Sampling Intervals Enlargement for a Class of Parabolic Sampled-Data Observers

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Abstract. The problem of state observation is addressed for a class of parabolic systems governed by linear diffusion PDEs. An observer is designed that provides online estimates of the system (spatially distributed) state, based on time sampled output measurements. The observer is a fixed-gain involving an inter-sample output predictor, making the state trajectories (at the different spatial positions) continuous in time. The observer convergence is analyzed using Lyapunov's direct method, Writtinger's inequalities and other tools. Sufficient conditions for exponential convergence are established in terms of LMIs involving the sampling period and the observer gain. Interestingly, the conditions entail no limitation on the spatial domain length and no persistent excitation requirement.

1. Introduction

State observers are resorted to get online estimates of system state variables that are not accessible to measurements. They are beneficial when suitable physical sensors are missing or when their implementation is prohibitive or entails reliability issues. State observers are widely used in system control and system fault detection and diagnostics. During the three past decades, the problem of system observability and observer design has intensively been investigated especially for finite-dimension systems (FDSs). For linear systems, the Luenberger observer [2] and the Kalman filter [3] have been proposed in the early sixties. Several extensions to nonlinear systems have been developed, over the thirty lat years, including the high-gain observer (e.g.[3]-[6]), sliding-mode observers (e.g. [7]-[9]), Luenberger-like observers (e.g. [10]).

The problem of distributed parameter system (DPS) observability and observer design has also been given a great deal of interest, especially in recent years. The earliest works have focused on linear IDSs and a relatively complete theoretical framework exists since the nineties, including the infinite dimensional Luenberger observer (e.g. [11],[12]) and reference list therein. Boundary observer design of bilinear DPSs have been studied, in e.g. [13]-[15]. A unifying study of both interior and boundary observation for linear and bilinear systems is found in [16]. In [17], backstepping techniques have been used to design exponentially convergent boundary observers for a class of parabolic partial integro-differential equations. The problem of initial state recovery has also been given interest. In [18], an iterative algorithm is proposed to recover the initial state of a linear infinite dimensional system. The proposed algorithm generalizes various algorithms, proposed earlier for specific classes of systems, and stands as an alternative to methods based on Gramian inversion [19]. The ideas of [18] have been extended to some nonlinear infinite dimensional systems, using LMI techniques [20].

In this paper, we are interested in parabolic systems governed by linear diffusion type PDEs. The problem of designing sampled-data controllers for this class of systems has been dealt with in e.g. [21], using H-infinity and LMIs. Presently, the focus is made on sampled-data observer design. It is assumed that a sensor provides the spatially averaged value of the system over the whole spatial domain. An observer is designed that provides online estimates of the full spatially distributed state, based on time sampled output measurements. The observer involves a fixed injection gain and an inter-sample output predictor subject to periodic resetting. Accordingly, the latter turns out to be the only hybrid part of the observer. In particular, the state estimator is continuous-time and all generated state estimate trajectories are continuously varying. The observer convergence is analyzed using Lyapunov's direct method, Writtinger's inequalities and other tools. Sufficient conditions for exponential convergence are established in terms of LMIs involving the sampling period and the output injection gain. Interestingly, the conditions entail no limitation on the length of the spatial domain. The observer thus developed may be viewed as an extension of the hybrid adaptive observer proposed by [22] for FDSs.

The paper is organised as follows: first, the observation problem under study is formulated in Section 2; then, the observer design and analysis are dealt with in Section 3; a conclusion and reference list end the paper.

2. Observation Problem Statement

The system under study is governed by the following linear diffusion type PDE:

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \quad 0 \le x \le 1, \; t \ge 0 \tag{1a}$$

with the boundary condition:

$$u(1,t) = 0, \; u_x(0,t) = 0 \tag{1b}$$

where the state variable $u(x,t) \in \mathbf{R}$ is assumed to be bounded and the scalar parameter $\lambda > 0$ is known. The system is observed via the output signal:

$$y(t) = \int_0^1 u(x,t)dx \tag{2}$$

The goal is to generate accurate online estimates $\hat{u}(x,t)$ of the system state $u(x,t)$ ($0 \le x \le 1$; $t \ge 0$), based on the sampled output measurements $y(t_k)$, where the t_k's ($k \in \mathbf{N}$) denote the sampling instants. Presently, the case of constant sampling period $\tau = t_k - t_{k-1}$ is considered.

Note that the assumption $\lambda > 0$ ensures the system observability and guarantee the existence of an observer solving the considered observation problem. The case $\lambda < 0$ is ruled out because the system is then only detectable.

3. Observer Design and Analysis

3.1 Observer Design

To estimate the state of the system (1a-c)-(2), the following observer is proposed:

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \lambda\hat{u}(x,t) - K\left(\int_0^1 \hat{u}(\xi,t)d\xi - w(t)\right),\ 0 \le x \le 1,\ t \ge 0 \tag{3a}$$

$$\hat{u}(1,t) = 0,\ \ \hat{u}_x(0,t) = 0 \tag{3b}$$

$$\dot{w}(t) = \int_0^1 \hat{u}_{xx}(x,t) + \lambda\hat{u}(x,t)dx,\ t_k < t < t_{k+1} \tag{3c}$$

$$w(t_k) = y(t_k) \tag{3d}$$

where $\hat{u}(x,t)$ denotes the state estimate, $w(t)$ the output prediction between two successive sampling times, and K >0 is the output-injection gain. Clearly, the observer is composed of two parts: the state estimator (3a-b) and the output predictor (3c-d). The former features a feedback structure while the latter does not. Nevertheless, the predictor open-loop structure is compensated for by its periodic reinitialization. It turns out that, the predictor is the only hybrid part of the observer. The state estimator (3a-b) enjoys time continuity whatever the spatial position $0 \le x \le 1$.

The inter-sample predictor based observer (3a-d) can also be expressed in terms of a time-varying gain involving a ZOH innovation

$$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \lambda\hat{u}(x,t) - Ke^{-K(t-t_k)}\left(\int_0^1 \hat{u}(\xi,t_k)d\xi - y(t_k)\right),\ 0 \le x \le 1,\ t_k \le t < t_{k+1} \tag{4a}$$

$$\hat{u}(1,t) = 0,\ \ \hat{u}_x(0,t) = 0 \tag{4b}$$

Compared with (Bar Am and Fridman, 2014), the observer (4a-b) enjoys its time-varying gain feature. Notice that the varying gain is exponentially decreasing, over any sampling interval $t_k < t < t_{k+1}$. Accordingly, the information obtained at the sampling time t_k is progressively forgotten.

3.2 Observer Analysis

A suitable choice of the output injection gain and the sampling period τ will now be determined based on the analysis of the following errors:

$$\tilde{u}(x,t) = \hat{u}(x,t) - u(x,t) \text{ (state estimation error)} \tag{4a}$$

$$e(t) = w(t) - y(t) \quad \text{(output prediction error)} \tag{4b}$$

Then, it follows subtracting (side-by-side) equation (1a) to (3a) that, $\tilde{u}(x,t)$ undergoes the following equation:

$$\tilde{u}_t(x,t) = \tilde{u}_{xx}(x,t) + \lambda\tilde{u}(x,t) - K\left(\int_0^1 \tilde{u}(\xi,t)d\xi - e(t)\right),\ \ 0 \le x \le 1,\ t \ge 0 \tag{5a}$$

where the last term on the right side is obtained adding, within the parenthesis, the quantity $y(t) - \int_0^1 u(x,t)dx = 0$ (due to (2)). Equation (5a) is completed with the boundary conditions:

$$\tilde{u}(1,t) = 0,\ \tilde{u}_x(0,t) = 0 \tag{5b}$$

Similar equations are established for the error $e(t)$. First, note that by (2) the system output $y(t)$ undergoes the differential equation:

$$\dot{y}(t)=\int_0^1 u_{xx}(x,t)+\lambda u(x,t)dx \text{ (using (2) and (1a))}$$

Subtracting this equation to (3c) yields, successively:

$$\dot{e}(t)=\int_0^1 \tilde{u}_{xx}(x,t)+\lambda\tilde{u}(x,t)dx\,,\quad t_k<t<t_{k+1}$$

$$=\tilde{u}_x(1,t)+\lambda\int_0^1\tilde{u}(x,t)dx\,,\quad t_k<t<t_{k+1} \tag{6a}$$

where the last equation is obtained using (5b). Equation (6a) is completed with the following reinitialization equations, obtained from (3d):

$$e(t_k)=0 \tag{6b}$$

The error system (5a-b)-(6a-b) will now be analyzed considering the following Lyapunov function candidate:

$$V(t)=\frac{1}{2}\int_0^1\tilde{u}^2(x,t)dx+\frac{p_1}{2}\int_0^1\tilde{u}_x^2(x,t)dx+\frac{1}{2}\int_{t_k}^t(t-s+\tau)e^2(s)ds\,,\quad t_k\le t<t_{k+1} \tag{7}$$

with $k=0,1,2\ldots.$ and $p_1>0$ a scalar arbitrarily chosen.

The main result is described in the following theorem:

Theorem 1. Let the observer described by (3a-d) be applied to the system (1a-b)-(2). Let the output-injection gain K (in equation (3a)), the sampling period $\tau=t_k-t_{k-1}$ and the weighting coefficient in (7) be selected such that

$$K>2\lambda\,,\ \tau<\min\left\{\frac{p_1}{8},\frac{\varepsilon}{4\lambda}\left(\frac{K}{2}-\lambda-p_1K^2\right),\frac{1}{2p_1K^2+K}\right\},\ p_1<\min\left\{\frac{1}{4\lambda},\frac{1}{2K^2}\left(\frac{K}{2}-\lambda\right)\right\} \tag{8a}$$

In the case where $8\lambda<\pi^2(1-\varepsilon)$, the following additional conditions on K is required:

$$K<K_0 \text{ or } K>K_1 \tag{8b}$$

with $K_0=\dfrac{\gamma-\sqrt{\gamma^2-8\gamma\lambda}}{2}$, $K_1=\dfrac{\gamma+\sqrt{\gamma^2-8\gamma\lambda}}{2}$, and $\gamma=\pi^2(1-\varepsilon)$. Then, the estimation error $|\tilde{u}(x,t)|$ is exponentially vanishing as $t\to\infty$, uniformly in x, whatever $\tilde{u}(x,0)$, $(0<x<1)$ ■

Proof. The derivative of (7) along the trajectory of (5a-b)-(6a-b) writes:

$$\dot{V}(t)=\int_0^1\tilde{u}\ (x,t)\tilde{u}_t\ (x,t)dx+p_1\int_0^1\tilde{u}_x(x,t)\tilde{u}_{xt}(x,t)dx+\frac{d}{dt}\left(\int_{t_k}^t(t-s+\tau)\dot{e}^2(s)ds\right) \tag{9}$$

The different terms on the right side of (9) are separately analysed in the sequel. The first term develops as follows:

$$\int_0^1\tilde{u}(x,t)\tilde{u}_t\ (x,t)dx=\int_0^1\tilde{u}(x,t)\tilde{u}_{xx}(x,t)dx+\lambda\int_0^1\tilde{u}^2(x,t)dx$$

$$-K\int_0^1 \tilde{u}(x,t)\int_0^1 \tilde{u}(\xi,t)d\xi dx + K\int_0^1 e(t)\tilde{u}(x,t)dx$$

$$= -\int_0^1 \tilde{u}_x^2(x,t)dx + \lambda\int_0^1 \tilde{u}^2(x,t)dx$$

$$-K\int_0^1 \tilde{u}(x,t)\int_0^1 \tilde{u}(\xi,t)d\xi dx + K\int_0^1 e(t)\tilde{u}(x,t)dx \qquad (10)$$

using (5b). The third term on the right side of (10) develops as follows:

$$-K\int_0^1 \tilde{u}(x,t)\int_0^1 \tilde{u}(\xi,t)d\xi dx = -K\int_0^1 \tilde{u}(x,t)\int_0^1 \tilde{u}(x,t)d\xi dx$$

$$-K\int_0^1 \tilde{u}(x,t)\left(\int_0^1 (\tilde{u}(\xi,t) - \tilde{u}(x,t))d\xi\right)dx$$

$$= -K\int_0^1 \tilde{u}^2(x,t)dx - K\int_0^1 \tilde{u}(x,t)\left(\int_0^1 (\tilde{u}(\xi,t) - \tilde{u}(x,t))d\xi\right)dx$$

$$= -K\int_0^1 \tilde{u}^2(x,t)dx + K\int_0^1 \tilde{u}(x,t)g(x)dx \qquad (11a)$$

where $g(x)$ is defined by:

$$g(x) = -\int_0^1 (\tilde{u}(\xi,t) - \tilde{u}(x,t))d\xi \qquad (11b)$$

It is readily seen that $\int_0^1 g(x)dx = 0$. Then, by one of Writtinger's inequalities (see e.g. Lemma 1.1 in [21]) one has:

$$\int_0^1 g^2(x)dx \le \frac{1}{\pi^2}\int_0^1 g_x^2(x)dx \qquad (11c)$$

It is also easily obtained from (11b) that, $g_x(x) = \tilde{u}_x$. Then, (11c) explicitly writes as follows:

$$\int_0^1 g^2(x)dx \le \frac{1}{\pi^2}\int_0^1 \tilde{u}_x^2 dx \qquad (11d)$$

Using (11a-d), one gets from (10):

$$\int_0^1 \tilde{u}(x,t)\tilde{u}_t(x,t)dx \le -\frac{1}{2}\int_0^1 \tilde{u}_x^2(x,t)dx - \frac{\pi^2}{2}\int_0^1 g^2(x,t)dx$$

$$+\lambda\int_0^1 \tilde{u}^2(x,t)dx - K\int_0^1 \tilde{u}^2(x,t)dx$$

$$+K\int_0^1 \tilde{u}(x,t)g(x)dx + \frac{K}{2}e^2(t)$$

$$+\frac{K}{2}\int_0^1 \tilde{u}^2(x,t)dx \text{ (using Young inequality)}$$

$$\le -\varepsilon\left(\frac{K}{2} - \lambda\right)\int_0^1 \tilde{u}^2(x,t)dx - \frac{1}{2}\int_0^1 \tilde{u}_x^2(x,t)dx + \frac{K}{2}e^2(t)$$

$$+\int_0^1 \left((1-\varepsilon)\left(\lambda - \frac{K}{2}\right)\tilde{u}^2(x,t) - \frac{\pi^2}{2}g^2(x) + K\tilde{u}(x,t)g(x)\right)dx \qquad (11e)$$

where $0 < \varepsilon < 1$ is arbitrary. The last term on the right side of (11e) can be condensed as follows:

$$(1-\varepsilon)(\lambda-\frac{K}{2})\tilde{u}^2(x,t)-\frac{\pi^2}{2}g^2(x)+K\tilde{u}(x,t)g(x)=\left[\tilde{u}(x,t)\ g(x)\right]\begin{bmatrix}(1-\varepsilon)(\lambda-\frac{K}{2}) & \frac{K}{2}\\ \frac{K}{2} & -\frac{\pi^2}{2}\end{bmatrix}\begin{bmatrix}\tilde{u}(x,t)\\ g(x)\end{bmatrix}$$

It is readily checked that the matrix on the right side of this equation is negative semidefinite iff:

$$\lambda-\frac{K}{2}<0 \quad \text{and} \quad -\pi^2(1-\varepsilon)(2\lambda-K)-K^2\leq 0 \tag{11f}$$

The second inequality in (11f) involves the quadratic function $K^2-\gamma K+2\lambda\gamma$ and the associated discriminant $\Delta=\gamma^2-8\gamma\lambda$ with $\gamma=\pi^2(1-\varepsilon)$. This discriminant is positive if $\pi^2(1-\varepsilon)>8\lambda$. Then, the second inequality in (10f) holds if $K<K_0$ or $K>K_1$. In the case where $\pi^2(1-\varepsilon)<8\lambda$, the second inequality in (11f) holds whatever $K>0$. That is, under conditions (8a-b) one has, $(1-\varepsilon)(\lambda-\frac{K}{2})\tilde{u}^2(x,t)-\frac{\pi^2}{2}g^2(x)+K\tilde{u}(x,t)g(x)\leq 0$, whatever x,t. Then, inequality (10e) yields:

$$\int_0^1\tilde{u}(x,t)\tilde{u}_t(x,t)dx\leq-\varepsilon(\frac{K}{2}-\lambda)\int_0^1\tilde{u}^2(x,t)dx-\frac{1}{2}\int_0^1\tilde{u}_x^2(x,t)dx+\frac{K}{2}e^2(t) \tag{12}$$

In turn, the second term on the right side of (9) develops as follows:

$$p_1\int_0^1\tilde{u}_x(x,t)\tilde{u}_{xt}(x,t)dx=p_1\int_0^1\tilde{u}_x(x,t)\tilde{u}_{tx}(x,t)dx$$

$$=p_1\int_0^1\tilde{u}_x(x,t)\tilde{u}_{tx}(x,t)dx$$

$$=p_1\int_0^1\tilde{u}_x(x,t)\tilde{u}_{xxx}(x,t)dx+p_1\lambda\int_0^1\tilde{u}_x^2(x,t)dx \quad \text{(using (5a))}$$

$$=p_1\left[\tilde{u}_x(x,t)\tilde{u}_{xx}(x,t)\right]_{x=0}^{x=1}-p_1\int_0^1\tilde{u}_{xx}^2(x,t)dx+p_1\lambda\int_0^1\tilde{u}_x^2(x,t)dx$$

$$=p_1\tilde{u}_x(1,t)\tilde{u}_{xx}(1,t)-p_1\int_0^1\tilde{u}_{xx}^2(x,t)dx+p_1\lambda\int_0^1\tilde{u}_x^2(x,t)dx \tag{13a}$$

where the two last equalities are obtained using an integration by part and (5b). Note that the first equality holds because $\tilde{u}$ is smooth (see a similar proof in Appendix A of [21]). Now, let us closely look at the term $\tilde{u}_{xx}(1,t)$. It follows from (5b) that $\tilde{u}_t(1,t)=0$. Then, one has letting $t=1$ in (5a):

$$0=\tilde{u}_{xx}(1,t)-K\left(\int_0^1\tilde{u}(\xi,t)d\xi-e(t)\right)$$

which gives:

$$\tilde{u}_{xx}(1,t)=K\left(\int_0^1\tilde{u}(\xi,t)d\xi-e(t)\right) \tag{13b}$$

Also, the quantity $\tilde{u}_x(1,t)$ rewrites as follows:

$$\tilde{u}_x(1,t)=\int_0^1\tilde{u}_{xx}(x,t)dx \quad \text{(using (5b))} \tag{13c}$$

Combining (13a-c) one gets:

$$p_1\int_0^1 \tilde{u}_x(x,t)\tilde{u}_{xt}(x,t)dx = p_1 K\left(\int_0^1 \tilde{u}(\xi,t)d\xi - e(t)\right)\int_0^1 \tilde{u}_{xx}(x,t)dx$$

$$- p_1\int_0^1 \tilde{u}_{xx}^2(x,t)dx + p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$= \frac{p_1 K}{\xi\sqrt{p_1}}\left(\int_0^1 \tilde{u}(\xi,t)d\xi - e(t)\right)\times \xi\sqrt{p_1}\int_0^1 \tilde{u}_{xx}(x,t)dx$$

$$- p_1\int_0^1 \tilde{u}_{xx}^2(x,t)dx + p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$\leq \frac{1}{2}\left(\frac{p_1 K}{\xi\sqrt{p_1}}\right)^2\left(\int_0^1 \tilde{u}(\xi,t)d\xi - e(t)\right)^2 + \frac{\left(\xi\sqrt{p_1}\right)^2}{2}\left(\int_0^1 \tilde{u}_{xx}(x,t)dx\right)^2$$

$$- p_1\int_0^1 \tilde{u}_{xx}^2(x,t)dx + p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx \quad \text{(using Young inequality)}$$

$$\leq \frac{1}{2}\left(\frac{p_1 K}{\xi\sqrt{p_1}}\right)^2\left(2\left(\int_0^1 \tilde{u}(\xi,t)d\xi\right)^2 + e^2(t)\right) + \frac{\left(\xi\sqrt{p_1}\right)^2}{2}\left(\int_0^1 \tilde{u}_{xx}(x,t)dx\right)^2$$

$$- p_1\int_0^1 \tilde{u}_{xx}^2(x,t)dx + p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx \quad \text{(using Young inequality)}$$

$$\leq \left(\frac{p_1 K^2}{\xi^2}\right)\left(\left(\int_0^1 \tilde{u}(\xi,t)d\xi\right)^2 + e^2(t)\right) + \frac{p_1\xi^2}{2}\left(\int_0^1 \tilde{u}_{xx}(x,t)dx\right)^2$$

$$- p_1\int_0^1 \tilde{u}_{xx}^2(x,t)dx + p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$\leq \frac{p_1 K^2}{\xi^2}\left(\int_0^1 \tilde{u}^2(\xi,t)d\xi + e^2(t)\right) + \frac{p_1\xi^2}{2}\int_0^1 \tilde{u}_{xx}^2(x,t)dx$$

$$- p_1\int_0^1 \tilde{u}_{xx}^2(x,t)dx + p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx \quad \text{(using Schwartz inequality)}$$

$$\leq \frac{p_1 K^2}{\xi^2}\left(\int_0^1 \tilde{u}^2(\xi,t)d\xi + e^2(t)\right) - p_1(1-\frac{\xi^2}{2})\int_0^1 \tilde{u}_{xx}^2(x,t)dx$$

$$+ p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx \qquad (14)$$

Combining (12) and (14) one gets:

$$\int_0^1 \tilde{u}(x,t)\tilde{u}_t(x,t)dx + p_1\int_0^1 \tilde{u}_x(x,t)\tilde{u}_{xt}(x,t)dx \leq -\varepsilon(\frac{K}{2}-\lambda)\int_0^1 \tilde{u}^2(x,t)dx - \frac{1}{2}\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$+ \frac{K}{2}e^2(t) + \frac{p_1 K^2}{\xi^2}\left(\int_0^1 \tilde{u}^2(\xi,t)d\xi + e^2(t)\right)$$

$$- p_1(1-\frac{\xi^2}{2})\int_0^1 \tilde{u}_{xx}^2(x,t)dx + p_1\lambda\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$\leq -\varepsilon(\frac{K}{2} - \lambda - \frac{p_1 K^2}{\xi^2})\int_0^1 \tilde{u}^2(x,t)dx - (\frac{1}{2} - p_1\lambda)\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$- p_1(1-\frac{\xi^2}{2})\int_0^1 \tilde{u}_{xx}^2(x,t)dx + \left(\frac{p_1 K^2}{\xi^2}+\frac{K}{2}\right)e^2(t) \quad (15)$$

Finally, the third term on the right side of (9) develops as follows:

$$\frac{d}{dt}\left(\int_{t_k}^t (t-s+\tau)\dot{e}^2(s)ds\right) = \tau\dot{e}^2(t) - \int_{t_k}^t \dot{e}^2(s)ds$$

$$= \tau(\tilde{u}_x(1,t)+\lambda\int_0^1 \tilde{u}(x,t)dx)^2 - \int_{t_k}^t \dot{e}^2(s)ds \quad \text{(using (6a))}$$

$$\le 2\tau\tilde{u}_x^2(1,t) + 2\tau\lambda\left(\int_0^1 \tilde{u}(x,t)dx\right)^2 - \int_{t_k}^t \dot{e}^2(s)ds \quad \text{(using Young inequality)}$$

$$\le 2\tau\left(\int_0^1 \tilde{u}_{xx}(x,t)dx\right)^2 + 2\tau\lambda\left(\int_0^1 \tilde{u}(x,t)dx\right)^2 - \int_{t_k}^t \dot{e}^2(s)ds \quad \text{(using (13c))}$$

$$\le 2\tau\int_0^1 \tilde{u}_{xx}^2(x,t)dx + 2\tau\lambda\int_0^1 \tilde{u}^2(x,t)dx - \int_{t_k}^t \dot{e}^2(s)ds \quad (16)$$

where the last inequality is obtained using Schwartz inequality twice. Using (15) and (16), it follows from (9) that:

$$\dot{V}(t) \le -\left(\varepsilon(\frac{K}{2}-\lambda-\frac{p_1K^2}{\xi^2})-2\tau\lambda\right)\int_0^1 \tilde{u}^2(x,t)dx - (\frac{1}{2}-p_1\lambda)\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$-\left(p_1(1-\frac{\xi^2}{2})-2\tau\right)\int_0^1 \tilde{u}_{xx}^2(x,t)dx - \int_{t_k}^t \dot{e}^2(s)ds + \left(\frac{p_1K^2}{\xi^2}+\frac{K}{2}\right)e^2(t) \quad (17)$$

for all $t_k \le t < t_{k+1}$. The last term on the right side of (17) is worked out as follows:

$$e^2(t) = \left(\int_{t_k}^t \dot{e}(s)ds\right)^2 \le \tau\int_{t_k}^t \dot{e}^2(s)ds$$

where we have used Schwartz inequality and the fact that $e(t_k)=0$. This together with (17) yields:

$$\dot{V}(\tilde{u}(.,t),e(t)) \le -\left(\varepsilon(\frac{K}{2}-\lambda-\frac{p_1K^2}{\xi^2})-2\tau\lambda\right)\int_0^1 \tilde{u}^2(x,t)dx - (\frac{1}{2}-p_1\lambda)\int_0^1 \tilde{u}_x^2(x,t)dx$$

$$-\left(p_1(1-\frac{\xi^2}{2})-2\tau\right)\int_0^1 \tilde{u}_{xx}^2(x,t)dx - \left(1-\tau(\frac{p_1K^2}{\xi^2}+\frac{K}{2})\right)\int_{t_k}^t \dot{e}^2(s)ds \quad (18)$$

for all $t_k \le t < t_{k+1}$. At this point, it is worth recalling that, the parameters $p_1 > 0, \xi > 0,$ and $\tau > 0$ are still arbitrary, while $0 < \varepsilon < 1$ and K satisfies (8a-b). Then, the third term on the right side of (17) suggests that ξ and τ must be such that $\xi < \sqrt{2}$ and $2\tau < p_1(1-\frac{\xi^2}{2})$. Accordingly, we simply let:

$$\xi = 1 \text{ and } \tau < \frac{p_1}{8} \quad (19a)$$

Doing so, one gets:

$$-\left(p_1(1-\frac{\xi^2}{2})-2\tau \right) < -\frac{p_1}{4} \tag{19b}$$

In turn, the second term on the right side of (17) suggests that p_1 must be selected so that $p_1\lambda < 1/2$. For simplicity, we let:

$$p_1 < \frac{1}{4\lambda} \tag{19c}$$

Doing so, one gets:

$$-(\frac{1}{2}-p_1\lambda) < -\frac{1}{4} \tag{19d}$$

The first term on the right side of (17) suggests that p_1 and τ must be selected so that $\varepsilon(\frac{K}{2}-\lambda-\frac{p_1K^2}{\xi^2})-2\tau\lambda > 0$. As $\xi = 1$ (by (19a)), a sufficient condition is that:

$$p_1 < \frac{1}{2K^2}(\frac{K}{2}-\lambda) \text{ and } \tau < \frac{\varepsilon}{4\lambda}(\frac{K}{2}-\lambda-p_1K^2) \tag{19e}$$

Doing so, one gets $\frac{K}{2}-\lambda-p_1K^2 > \frac{1}{2}(\frac{K}{2}-\lambda)$ and

$$-\left(\varepsilon(\frac{K}{2}-\lambda-\frac{p_1K^2}{\xi^2})-2\tau\lambda \right) < -\frac{\varepsilon}{4}(\frac{K}{2}-\lambda) \tag{20f}$$

Finally, the last term on the right side of (17) suggest that τ must be selected such that $1-\tau(p_1K^2+\frac{K}{2}) > 0$, using the fact that $\xi = 1$. A sufficient condition is to let:

$$\tau < \frac{1}{2}\frac{1}{p_1K^2+\frac{K}{2}} \tag{20g}$$

Doing so, one ensures that:

$$-\left(1-\tau(\frac{p_1K^2}{\xi^2}+\frac{K}{2}) \right) < \frac{1}{2} \tag{20h}$$

Combining (17) and (18a-h), it follows that, if

$$p_1 < \min\left\{ \frac{1}{4\lambda}, \frac{1}{2K^2}(\frac{K}{2}-\lambda) \right\} \quad \text{and} \quad \tau < \min\left\{ \frac{p_1}{8}, \frac{\varepsilon}{4\lambda}(\frac{K}{2}-\lambda-p_1K^2), \frac{1}{2p_1K^2+K} \right\} \tag{21a}$$

then,

$$\begin{aligned} \dot{V}(\tilde{u}(.,t),e(t)) &\le -\left(\frac{\varepsilon}{4}(\frac{K}{2}-\lambda) \right) \int_0^1 \tilde{u}^2(x,t)dx - \frac{1}{4}\int_0^1 \tilde{u}_x^2(x,t)dx - \frac{p_1}{4}\int_0^1 \tilde{u}_{xx}^2(x,t)dx - \frac{1}{2}\int_{t_k}^t \dot{e}^2(s)ds \\ &\le -\sigma V(\tilde{u}(.,t),e(t)), \quad t_k \le t < t_{k+1} \end{aligned} \tag{21b}$$

with

$$\sigma = \min\left\{\frac{1}{4}, \frac{p_1}{4}, \frac{\varepsilon}{4}(\frac{K}{2} - \lambda)\right\} \qquad (21c)$$

It readily follows from (21) that:

$$V(\tilde{u}(.,t), e(t)) = V(t_k)e^{-\sigma(t-t_k)}, \quad t_k \le t < t_{k+1} \qquad (22a)$$

On the other hand, it readily follows from (7):

$$V(t_{k+1}) = \frac{1}{2}\int_0^1 \tilde{u}^2(x, t_{k+1})dx + \frac{p_1}{2}\int_0^1 \tilde{u}_x^2(x, t_{k+1})dx, \qquad (22b)$$

$$V(t_{k+1}^-) \overset{def}{=} \lim_{\substack{t \to t_{k+1} \\ t < t_{k+1}}} V(t) = \frac{1}{2}\int_0^1 \tilde{u}^2(x, t_{k+1})dx + \frac{p_1}{2}\int_0^1 \tilde{u}_x^2(x, t_{k+1})dx + \frac{1}{2}\int_{t_k}^{t_{k+1}} (t_{k+1} - s + \tau)e^2(s)ds \qquad (22c)$$

Comparing (22c) and (22b), one gets that $V(t_{k+1}) \le V(t_{k+1}^-)$, for all k. This, together with (22a), implies that:

$$V(t) = V(t_0)e^{-\sigma(t-t_0)}, \text{ for all } t \ge t_0 \qquad (23)$$

Theorem 1 is established ■

4. Conclusion

We have addressed the problem of estimating the state of the class of PDSs described by the model (1a-b)-(2) which is basically a diffusive parabolic PDE. The main features of the observer (3a-d) are its time sampled-data nature, its constant output-injection gain, and an inter-sample output predictor. The last feature is crucial in ensuring time-continuity of all state estimate trajectories (at all spatial positions). The analysis made in Theorem 1 has lead to sufficient conditions for the observer to be exponentially convergent. The conditions involve the sampling period and the injection gain. Interestingly, the conditions entail no limitation on the spatial domain length and no persistent excitation requirement. The present study can be extended to the case of multiple sensors, each one of them providing the averaged value of the spatial state over a subdomain $[x_i, x_{i+1}]$, i.e.

$$y_i(t_k) = \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} u(x, t_k)dx; \quad (i = 0 \cdots p-1).$$

where $0 = x_0 < x_1 < \cdots < x_p = 1$ for some finite integer p.

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