Smooth stabilisation of nonholonomic robots subject to disturbance

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Alejandro Donaire[1], Jose Guadalupe Romero[2], Tristan Perez[3] and Romeo Ortega[2]

***Abstract*— In this paper, we address the problem of stabilisation of robots subject to nonholonommic constraints and external disturbances using port-Hamiltonian theory and smooth time-invariant control laws. This should be contrasted with the commonly used switched or time-varying laws. We propose a control design that provides asymptotic stability of an manifold (also called relative equilibria)—due to the Brockett condition this is the only type of stabilisation possible using smooth time-invariant control laws. The equilibrium manifold can be shaped to certain extent to satisfy specific control objectives. The proposed control law also incorporates integral action, and thus the closed-loop system is robust to unknown constant disturbances. A key step in the proposed design is a change of coordinates not only in the momentum, but also in the position vector, which differs from coordinate transformations previously proposed in the literature for the control of nonholonomic systems. The theoretical properties of the control law are verified via numerical simulation based on a robotic ground vehicle model with differential traction wheels and non co-axial centre of mass and point of contact.**

I. Introduction

The study of systems subject to nonholonomic constraints have been developed within the realm of analytical mechanics [14], [5], [4]. The complexity and highly nonlinear dynamics of nonholonomic-mechanical systems (NHMS) make the motion control problem challenging [8], [7], [4]. A key feature that distinguishes the control of NHMS from that of holonomic systems is that in the former, it is not possible to stabilise an isolated equilibrium with a smooth state-feedback control law. The best one can achieve with smooth control laws is to stabilise an equilibrium manifold also known as relative equilibria. This fact follows from Brockett's necessary condition—see for example [4] (p. 303).

Wheeled robots are typical examples nonholonomic mechanical systems. The dynamics of these systems can be described using either Euler-Lagrange or Hamiltonian formulations [12], [21], [4]. Mechanical systems with nonholonomic constraints may also be represented as driftless systems, where the input to these systems are usually velocities instead of forces. This leads to kinematic models for which the control law is designed. Another approach considers the open-loop system in a canonical chained form for control design. In this paper, we adopt the Hamiltonian representation. For control designs based on driftless and canonical chained forms see [20], [8], [1], [4]. The survey in [13] provides a general picture on control of NHMS.

The natural approach to control port-Hamiltonian (pH) systems is the classical interconnection and damping assignment passivity based control (IDA-PBC)—see [15] for a survey. In the case of mechanical pH systems with nonholonomic constraints IDA-PBC has been used in [3], [19], [11]. In this paper, we follow the Hamiltonian formulation proposed in [21] to describe NHMS with disturbances. Then, we design a dynamic controller to stabilise the positions of the nonholonomic system to an equilibrium manifold. The design is robust to unknown constant disturbances in the sense that these disturbances do not modify the relative equilibria. In this way, we extend previous results on smooth stabilisation of nonholonomic pH mechanical systems by considering the disturbances rejection problem. The development here also extends results on the use of integral action for unconstrained pH proposed in [9], [16] to pH mechanical systems with nonholonomic constraints. In particular, we use a change of coordinate to to assign a full rank dissipation matrix first proposed in [10], and then generalised for mechanical systems in [17], [18].

The remaining of the paper is organised as follows. Section 3 presents the port-Hamiltonian models of NHMS. The control design of the smooth control law is developed in Section 4. In section 5, we present a case study with numerical simulations to illustrate the application of the developed theory. The paper is concluded in Section 6.

II. Notation

We denote the function $|x|^2 := x^\top x$ for $x \in \mathbb{R}^n$. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, we define the differential operators

$$\nabla f := \left(\frac{\partial f}{\partial x}\right)^\top, \; \nabla^2 f := \left(\frac{\partial^2 f}{\partial x^2}\right)^\top, \; \nabla_{x_i} f := \left(\frac{\partial f}{\partial x_i}\right)^\top,$$

where ∇f a column vector, $x_i \in \mathbb{R}^p$, with $p \le n$, x_i is an subset of components of the vector x. For a mapping $g : \mathbb{R}^n \to \mathbb{R}^m$, its Jacobian matrix ($m \times n$) is defined as

$$\nabla g := \begin{bmatrix} (\nabla g_1)^\top \\ \vdots \\ (\nabla g_m)^\top \end{bmatrix},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ is the i-th element of g.

III. Hamiltonian formulation of nonholonomic mechanical systems

This section follows the development in [21]. We proposed, however, a different coordinate transformation to

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obtain a Hamiltonian function with an identity inertia matrix. This novel coordinate transformation is inspired by the design of observers proposed in [22].

Consider the NHMS described by

$$\begin{aligned}\dot{q} &= \nabla_p H \\ \dot{\mathbf{p}} &= -\nabla_q H + A(q)\lambda + G(q)(u+d),\end{aligned} \tag{1}$$

with Hamiltonian

$$H(q,\mathbf{p}) = \frac{1}{2}\mathbf{p}^\top M^{-1}(q)\mathbf{p} + V(q), \tag{2}$$

and Pfaffian nonholonomic constraints

$$A^\top(q)\,\dot{q} = 0. \tag{3}$$

where $\mathbf{p} \in \mathbb{R}^n$ and $q \in \mathbb{R}^n$ are the generalised momentum and position variables, and $A : \mathbb{R}^n \to \mathbb{R}^{n\times k}$, with $k < n$ and rank $(A) = k$. The constraint forces $A(q)\lambda$ are computed such that (3) is satisfied $\forall t$. The vector $u \in \mathbb{R}^m$ represents the control inputs and $d \in \mathbb{R}^m$ represents disturbances. The inclusion of disturbances on the model and its rejection is a novel contribution of this paper.
We next introduce a momentum transformation that in addition to the elimination of nonholonomic constraints developed in [21] leads to an identity mass matrix. The resulting transformed system simplifies the subsequent control design.

Proposition 3.1: Consider the full-rank matrix $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(q) = \begin{bmatrix} T^{-\top}(q)S^\top(q) \\ A^\top(q)M^{-1}(q)\end{bmatrix}, \tag{4}$$

where $S : \mathbb{R}^n \to \mathbb{R}^{n\times(n-k)}$ satisfies that rank $(S) = n-k$, and $A^\top(q)S(q) = 0$. The matrix $T : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is such that

$$T^\top(q)T(q) = S^\top(q)M(q)S(q).$$

Consider also the momentum transformation

$$\tilde{p} = \Phi(q)\mathbf{p},$$

and the partition of the new momenta

$$\tilde{p} = \begin{bmatrix} p \\ p_o \end{bmatrix}$$

where $p \in \mathbb{R}^{n-k}$ and $p_o \in \mathbb{R}^k$.

Then, the dynamics of the nonholonomic system (1) can be written in pH form

$$\begin{aligned}\begin{bmatrix}\dot{q} \\ \dot{p}\end{bmatrix} &= \begin{bmatrix} 0 & L(q) \\ -L^\top(q) & N(q,p)\end{bmatrix}\nabla W(q,p) \\ &+ \begin{bmatrix} 0 \\ G_c(q)\end{bmatrix}(u+d),\end{aligned} \tag{5}$$

where

$$W(q,p) = \frac{1}{2}|p|^2 + V(q) \tag{6}$$

is the new Hamiltonian function, and the matrices $L(q)$, $N(q,p)$ and $G_c(q)$ are as follows

$$\begin{aligned} L(q) &= S(q)T^{-1}(q), && (7)\\ N(q,p) &= \sum_{i=1}^n \nabla_{q_i}(T^{-\top}S^\top)\mathbf{p}e_i^\top ST^{-1} \\ &- \sum_{i=1}^n (ST^{-1})^\top e_i \mathbf{p}^\top \nabla_{q_i}^\top (T^{-\top}S^\top)\Big|_{\mathbf{p}=\Phi^{-1}p} && (8)\\ G_c(q) &= T^{-\top}(q)S^\top(q)G(q). && (9)\end{aligned}$$

Proof: We first write the Hamiltonian H in (2) as a function of the new momenta $\tilde{p}$:

$$\begin{aligned} H(q,\mathbf{p})\Big|_{\mathbf{p}=\Phi^{-1}\tilde{p}} &= \frac{1}{2}\tilde{p}^\top\Phi^{-\top}M^{-1}\Phi^{-1}\tilde{p} + V(q) \\ &= \frac{1}{2}\tilde{p}^\top \begin{bmatrix} I_{n-k} & 0 \\ 0 & A^\top M^{-1}A\end{bmatrix}^{-1}\tilde{p} \\ &+ V(q) = \tilde{H}(q,\tilde{p}).\end{aligned} \tag{10}$$

The constraint (3) in the new momenta leads to

$$\begin{aligned} A^\top M^{-1}\mathbf{p} &= A^\top\Phi^\top\Phi^{-\top}M^{-1}\Phi^{-1}\tilde{p} = A^\top\Phi^\top\nabla_{\tilde{p}}\tilde{H} \\ &= \begin{bmatrix} 0 & A^\top M^{-1}A\end{bmatrix}\nabla_{\tilde{p}}\tilde{H} = 0,\end{aligned} \tag{11}$$

which implies that $\nabla_{p_o}\tilde{H} = 0$. Then, it follows that the Hamiltonian (2) in the new coordinates is (6).

We can then write the dynamics of q as follows:

$$\begin{aligned}\dot{q} &= M^{-1}p = \Phi^\top\Phi^{-\top}M^{-1}\Phi^{-1}\tilde{p} = \Phi^\top\nabla_{\tilde{p}}\tilde{H} \\ &= \begin{bmatrix} ST^{-1} & M^{-1}A\end{bmatrix}\nabla_{\tilde{p}}\tilde{H} \\ &= ST^{-1}\nabla_p W(q,p) = L\nabla_p W(q,p),\end{aligned} \tag{12}$$

which is the state equation for q in (5).

We now write the constraint forces in (1) as a function of the states to build the dynamics (5). We compute the time derivative of $A^\top\dot{q} = 0$, namely,

$$\begin{aligned}\frac{d}{dt}[A^\top M^{-1}\mathbf{p}] &= \nabla_q(A^\top M^{-1}\mathbf{p})\dot{q} + \nabla_{\mathbf{p}}(A^\top M^{-1}\mathbf{p})\dot{\mathbf{p}} \\ 0 &= \nabla_q(A^\top M^{-1}\mathbf{p})\nabla_{\mathbf{p}}H + A^\top M^{-1} \\ &\quad [-\nabla_q H + A\lambda + G(u+d)],\end{aligned}$$

from which we obtain

$$\begin{aligned}\lambda &= -(A^\top M^{-1}A)^{-1}\Big[\nabla_q(A^\top M^{-1}\mathbf{p})M^{-1}\mathbf{p} \\ &\quad -A^\top M^{-1}\nabla_q H + A^\top M^{-1}G(u+d)\Big].\end{aligned} \tag{13}$$

The state equation for the new momentum p_o is as follows

$$
\begin{aligned}
\dot{p}_o &= \frac{d}{dt}(A^\top M^{-1})\mathbf{p} + A^\top M^{-1}\dot{\mathbf{p}} = \frac{d}{dt}(A^\top M^{-1})\mathbf{p} \\
&+ A^\top M^{-1}\left[-\nabla_q H + A\lambda + G(u+d)\right] \\
&= \frac{d}{dt}(A^\top M^{-1})\mathbf{p} - A^\top M^{-1}\nabla_q H + A^T M^{-1}A\lambda \\
&+ A^\top M^{-1}G(u+d) \\
&= \frac{d}{dt}(A^\top M^{-1})\mathbf{p} - A^\top M^{-1}\nabla_q H \\
&+ \Big[-\frac{d}{dt}(A^\top M^{-1})\mathbf{p} + A^\top M^{-1}\nabla_q H \\
&- A^\top M^{-1}G(u+d)\Big] + A^\top M^{-1}G(u+d) = 0,
\end{aligned}
$$

which implies that there is no motion along the coordinates p_o. The state equation for the new momentum variable p becomes

$$
\begin{aligned}
\dot{p} &= \frac{d}{dt}(T^{-\top}S^\top)\mathbf{p} + T^{-\top}S^\top\dot{\mathbf{p}} = \frac{d}{dt}(T^{-\top}S^\top)\mathbf{p} \\
&- T^{-\top}S^\top\nabla_q H + T^{-\top}S^\top A\lambda + T^{-\top}S^\top G(u+d) \\
&= \frac{d}{dt}(T^{-\top}S^\top)\mathbf{p} - T^{-\top}S^\top\nabla_q H + T^{-\top}S^\top G(u+d) \\
&= \sum_{i=1}^{n}\nabla_{q_i}(T^{-\top}S^\top)e_i^\top\dot{q}\mathbf{p} - \frac{1}{2}T^{-\top}S^\top\nabla_q[\mathbf{p}^\top M^{-1}\mathbf{p}] \\
&- T^{-\top}S^\top\nabla_q V(q) + T^{-\top}S^\top G(u+d) \\
&= \Bigg[\sum_{i=1}^{n}\nabla_{q_i}(T^{-\top}S^\top)\mathbf{p}e_i^\top ST^{-1} \\
&- \sum_{i=1}^{n}(ST^{-1})^\top e_i\mathbf{p}^\top\nabla_{q_i}^\top(T^{-\top}S^\top)\Bigg]_{\mathbf{p}=\Phi^{-1}p}\nabla_p W \\
&- T^{-\top}S^\top\nabla_q W(q,p) + T^{-\top}S^\top G(u+d) \\
&= N(q,p)\nabla_p W(q,p) - L^\top\nabla_q W(q,p) + G_c(u+d),
\end{aligned}
$$

which is the state equation for p in (5). Then, the nonholonomic dynamics (1) can be written in terms of q and p as the Hamiltonian system (5). ■

We next use the transformed pH model for control design.

IV. CONTROL DESIGN FOR SMOOTH STABILISATION OF AN EQUILIBRIUM MANIFOLD.

From Brockett's necessary condition, it follows that for NHMS it is impossible to stabilise equilibrium points asymptotically with a $\mathcal{C}^1$-control law. With such control, however, it is possible to stabilise the system to an equilibrium manifold—see for example [8], [4]. In this section, we propose a control law to stabilise an equilibrium manifold for NHMS with disturbances. This control law is robust to unknown constant disturbances in the sense that the convergence of the system to the target equilibrium manifold is ensured despite the presence of this kind of disturbances.

We consider the problem of finding a smooth control law that stabilises the system to an equilibrium manifold $\mathcal{M}_s = \{(q,p)|p=0\}$ for the system (1).

A. Assumptions

The following assumptions are made.

A1 The matrix G_c is invertible, which is satisfied if $\mathrm{rank}(G) = n-k$.

A2 Consider the partition of $L = [L_1^\top, L_2^\top]^\top$ in the system (5), where the matrix $L_1 : \mathbb{R}^n \to \mathbb{R}^{(n-k)\times(n-k)}$ and the matrix $L_2 : \mathbb{R}^n \to \mathbb{R}^{k\times(n-k)}$. We further assume that this partition of L is such that L_1 is invertible.

If L_1 is non-invertible, we can assume that there exists a mapping $\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that the coordinate transformation $w = \pi(q)$ satisfies

$$\dot{w} = \nabla_q\pi(q)\,L(q)\big|_{q=\pi^{-1}(w)}p = Q(w)p \tag{14}$$

and the partition of $Q = [Q_1^\top, Q_2^\top]^\top$ is such that Q_1 invertible. Then, the dynamics (5) in closed loop with the control law

$$u = \hat{u} + v = G_c^{-1}\left[L^\top\nabla_q W - L^\top\nabla_q\pi\nabla_w U\right] + v, \tag{15}$$

can be written in coordinates w as follows

$$\begin{bmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0_{n-k} & 0 & Q_1\\ 0 & 0_k & Q_2\\ -Q_1^\top & -Q_2^\top & J\end{bmatrix}\nabla U + \begin{bmatrix}0\\0\\G_w\end{bmatrix}(v+d), \tag{16}$$

where $w = (w_1^\top, w_2^\top)^\top$, with $w_1 \in \mathbb{R}^{n-k}$, $w_2 \in \mathbb{R}^k$, $Q_1 : \mathbb{R}^n \to \mathbb{R}^{(n-k)\times(n-k)}$ is a full rank matrix, and $Q_2 : \mathbb{R}^n \to \mathbb{R}^{k\times(n-k)}$. The function $U : \mathbb{R}^{2n-k} \to \mathbb{R}$ is the new Hamiltonian defined as $U(w,p) = W(q,p)|_{q=\pi^{-1}(w)}$, $J(w,p) = N(q,p)|_{q=\pi^{-1}(w)}$, and $G_w = G_c(q)|_{q=\pi^{-1}(w)}$.

The assumption A2 is satisfied for mechanical systems that can be written in canonical form. Indeed, for these systems the matrix L_1 is the identity matrix, therefore non-singular [8], [11].

B. A Robust Passivity-Based Control

Proposition 4.1: Consider the transformed pH systems (16) in closed loop with the control law

$$
\begin{aligned}
v &= (R_2+R_3)^{-1}\Big\{-2J_{13}^\top\nabla_{w_1}V_d - 2J_{23}^\top\nabla_{w_2}V_d \\
&+ (2J - R_2 - R_3)J_{13}^{-1}(Q_1 p + R_1\nabla_{w_1}V_d) \\
&- \frac{d}{dt}(J_{13}^{-1}Q_1)p - J_{13}^{-1}Q_1(-Q^\top\nabla_w V + Jp) \\
&- \frac{d}{dt}(J_{13}^{-1})R_1\nabla_{w_1}V_d - J_{13}^{-1}R_1\nabla_{w_1}^2 V_d Q_1 p\Big\} - z_2,
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\dot{z}_2 &= J_{14}^\top\nabla_{w_1}V_d + J_{24}^\top\nabla_{w_2}V_d \\
&+ (R_3 + J^\top)J_{13}^{-1}(Q_1 p + R_1\nabla_{w_1}V_d).
\end{aligned} \tag{18}
$$

The matrices J_{13}, and J_{14} have dimension $(n-k)\times(n-k)$; J_{23} and J_{24} have dimension $k\times(n-k)$; $R_1 = R_1^\top \geq 0$, $R_2 = R_2^\top > 0$ and $R_3 = R_3^\top > 0$ are parameters of the controller, which satisfy

$$J_{13} = J_{14} = Q_1 G_c(R_2+R_3)^{-1}, \tag{19}$$

$$J_{23} = J_{24} = Q_2 G_c(R_2+R_3)^{-1}. \tag{20}$$

The matrices R_2 and R_3 are free design parameters, and the matrix R_1 has to satisfy the constraint

$$Q_2 Q_1^{-1}(R_1 \nabla_{w_1} V_d) = 0. \tag{21}$$

i) The closed-loop dynamics with

$$z_1 := J_{13}^{-1}(Q_1 p + R_1 \nabla_{w1} V_d) + z_2 - d, \tag{22}$$

takes the following pH form

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 & J_{13} & -J_{14} \\ 0 & 0_k & J_{23} & -J_{24} \\ -J_{13}^\top & -J_{23}^\top & J - R_2 & -J - R_3 \\ J_{14}^\top & J_{24}^\top & R_3 + J^\top & -R_3 - J^\top \end{bmatrix} \nabla W_z \tag{23}$$

with Hamiltonian

$$W_z(w, z_1, z_2) = \frac{1}{2}|z_1|^2 + \frac{1}{2}|z_2 - d|^2 + V_d(w), \tag{24}$$

where $V_d(w)$ is chosen such that it has a minimum at a desired state.

ii) The system has an (almost-globally) asymptotically stable equilibrium manifold given by

$$\mathcal{M}_s = \left\{ (w, z) \middle| \begin{array}{c} \nabla_{w_1}^\top V_d R_1 \nabla_{w_1} V_d = 0, \\ \nabla_{w_1} V_d + Q_1^{-\top} Q_2^\top \nabla_{w_2} V_d = 0, \\ z_1 = 0, z_2 = d. \end{array} \right\}.$$

Proof: The pH closed-loop system (23) results if the matching conditions detailed in the following are satisfied.

i) The first row of the closed-loop (23) follows from the equation of $\dot{w}_1$ in (16) and the change of coordinates (22) together with the condition $J_{13} = J_{14}$.

From the matching the equations of $\dot{w}_2$ in (16) and (23), it follows that

$$J_{23} z_1 - J_{24} \tilde{z}_2 = Q_2 Q_1^{-1}\Big(- R_1 \nabla_{w_1} V_d + J_{13} z_1 - J_{14} \tilde{z}_2 \Big),$$

where $\tilde{z}_2 := z_2 - d$. This matching equation can be separated into two conditions that must be satisfied jointly:

$$Q_2 Q_1^{-1}(R_1 \nabla_{w_1} V_d) = 0, \tag{25}$$
$$J_{24} = J_{23} := Q_2 Q_1^{-1} J_{14}. \tag{26}$$

Since the Hamiltonian function W_z has been adopted, the matching equation (25) determines the total damping terms that can be added to the proposed closed loop (23).

The control law is computed by matching the time derivative of the coordinate transformation (22) and the third row of (23). Solving this matching equation for u it gives the control law (17). A requirement for robustness is that the control law is independent of the disturbance. This condition is satisfied by choosing

$$J_{14} = Q_1 G_c (R_2 + R_3)^{-1}. \tag{27}$$

ii) Taking (24) as a Lyapunov candidate function and making its time derivative along the solution of the system (23) gives

$$\dot{W}_z = -\nabla_{w_1}^\top V_d R_1 \nabla_{w_1} V_d - z_1^\top R_z z_1 - \tilde{z}_2^\top R_3 \tilde{z}_2. \tag{28}$$

Furthermore, the trajectories will converge to the largest invariant set included in

$$\begin{aligned} \mathcal{S} &= \{(w, z) | \dot{W}_z = 0\} \\ &= \left\{ (w, z) \middle| \begin{array}{c} \nabla_{w_1}^\top V_d R_1 \nabla_{w_1} V_d = 0, \\ z_1 = 0, z_2 = d, w_2 \in \mathbb{R}^k \end{array} \right\}. \end{aligned}$$

From (23), we can conclude that the largest invariant set in $\mathcal{S}$ is the manifold

$$\mathcal{M}_s = \left\{ (w, z) \middle| \begin{array}{c} \nabla_{w_1}^\top V_d R_1 \nabla_{w_1} V_d = 0, \\ \nabla_{w_1} V_d + Q_1^{-\top} Q_2^\top \nabla_{w_2} V_d = 0, \\ z_1 = 0, z_2 = d. \end{array} \right\}. \tag{29}$$

This proves that the equilibrium manifold $\mathcal{M}_s$ of the target dynamics (23) is asymptotically stable.

■

Note that the control law is given by (17)-(18), and information about the disturbance d is not required to implement this control law.

The condition (21) constraints the damping injection in coordinates w. Indeed, if we choose the desired potential energy V_d, then R_1 is selected to satisfy (21). The functions V_d and R_1 characterise the equilibrium manifold.

V. Case Study - Wheeled Robot

In this section, we consider an application of the proposed control design and present simulations to assess the performance of the proposed control system. As an example we consider the configuration of an autonomous wheeled robot shown in Figure 1. The two front wheels with axis through the point P are traction wheels with independent torque control, and the two rear wheels are free castor wheels. The robot has a mass m and the centre of mass is at the point C. The mass of the rear wheels and their friction about the vertical axis of rotation are considered negligible. Under these assumptions, the motion control of the robot can be considered analogous to that of the classic Chaplygin sleigh, proposed by [6]—see also the model in [2].

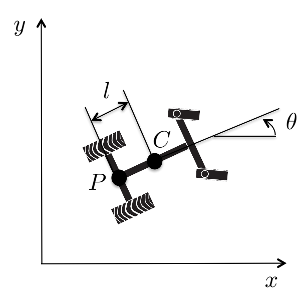

Fig. 1. Wheeled robot moving on the horizontal plane. The two front weeks with axis through the point P are traction wheels with independent torque control. The two rear wheels are free castor wheels.

The dynamics of the robot can be written in the from (1) using coordinates $q = [x, y, \theta]^\top$, where x and y are the cartesian coordinates of the point P. The Hamiltonian function is

$$H(q, \mathbf{p}) = \frac{1}{2}\mathbf{p}^\top M^{-1}(q)\mathbf{p} \tag{30}$$

and the mass matrix is

$$M(q)=\begin{bmatrix} m & 0 & -m\,l\sin(q_3) \\ 0 & m & m\,l\cos(q_3) \\ -m\,l\sin(q_3) & m\,l\cos(q_3) & I_c+ml^2 \end{bmatrix}, \tag{31}$$

where I_c is the moment of inertia about C. The nonholonomic constraint (3) takes the form

$$\begin{bmatrix} -\sin(q_3) & \cos(q_3) & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}=0. \tag{32}$$

We simulate the robot in close loop with the control law (17). The parameters considered for the simulation are $m=1$ Kg, $l=0.2$ m, $I_c=0.5$ Kg m^2. We consider a control force in the longitudinal direction and a control torque about the vertical axis through P. Then,

$$G(q)=\begin{bmatrix} \cos(q_3) & 0 \\ \sin(q_3) & 0 \\ 0 & 1 \end{bmatrix}. \tag{33}$$

The matrices S and T are as follows

$$S(q)=\begin{bmatrix} 0 & \cos(q_3) \\ 0 & \sin(q_3) \\ 1 & 0 \end{bmatrix}, \quad T=\mathrm{diag}(\sqrt{I_c+ml^2},\sqrt{m}). \tag{34}$$

With these, we obtain the port-Hamiltonian form (5). Assumption A3 is not trivially satisfied since the sub-matrix L_1 of L is not invertible

$$L(q)=\begin{bmatrix} 0 & \frac{\cos(q_3)}{\sqrt{m}} \\ 0 & \frac{\sin(q_3)}{\sqrt{m}} \\ \frac{1}{\sqrt{I_c+ml^2}} & 0 \end{bmatrix}. \tag{35}$$

However, the change of coordinates

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ \cos(q_3) & \sin(q_3) & 0 \\ -\sin(q_3) & \cos(q_3) & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{36}$$

readily satisfies A3 with $\hat{v}=0$. Indeed, the dynamics of the Chaplyging sleigh can be written in the form (16) with

$$Q(w)=\begin{bmatrix} \frac{1}{\sqrt{I_c+ml^2}} & 0 \\ \frac{w_3}{\sqrt{I_c+ml^2}} & \frac{1}{\sqrt{m}} \\ \frac{-w_2}{\sqrt{I_c+ml^2}} & 0 \end{bmatrix}, \tag{37}$$

and since there is no potential energy, the control law $\hat{u}$ in (15) is zero.

The objective is to asymptotically stabilise the system to the target manifold $\mathcal{M}_t=\{(x,y,\theta)\big|\,x\cos\theta^*+y\sin\theta^*=0$ and $\theta=\theta^*\}$ with $z_1=0$ and $z_2=d$, where we choose the desired heading angle θ^*. In our simulations, we choose $\theta^*=-\pi/2$, and then the manifold $\mathcal{M}_t$ is the x-coordinate axis. The control law is

$$\begin{aligned} u &= \begin{bmatrix} -\frac{2}{m\rho_1^2} & -r_1\rho_1 m \\ -\frac{2}{I_p\rho_2^2} & \frac{2r_1\sqrt{m^3}\rho}{I_p}p_1-\frac{2w_3}{\rho_2} \end{bmatrix}\begin{bmatrix} k_{w_1}\tilde{w}_1 \\ k_{w_1}\tilde{w}_2 \end{bmatrix}+z_2 \\ &+ \begin{bmatrix} -a_1p_1-\frac{mr_1k_{w_1}w_3}{\sqrt{I_p}} & -a_2 \\ -\sqrt{I_p}\rho_2 & a_3p_1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ \frac{2w_2}{I_p\rho_2^2} \end{bmatrix}k_{w_3}\tilde{w}_3, \end{aligned}$$

$$\begin{aligned} \dot{z}_2 &= \begin{bmatrix} \frac{1}{m\rho_1} & r_1r_{31}\rho_1 m \\ \frac{1}{I_p\rho_2} & w_3-\frac{r_1\sqrt{m^3}\rho_1}{I_p}p_1 \end{bmatrix}\begin{bmatrix} k_{w_1}\tilde{w}_1 \\ k_{w_1}\tilde{w}_2 \end{bmatrix} \\ &+ \begin{bmatrix} \frac{\sqrt{m}}{\sqrt{I_p}}\rho_2p_1 & \sqrt{m}r_{31}\rho_1 \\ \sqrt{I_p}r_{32}\rho_2 & -\frac{m}{I_p}\rho_1p_1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ \frac{-w_2}{I_p\rho_2} \end{bmatrix}k_{w_3}\tilde{w}_3, \end{aligned}$$

where $\tilde{w}_i=w_i-w_i^*$ for $i=1,2,3$; $R_1=\mathrm{diag}(0,r_1)$, $R_2=\mathrm{diag}(r_{21},r_{22})$, $R_3=\mathrm{diag}(r_{31},r_{32})$, $I_p=I_c+ml^2$, $\rho_1=r_{21}+r_{31}$, $\rho_2=r_{22}+r_{32}$, $\rho=\rho_1/\rho_2$, $a_1=\frac{2\sqrt{m}}{\rho\sqrt{I_p}}+\frac{m}{I_p}$, $a_2=\sqrt{m}\frac{(r_{22}+r_{32})+r_1k_{w_2}\rho}{\rho}$, $a_3=\frac{2m\rho}{I_p}+\frac{\sqrt{m}}{\sqrt{I_p}}$. The potential energy is $V_d=\frac{1}{2}(w-w^*)^\top K_w(w-w^*)$, with $K_w=\mathrm{diag}(k_{w_1},k_{w_2},k_{w_3})$. The parameters of the controller are as follows: $K_w=\mathrm{diag}(50,10,10)$ and $w^*=(-\pi/2,0,0)$, $R_1=\mathrm{diag}(0,0.1)$, $R_2=\mathrm{diag}(1,5)$, $R_3=\mathrm{diag}(5,5)$.

The system is simulated from different initial conditions (ic), and the state converges to the manifold $\mathcal{M}_t$ even under the action of unknown disturbances as shown in Figure 2 and 3. The control inputs and the states of the controller are shown in Figure 4, which details how the controller states produce the input needed to compensate the unknown disturbances. The trajectories of the system in the xy-plane are shown in Figure 5. In this figure, the system starts in different positions x and y and heading angles θ and converge to axis x with the desired heading (dashed line). Then, the disturbances shift the target position of the system, and the controller drives again the system to the target manifold $\mathcal{M}_t$.

The line, which is a projection of the manifold onto the x - y plane, is motivated by an application in agriculture where a robot used for weed and crop management is deployed and must position itself at the end of the paddock in order to commence the operation.

VI. Conclusions

We propose a control design based on IDA-PBC for smooth stabilisation of mechanical systems in pH form subject to nonholonomic constraints and disturbances. The continuous control law stabilises the system to an equilibrium manifold, and the closed loop is robust to unknown constant disturbances in the sense that the equilibrium manifold does not change due to the presence of disturbances—integral action. A wheeled robot is used an example to illustrate the design and evaluate its performance. By adopting various functions, one can change the shape of the target namifold. A

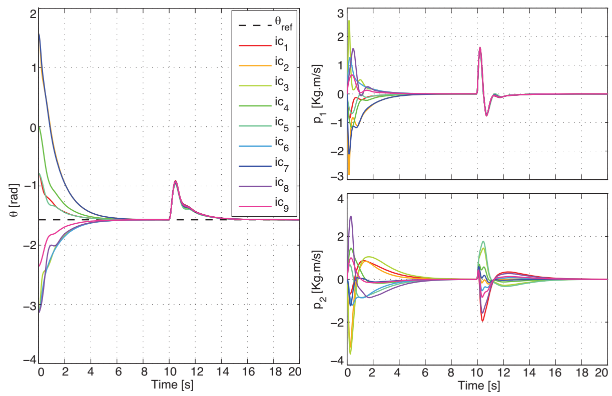

Fig. 2. Time histories of the angle θ and momenta p_1 and p_2 from different initial conditions.

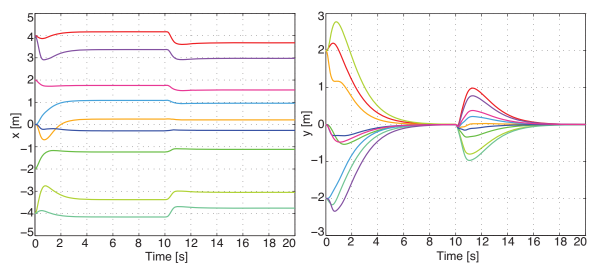

Fig. 3. Time histories of the positions x and y.

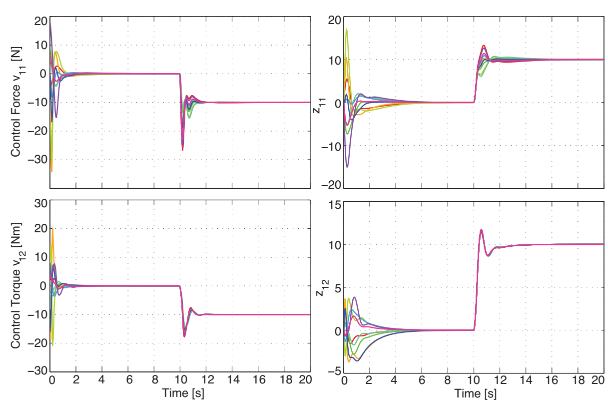

Fig. 4. Time histories of force and torque control inputs and controller states.

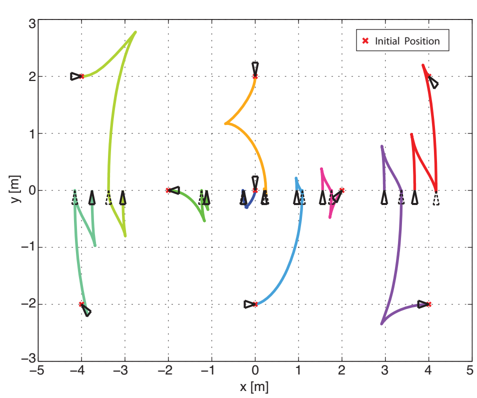

Fig. 5. Trajectories of the wheeled robot in the x-y plane. A solid robot with a cross indicates the initial position, and the dashed one indicates the position at the target manifold before the disturbances.

natural extension of the results in this paper, to be considered as part of future work is the study of stabilisation of the formation of multiple vehicles with a particular prescribed final distribution on the manifold. Such application may require non-smooth control laws.

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