}\exp\left(-\theta\, t\right)\sup_{0\leq s\leq \tau+r+T_s}\left(\|x_s\| + \|\breve{u}_s\| + \textstyle\sum_{j=0}^{N}\|z_{j,s}\|\right) \\ &+\bar{\kappa}\left(\sup_{0\leq s\leq t}\left(|\xi\,(s)|\right) + \sup_{0\leq s\leq t}\left(|v\,(s)|\right)\right)\end{aligned} \tag{49}$$

To show (16), we use inequality (49) and the following claim:

Claim 2 For every integer $i \geq 0$ there exists a constant $M_i > 0$ such that

$$\begin{aligned}
&\|x_t\| + \|\breve{u}_t\| + \textstyle\sum_{j=0}^{N} \|z_{j,t}\| \leq \\
&M_i \left(\|x_0\| + \textstyle\sum_{j=0}^{N} \|z_{j,0}\| + \|\breve{u}_0\| + \sup_{0 \leq s \leq t} (|v(s)|) + \sup_{0 \leq s \leq t} (|\xi(s)|)\right) \\
&\text{for all } t \in [0, iT_H]
\end{aligned} \tag{50}$$

Indeed, if Claim 2 holds, then inequality (16) is a direct consequence of (49). Therefore, the rest of the proof is devoted to the proof of Claim 2.

Using Lemma 2 and inequality (29) we obtain for all $t \geq 0$:

$$\begin{aligned}
\sup_{0 \leq s \leq t} (|z_0(s)| \exp(\theta s)) &\leq \sqrt{\tfrac{K_4}{K_3}}\, |z_0(0)| \\
&+ \sqrt{\tfrac{K_4}{2\omega K_3(\omega - \theta K_4)}}\, |Q|\, L_2 \sup_{0 \leq s \leq t} (|u(s-r-\tau)| \exp(\theta s)) \\
&+ \sqrt{\tfrac{K_4}{2\omega K_3(\omega - \theta K_4)}}\, |QL| \sup_{0 \leq s \leq t} (|w(s)| \exp(\theta s))
\end{aligned} \tag{51}$$

Using (4), (5) and (12) we obtain the following for all $t \in [\tau_i, \tau_{i+1})$ and $i = 0, 1, 2, \ldots$:

$$\begin{aligned}
&|w(t)| \leq |w(\tau_i)| + |H|\, L_1 \tfrac{\exp(-\theta \tau_i) - \exp(-\theta t)}{\theta} \sup_{\tau_i \leq s \leq t} (|z_0(s)| \exp(\theta s)) \\
&+ |H|\, L_2 \tfrac{\exp(-\theta \tau_i) - \exp(-\theta t)}{\theta} \sup_{\tau_i \leq s \leq t} (|u(s-r-\tau)| \exp(\theta s))
\end{aligned} \tag{52}$$

Combining the estimate (52) with (13) and using the facts that $t \in [\tau_i, \tau_{i+1})$ and $\sup_{i \geq 0} (\tau_{i+1} - \tau_i) \leq T_s$, we obtain:

$$\begin{aligned}
\sup_{0 \leq s \leq t} (|w(s)| \exp(\theta s)) &\leq \sup_{0 \leq s \leq t} (|\xi(s)|) \exp(\theta t) + \\
&|H| \exp(\theta (r + T_s)) \sup_{-r \leq s \leq t-r} (|x(s)| \exp(\theta s)) \\
&+ |H|\, L_1 \tfrac{\exp(\theta T_s) - 1}{\theta} \sup_{0 \leq s \leq t} (|z_0(s)| \exp(\theta s)) \\
&+ |H|\, L_2 \tfrac{\exp(\theta T_s) - 1}{\theta} \sup_{0 \leq s \leq t} (|u(s-r-\tau)| \exp(\theta s))
\end{aligned} \tag{53}$$

for all $t \geq 0$. Using our bound (45), the combination of (51) and (53) implies the existence of constants $\Theta_i > 0$ $(i = 1, 2, 3, 4)$ such that the following estimate holds for all $t \geq 0$:

$$\begin{aligned}
&\sup_{0 \leq s \leq t} (|z_0(s)| \exp(\theta s)) \leq \\
&\Theta_1 |z_0(0)| + \Theta_2 \sup_{0 \leq s \leq t} (|u(s-r-\tau)| \exp(\theta s)) \\
&+ \Theta_3 \sup_{-r \leq s \leq t-r} (|x(s)| \exp(\theta s)) + \Theta_4 \exp(\theta t) \sup_{0 \leq s \leq t} (|\xi(s)|)
\end{aligned} \tag{54}$$

Combining (53) and (54), we obtain:

$$\begin{aligned}
&\sup_{0 \leq s \leq t} (|z_0(s)| \exp(\theta s)) + \sup_{0 \leq s \leq t} (|w(s)| \exp(\theta s)) \leq \\
&\tilde{\Theta}_1 |z_0(0)| + \tilde{\Theta}_2 \sup_{0 \leq s \leq t} (|u(s-r-\tau)| \exp(\theta s)) \\
&+ \tilde{\Theta}_3 \sup_{-r \leq s \leq t-r} (|x(s)| \exp(\theta s)) + \tilde{\Theta}_4 \exp(\theta t) \sup_{0 \leq s \leq t} (|\xi(s)|)
\end{aligned} \tag{55}$$

for all $t \geq 0$, for appropriate constants $\tilde{\Theta}_i > 0$ $(i = 1, 2, 3, 4)$.

Inequality (4) implies the following estimates for all $j = 1, ..., N$ and $t \geq 0$:

$$\begin{aligned} \|z_{j,t}\| \leq & (1 + 2L_1\delta) \|z_{j,0}\| + 2\sup_{0\leq s\leq t} \left(|z_{j-1}(s)|\right) \\ & + L_1 \int_0^t \|z_{j,s}\| \, ds + 2L_2\delta \sup_{0\leq s\leq t} \|\breve{u}_s\| \end{aligned} \tag{56}$$

Using (56) and the Gronwall lemma, we get this for all $j = 1, ..., N$ and $t \geq 0$:

$$\begin{aligned} &\|z_{j,t}\| \leq \\ &\exp(L_1 t) \left((1 + 2L_1\delta) \|z_{j,0}\| + 2\sup_{0\leq s\leq t} \left(|z_{j-1}(s)|\right) + 2L_2\delta \sup_{0\leq s\leq t} \|\breve{u}_s\|\right) \end{aligned} \tag{57}$$

Using (55) and (57) repeatedly, we obtain the following for all $t \geq 0$:

$$\begin{aligned} \sum_{j=0}^{N} \|z_{j,t}\| \leq \bar{\Theta} \exp(\sigma t) \Big(& \sum_{j=0}^{N} \|z_{j,0}\| + \sup_{-r-\tau\leq s<t} (|u(s)|) \\ & + \sup_{-r\leq s\leq t} (|x(s)|) + \sup_{0\leq s\leq t} (|\xi(s)|)\Big) \end{aligned} \tag{58}$$

for appropriate constants $\bar{\Theta} > 0$ and $\sigma > 0$. We are now ready to prove Claim 2.

Proof (Claim 2) We use induction. The claim holds automatically for $i = 0$. Using (3), (14), and (50) for a certain integer $i \geq 0$ and the fact that $k(0) = 0$, we get:

$$\begin{aligned} &\sup_{-r-\tau\leq s<t} (|u(s)|) \leq (1 + K)M_i \Big(\|x_0\| + \sum_{j=0}^{N} \|z_{j,0}\| + \|\breve{u}_0\| \\ &+ \sup_{0\leq s\leq t} (|v(s)|) + \sup_{0\leq s\leq t} (|\xi(s)|)\Big) \end{aligned} \tag{59}$$

for all $t \in [0, (i + 1)T_H]$. Using (4), (5), Lemma 1 and (59), we obtain

$$\begin{aligned} &\sup_{-r\leq s\leq t} (|x(s)|) \leq R_i \Big(\|x_0\| + \sum_{j=0}^{N} \|z_{j,0}\| + \|\breve{u}_0\| \\ &+ \sup_{0\leq s\leq t} (|v(s)|) + \sup_{0\leq s\leq t} (|\xi(s)|)\Big) \end{aligned} \tag{60}$$

for all $t \in [0, (i + 1)T_H]$ for an appropriate constant $R_i > 0$. Inequality (50) for $i + 1$ and an appropriate constant $M_{i+1} > 0$ is a direct consequence of (58), (59), and (60). The proof of Claim 2 is complete, so the proof of Theorem 1 is complete.

6 Concluding Remarks

In this chapter, we proposed a novel control scheme for nonlinear globally Lipschitz systems for which the input is delayed and applied with zero order hold, the measurements are sampled and delayed, and only an output is measured. The novelty of our work is in the use of a chain of approximate (dynamic) predictors to handle long delays in both input and output. Using small gain arguments, sufficient conditions on both the upper diameter of the sampling partition and the holding period, and a sufficient number of predictors in the chain, we proved that the closed loop is robust with respect to measurements and modelling errors. We also provided explicit estimates of the asymptotic gains of the external inputs v and ξ. This can be extended easily o networked control systems with uniformly globally exponentially stable scheduling protocols.

References

1. Bekiaris-Liberis, N., Krstic, M.: Robustness of nonlinear predictor feedback laws to time-and state-dependent delay perturbations. Automatica **49**(6), 1576–1590 (2013)
2. Bekiaris-Liberis, N., Krstic, M.: Nonlinear Control Under Nonconstant Delays. SIAM, Philadelphia (2013)
3. Karafyllis, I., Krstic, M.: Nonlinear stabilization under sampled and delayed measurements, and with inputs subject to delay and zero-order hold. IEEE Trans. Autom. Control **57**(5), 1141–1154 (2012)
4. Krstic, M.: Delay Compensation for Nonlinear, Adaptive, and PDE Systems. Birkhauser, Boston (2009)
5. Krstic, M.: Input delay compensation for forward complete and strict-feedforward nonlinear systems. IEEE Trans. Autom. Control **55**(2), 287–303 (2010)
6. Karafyllis, I.: Stabilization by means of approximate predictors for systems with delayed input. SIAM J. Control Optim. **49**(3), 1100–1123 (2011)
7. Karafyllis, I., Krstic, M.: Stabilization of nonlinear delay systems using approximate predictors and high-gain observers. Automatica **49**(12), 3623–3631 (2013)
8. Karafyllis, I., Krstic, M.: Numerical schemes for nonlinear predictor feedback. Math. Control Signals Systems **26**(4), 519–564 (2014)
9. Karafyllis, I., Malisoff, M., de Queiroz, M., Krstic, M., Yang, R.: Predictor-based tracking for neuromuscular electrical stimulation. Int. J. Robust Nonlinear Control doi:10.1002/rnc.3211
10. Ahmed-Ali, T., Karafyllis, I., Lamnabhi-Lagarrigue, F.: Global exponential sampled-data observers for nonlinear systems with delayed measurements. Syst. Control Lett. **62**(7), 539–549 (2013)
11. Germani, G., Manes, C., Pepe, P.: A new approach to state observation of nonlinear systems with delayed output. IEEE Trans. Autom. Control **47**(1), 96–101 (2002)
12. Karafyllis, I., Krstic, M., Ahmed-Ali, T., Lamnabhi-Lagarrigue, F.: Global stabilization of nonlinear delay systems with a compact absorbing set. Int. J. Control **87**(5), 1010–1027 (2014)
13. Sun, X.-M., Liu, K.-Z., Wen, C., Wang, W.: Predictive control of nonlinear continuous networked control systems with large time-varying transmission delays and transmission protocols. Personal Communication (2014)
14. Karafyllis, I., Jiang, Z.-P.: Stability and Stabilization of Nonlinear Systems. Springer-verlag, London (2011)
15. Heemels, W., Teel, A., van de Wouw, N., Nesic, D.: Networked control systems with communication constraints: tradeoffs between transmission intervals, delays and performance. IEEE Trans. Autom. Control **55**(8), 1781–1796 (2010)