Shaping the energy of port-Hamiltonian systems without solving PDE's

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Abstract— **Equilibrium stabilisation of nonlinear systems via energy shaping is a well–established, robust, passivity–based controller design technique. Unfortunately, its application is often stymied by the need to solve partial differential equations. In this paper a new, fully constructive, procedure to shape the energy for a class of *port–Hamiltonian* systems that *obviates* the solution of partial differential equations is proposed. Proceeding from the well–known passive, *power shaping* output we propose a nonlinear static state–feedback that preserves passivity of this output but with a *new storage function*. This function contains some tuning gains used to ensure it is positive definite, hence a suitable Lyapunov function for the closed–loop. Connections with other standard passivity–based controllers are indicated and it is shown that the new controller design is applicable to two benchmark examples.**

Index Terms— **Passivity, nonlinear systems, passivity–based control, Hamiltonian systems.**

I. Introduction

Control of physical systems via energy shaping is a well–established technique whose roots can be traced back to Lagrange's and Dirichlet's work [1]. In the control community this fundamental concept was first introduced by Takegaki and Arimoto in [2] for mechanical systems and by Jonckheere in [3] for electromechanical systems. In [2] the Hamiltonian formalism is used to describe the system dynamics while the Euler–Lagrange formulation is used in [3]. To date, these two mathematical descriptions of the system dynamics are favoured to carry out the energy–shaping task with a lot of research devoted to the particular case of mechanical systems, see *e.g.*, [4], [5], [6] and references therein.

One of the central features of existing energy shaping techniques is the preservation in closed–loop of the original systems structure—either Lagrangian or Hamiltonian—and a key step is the identification of the energy (or Lagrangian) functions that can be assigned via feedback. This assignable energy functions are the solutions of the so–called *matching equations*, which are a set of partial differential equations (PDEs) that need to be solved to complete the controller design. A lot of research effort has been devoted to the solution of the matching equations—see *e.g.*, [7], [8], [9], [10], [11], [12], [13]. Also, there is a large list of applications where it has been possible to solve these equations, including (almost) all basic pendular systems considered in the literature, motors, generators, power systems, power converters, level control systems, etc.—see [14] for a partial list. In spite of that, this difficult key step remains the main stumbling block for the wider application of these methods.

In the recent paper [15] it has been proposed *for mechanical systems* to abandon the objective of structure preservation and attention has been concentrated on the energy shaping objective only. That is, we look for a static state–feedback that stabilizes the desired equilibrium assigning to the closed–loop a Lyapunov function of the same form as the energy function of the open–loop system but with new, desired inertia matrix and potential energy function. However, it was *not required* that the closed–loop system is a mechanical system with this Lyapunov function qualifying as its energy function. In this way, the need to solve the matching equations is avoided.

In this paper we pursue the same research line considering the more general case of *port–Hamiltonian* (pH) systems [16] of the form $\dot{x} = F(x)\nabla H(x) + g(x)u$, where $F(x) + F^{\top}(x) \leq 0$ and $F(x)$ is full rank. The starting point of the design is the well–known *power shaping* output [17], which is known to define a passive output for the pH system with storage function its energy function $H(x)$. Assuming that this output is "integrable", the next step is the design of a nonlinear static state–feedback that preserves this passive output but now with a *new storage function*. The latter is constructed as the weighted sum of the original energy function and a quadratic term of the shifted "integral" of the power shaping output. The weighting gains and the aforementioned shifting constant are the degrees of freedom introduced to ensure *positive definiteness* of the new energy function, which then qualifies as a Lyapunov function for the closed–loop system. The condition of integrability of the power shaping output boils down to a classical integrability condition of the vector fields $F^{-1}(x)g_i(x)$, with $g_i(x)$ the columns of the matrix $g(x)$, hence it can be readily verified.

The remaining of the paper is organized as follows. Section II presents the problem formulation and main assumptions on the pH system. The design of the energy shaping feedback is carried out in Section III. Section IV contains the main stabilization result. Section V explores the relationship between the new controller design and the two well–known passivity–based control (PBC) techniques of energy balancing (EB–PBC) [18], [19] and interconnection and damping assignment (IDA–PBC) [14]. In Section VI we show that the method is applicable to two benchmark examples and, to study the limitations of the technique, consider the case of LTI systems. In particular, we prove that, unlike general IDA–PBC of LTI systems but similarly to the case of IDA–PBC of mechanical systems [20], stabilisability of the LTI system is not enough to ensure that the proposed controller yields a stable closed–loop system. The paper is wrapped–up with concluding remarks and future research in Section VII.

Notation For $x \in \mathbb{R}^n$, $S \in \mathbb{R}^{n\times n}$, $S = S^\top > 0$, we denote the Euclidean norm $|x|^2 := x^\top x$, and the weighted–norm $\|x\|_S^2 := x^\top S x$. For a scalar function $H : \mathbb{R}^n \to \mathbb{R}$ we denote $\nabla_x H(x) = \left(\frac{\partial H(x)}{\partial x}\right)^\top$. For vector functions $\mathcal{C} : \mathbb{R}^n \to \mathbb{R}^m$, we define its (transposed) Jacobian matrix $\nabla\mathcal{C}(x) = [\nabla\mathcal{C}_1(x), \ldots, \nabla\mathcal{C}_m(x)]$. For any mapping $F : \mathbb{R}^n \to \mathbb{R}^{p\times q}$ and the distinguished element $x_* \in \mathbb{R}^n$ we denote the constant matrix $F_* := F(x_*)$. We define the pseudoinverse of a full-rank matrix $G \in \mathbb{R}^{n\times m}$ as $G^\dagger := (G^\top G)^{-1}G^\top$. When clear from the context the subindex of the operator ∇ and the arguments of the functions will be omitted. All the functions in the paper are assumed sufficiently smooth.

II. Problem Formulation and Main Assumptions

The standard representation of pH systems is of the form

$$\dot{x} = [\mathcal{J}(x) - \mathcal{R}(x)]\nabla H(x) + g(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$, $m \leq n$, is the control vector, $H : \mathbb{R}^n \to \mathbb{R}$ is the systems stored energy, $\mathcal{J}, \mathcal{R} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $\mathcal{J}(x) = -\mathcal{J}^\top(x)$ and $\mathcal{R}(x) = \mathcal{R}^\top(x) \geq 0$, are the interconnection and damping matrices, respectively, and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ is the input matrix, which is full rank. To simplify the notation in the sequel we define the matrix $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$

$$F(x) := \mathcal{J}(x) - \mathcal{R}(x).$$

The *control objective* is to stabilise an equilibrium x_*, which is an element of the set of assignable equilibria given by

$$\mathcal{E} := \left\{x \in \mathbb{R}^n \mid g^\perp(x)F(x)\nabla H(x) = 0\right\}. \tag{2}$$

The following assumptions identify the class of pH systems for which the proposed control strategy is applicable.

Assumption 1: The matrix $F(x)$ is *full rank*.

Assumption 2: The vector fields $F^{-1}(x)g_i(x)$, with $g_i(x)$, $i = 1, \ldots, m$, the columns of the matrix $g(x)$, are *gradient vector fields.* That is,

$$\nabla\left(F^{-1}(x)g_i(x)\right) = \left[\nabla\left(F^{-1}(x)g_i(x)\right)\right]^\top.$$

If Assumption 1 holds, it is possible to define the *power shaping output* as follows

$$y_{\tt PS} := -g^\top(x)F^{-\top}(x)\left[F(x)\nabla H(x) + g(x)u\right]. \tag{3}$$

As shown in [17], [19] $y_{\tt PS}$ is a cyclo–passive output for the pH system (1) with storage function $H(x)$. More precisely, the following dissipation inequality holds

$$\dot{H} \leq u^\top y_{\tt PS}. \tag{4}$$

Noting that $y_{\tt PS}$ may be written as

$$y_{\tt PS} = -g^\top(x)F^{-\top}(x)\dot{x} \tag{5}$$

and recalling Poincare's Lemma it is easy to see that Assumption 2 ensures the existence of a function $\gamma : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\dot{\gamma} = (\nabla\gamma)^\top\dot{x} = y_{\tt PS}, \tag{6}$$

with $y_{\tt PS}$ defined in (3).

III. Energy Shaping

In this section we define a static state–feedback such that the system (1) in closed–loop with this control preserves passivity of the mapping $v \mapsto y_{\tt PS}$ but with a suitably *modified storage function*.

Proposition 1: Suppose Assumptions 1 and 2 hold. Define the mapping $u_{\tt BC} : \mathbb{R}^n \to \mathbb{R}^m$

$$u_{\tt BC}(x) := [k_e I - k_a g^\top(x)F^{-\top}(x)g(x)]^{-1}\,[v - K_I(\gamma(x) + C) + k_a g^\top(x)F^{-\top}(x)F(x)\nabla H(x)] \tag{7}$$

where[1]

$$\nabla\gamma(x) := -F^{-1}(x)g(x) \tag{8}$$

and $k_a, k_e \in \mathbb{R}_+$, $C \in \mathbb{R}^m$, $K_I \in \mathbb{R}^{m\times m}$ with $K_I = K_I^\top > 0$ are *free parameters*. The system (1) in closed–loop with the control $u = u_{\tt BC}(x)$ defines a cyclo–passive mapping $v \mapsto y_{\tt PS}$ with storage function

$$H_d(x) = k_e H(x) + \frac{1}{2}\|\gamma(x) + C\|_{K_I}^2. \tag{9}$$

Proof: From (9) we get

$$\begin{aligned}
\dot{H}_d &= k_e\dot{H} + y_{\tt PS}^\top K_I(\gamma + C)\\
&\leq y_{\tt PS}^\top\left[k_e u + K_I(\gamma + C)\right]\\
&= y_{\tt PS}^\top\left[v + k_a g^\top F^{-\top}F\nabla H + k_a g^\top F^{-\top}g u\right]\\
&= y_{\tt PS}^\top(v - k_a y_{\tt PS})\\
&\leq y_{\tt PS}^\top v,
\end{aligned}$$

where we used (6) in the first equality, (4) in the first inequality and (7) and (3) for the second and third equality, respectively.

■

From (6) we have that

$$\gamma(x(t)) = \int_0^t y_{\tt PS}(x(\tau))d\tau + \gamma(x(0)),$$

where we recall that $y_{\tt PS}$ is defined in (3). Hence, the second term in the new storage function (9) may be interpreted as the *integral* of $y_{\tt PS}$. This establishes a connection with the PI–like controllers proposed for mechanical systems in [15].

[1] Notice that the existence of $\gamma(x)$ is ensured by Assumption 2.

IV. STABILIZATION

From Proposition 1 it is clear that if the new storage function $H_d(x)$ is positive definite (with respect to the desired equilibrium x_*) it qualifies as a *bona fide* Lyapunov function for the closed–loop system (with $v = 0$) that ensures stability of x_*. This fact is enunciated in the proposition below where we also give easily verifiable conditions for .

Proposition 2: Consider the system (1), verifying Assumptions 1 and 2, in closed–loop with the control $u = u_{\mathtt{BC}}(x)$, with $v = 0$, where $u_{\mathtt{BC}}(x)$ is given by (7).

(i) If $x_* \in \mathcal{E}$ and

$$x_* = \arg\min H_d(x), \tag{10}$$

with $H_d(x)$ defined in (9), then x_* is *stable* (in the sense of Lyapunov) with Lyapunov function $H_d(x)$. It is *asymptotically* stable If $y_{\mathtt{PS}}$, defined in (3) is detectable, that is, if the following implication is true

$$\left[y_{\mathtt{PS}}(t) \equiv 0 \Rightarrow \lim_{t\to\infty} x(t) = x_* \right].$$

(ii) Condition (10) holds if the following inequality is satisfied

$$(\nabla^2 H_d)_* > 0. \tag{11}$$

Proof: The proof of (i) follows from the fact that $H_d(x)$ is positive definite with

$$\dot{H}_d = -k_a |y_{\mathtt{PS}}|^2 \leq 0,$$

and classical Lyapunov theory [21].

To establish (ii) first notice that from (5) and (6) the control (7) reduces to

$$u = -\frac{1}{k_e}\left[K_I(\gamma + C) + k_a y_{\mathtt{PS}}\right]. \tag{12}$$

Since at the equilibrium $y_{\mathtt{PS}_*} = 0$, then (12) becomes

$$u_* = -\frac{1}{k_e} K_I(\gamma_* + C) \tag{13}$$

also, from (1)

$$\begin{aligned}(\nabla H)_* &= -F_*^{-1} g_* u_* \\ &= \frac{1}{k_e} F_*^{-1} g_* K_I(\gamma_* + C)\end{aligned} \tag{14}$$

where the last equation was obtained from the substitution of (13). On the other hand,

$$\begin{aligned}(\nabla H_d)_* &= k_e(\nabla H)_* + (\nabla\gamma)_* K_I(\gamma_* + C) \\ &= k_e(\nabla H)_* - F_*^{-1} g_* K_I(\gamma_* + C).\end{aligned} \tag{15}$$

Replacing (14) in (15) we have $(\nabla H_d)_* = 0$ which ensures that x_* is a critical point of the closed–loop system. The proof is completed noting that, the inequality (11) implies that (10) holds. ∎

Remark 1: From (15) it is clear that—for the purpose of stabilization—the constant vector C, which ensures x_* is an equilibrium point of the closed–loop, is *uniquely defined* by

$$C := k_e K_I^{-1} g_*^{\dagger} F_* (\nabla H)_* - \gamma_*. \tag{16}$$

V. RELATION WITH CLASSICAL PBCS

In this section we discuss the relationship between the new controller and the classical PBC techniques of EB and IDA.[2] It should be underscored that, in contrast with these PBC design techniques, the proposed method does not preserve—in general—the pH structure in closed–loop, but we will show that for some particular choice of the tuning gains it does.

A. Energy–balancing PBC

The basic idea of EB–PBC (with the output $y_{\mathtt{PS}}$) is to look for state feedbacks $u_{\mathtt{EB}} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\dot{H}_a = -u_{\mathtt{EB}}^\top y_{\mathtt{PS}},$$

for some "added" energy function $H_a : \mathbb{R}^n \to \mathbb{R}$. In this case, setting $u = u_{\mathtt{EB}}(x)$ transforms the cyclo–pasivity inequality (4) into

$$\dot{H} + \dot{H}_a \leq 0,$$

and if $H(x) + H_a(x)$ is positive definite the closed–loop system will have a stable equilibrium at x_*. The following proposition states that, for a suitable choice of the tuning gains, the new controller.

Proposition 3: Consider the pH system (1) verifying Assumptions 1 and 2. Fix the gains of the mapping $u_{\mathtt{BC}}(x)$ as $k_e = 1$ and $k_a = 0$. Then, the control $u = u_{\mathtt{BC}}(x)$, with $v = 0$, is an EB–PBC with added energy function

$$H_a(x) := \frac{1}{2}\|\gamma(x) + C\|^2_{K_I}. \tag{17}$$

Proof: For $v = 0, k_a = 0$ and $k_e = 1$ the mapping $u_{\mathtt{BC}}(x)$ reduces to

$$u_{\mathtt{BC}}(x) = -K_I[\gamma(x) + C].$$

On the other hand, from (17) we have

$$\dot{H}_a = y_{\mathtt{PS}}^\top K_I(\gamma + C) = -y_{\mathtt{PS}}^\top u_{\mathtt{BC}},$$

completing the proof. ∎

B. Interconnection and damping assignment PBC

In IDA–PBC we fix desired interconnection and damping matrices, hence, fix a matrix $F_d : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that

$$F_d(x) + F_d^\top(x) \leq 0,$$

and look for a control $u = u_{\mathtt{IDA}}(x)$ such that the closed–loop has the form

$$\dot{x} = F_d(x)\nabla H_{\mathtt{IDA}}(x);$$

for some energy function $H_{\mathtt{IDA}} : \mathbb{R}^n \to \mathbb{R}$, which has a minimum at the desired equilibrium. It is easy to show that the assignable energy functions $H_{\mathtt{IDA}}(x)$ are characterised by the solutions of the following PDE

$$g^\perp(x)\left[F_d(x)\nabla H_{\mathtt{IDA}}(x) - F(x)\nabla H(x)\right] = 0, \tag{18}$$

and the control is univocally defined as

$$u_{\mathtt{IDA}}(x) := g^\dagger(x)\left[F_d(x)\nabla H_{\mathtt{IDA}}(x) - F(x)\nabla H(x)\right] \tag{19}$$

[2]The interested reader is referred to [18], [19] for further details on EB–PBC and IDA–PBC.

The proposition below establishes the relation between IDA–PBC and Proposition 1.

Proposition 4: Consider the pH system (1) verifying Assumptions 1 and 2. Fix the gain of the mapping $u_{\mathtt{BC}}(x)$ as $k_a = 0$ and select the desired interconnection and damping matrices as

$$F_d(x) = \frac{1}{k_e} F(x).$$

Then, the energy function $H_d(x)$ defined in (9) and the control $u = u_{\mathtt{BC}}(x)$, with $v = 0$, given in (7) satisfy the IDA–PBC equations (18) and (19), respectively.

Proof: Replacing the gradient of $H_d(x)$, given by

$$\nabla H_d(x) = k_e \nabla H(x) - F^{-1}(x) g(x) K_I(\gamma(x) + C),$$

in the PDE (18) we get

$$g^{\perp} \left\{ F_d \left[k_e \nabla H - F^{-1} g K_I(\gamma + C) \right] - F \nabla H \right\} = g^{\perp} \left\{ \frac{1}{k_e} F \left[k_e \nabla H - F^{-1} g K_I(\gamma + C) \right] - F \nabla H \right\} = 0.$$

On the other hand, the control law (7) is given by

$$u_{\mathtt{BC}}(x) = -\frac{1}{k_e} K_I(\gamma(x) + C), \tag{20}$$

furthermore

$$\begin{aligned} u_{\mathtt{IDA}} &= g^{\dagger} \left\{ F_d \left[k_e \nabla H - F^{-1} g K_I(\gamma + C) \right] - F \nabla H \right\} \\ &= g^{\dagger} \left\{ \frac{1}{k_e} F \left[k_e \nabla H - F^{-1} g K_I(\gamma + C) \right] - F \nabla H \right\} \\ &= -\frac{1}{k_e} g^{\dagger} g K_I(\gamma + C) \end{aligned} \tag{21}$$

which coincides with (20). ■

VI. Examples

In this section we apply the proposed controller to two physical systems and investigate, with the example of LTI systems, the limitations of the method.

A. Micro electro–mechanical optical switch [22]

Consider the optical switch system with pH model

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -b & 0 \\ 0 & 0 & -\frac{1}{r} \end{bmatrix} \nabla H(x) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{r} \end{bmatrix} u$$

and energy function

$$H(x) = \frac{1}{2m} x_2^2 + \frac{1}{2} a_1 x_1^2 + \frac{1}{4} a_2 x_1^4 + \frac{1}{2c_1(x_1 + c_0)} x_3^2,$$

where $a_1 > 0, a_2 > 0$ are the spring constants, $b > 0, r > 0$ are resistive elements, $c_0 > 0, c_1 > 0$ are constants that determine the capacitance function and, finally, $m > 0$ denotes the mass of the actuator. It is important to underscore the physical constraint $x_1 > 0$. The set of equilibria for this system is

$$x_{2_*} = 0, \qquad x_{3_*} = (c_0 + x_{1_*}) \sqrt{2c_1 x_{1_*}(a_1 + a_2 x_{1_*}^2)}$$

and the goal is to stabilize at $x_{1_*} > 0$.

Some simple calculations prove that $y_{\mathtt{PS}} = \dot{x}_3$, therefore $\gamma(x) = x_3$. Also, F is clearly full rank. Hence, Assumptions 1 and 2 hold. It only remains to show that (ii) of Proposition 2 holds. Since at the equilibrium

$$(\nabla^2 H_d)_* = k_e \begin{bmatrix} a_1 + 3a_2 x_{1_*}^2 + d_1^2 d_2 & 0 & -d_1 d_2 \\ 0 & \frac{1}{m} & 0 \\ -d_1 d_2 & 0 & d_2 \end{bmatrix} + K_I \mathrm{diag}(0, 0, 1)$$

where

$$d_1 := \sqrt{2c_1 x_{1_*}(a_1 + a_2 x_{1_*}^2)}, \qquad d_2 := \frac{1}{c_1(c_0 + x_{1_*})},$$

then for all $K_I > 0, k_e > 0$ the condition (11) holds. Hence, x_* is a stable equilibrium for the closed–loop system.

B. Magnetic levitation system [23]

In this example we show that, including coordinate changes—as proposed in [5]—it is possible to add a new degree of freedom to the design. First, we will prove that in the natural cordinates considered in [23] the proposed method is not applicable. The pH model is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -r_2 & \frac{1}{c} \\ 0 & 0 & -\frac{1}{c} & -\frac{1}{r_1 c^2} \end{bmatrix} \nabla H(x) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r_1 c} \end{bmatrix} u,$$

with energy function

$$H(x) = \frac{1}{2m} x_2^2 + mgx_1 + \frac{1}{2n} x_3^2 (1 - x_1) + \frac{c}{2} x_4^2,$$

where $m > 0$ is the mass of the ball, $r_1 > 0$ and $c > 0$ are the parasite resistance and capacitance, respectively; $n > 0$ is constant which depends on the number of coil turns, $r_2 > 0$ is the coil resistance and g is the gravitational acceleration.

A physical constraint of this model is that $1 > x_1 > 0$. The set of equilibria is given by

$$x_{2_*} = 0, \quad x_{3_*} = \sqrt{2d_0}, \quad x_{4_*} = r_2(1 - x_{1_*}) \sqrt{\frac{2mg}{n}}$$

where $d_0 = nmg$ and the control goal is to stabilize the system at an assignable equilibrium with a desired ball position, x_{1_*}, satisfying the constraints.

In this example

$$y_{\mathtt{PS}} = \frac{1}{r}(\dot{x}_3 + r_2 c \dot{x}_4), \tag{22}$$

whose time integration defines $\gamma(x)$ as follows

$$\gamma = \frac{1}{r}(x_3 + r_2 c x_4)$$

where $\bar{r} = r_1 + r_2$. The Hessian of the energy function (9) is, then, given by

$$\nabla^2 H_d = k_e \begin{bmatrix} 0 & 0 & -\frac{1}{n}x_3 & 0 \\ 0 & \frac{1}{m} & 0 & 0 \\ -\frac{1}{n}x_3 & 0 & \frac{1}{n}(1-x_1) & 0 \\ 0 & 0 & 0 & c \end{bmatrix} + \frac{K_I}{\bar{r}^2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & r_2 c \\ 0 & 0 & r_2 c & r_2^2 c^2 \end{bmatrix}$$

which clearly is *not positive definite*, consequently the proposed method is not applicable to this system.

The source of the problem stems from the fact that with the power shaping output given by (22) the respective γ cannot shape the energy in the first coordinate, which is needed to ensure (11). To overcome this problem we propose the coordinate change

$$z = \mathrm{col}(x_1, x_2, x_3\sqrt{1-x_1}, x_4). \tag{23}$$

The system in the new coordinates is given by

$$\dot{z} = \bar{F}\nabla\bar{H}(z) + \bar{g}u$$

where $\bar{g} := \mathrm{col}(0,0,0,\frac{1}{r_1 c})$ and

$$\bar{F} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \frac{z_3}{2(1-z_1)} & 0 \\ 0 & -\frac{z_3}{2(1-z_1)} & -r_2(1-z_1) & \frac{\sqrt{1-z_1}}{c} \\ 0 & 0 & -\frac{\sqrt{1-z_1}}{c} & -\frac{1}{r_1 c^2} \end{bmatrix}.$$

Its energy function and power shaping output are, respectively,

$$\bar{H}(z) = \frac{1}{2m}z_2^2 + mgz_1 + \frac{1}{2n}z_3^2 + \frac{c}{2}z_4^2,$$
$$\bar{y}_{\mathtt{PS}} = \frac{1}{\bar{r}}\left[\frac{1}{2}\frac{z_3}{(1-z_1)^{\frac{3}{2}}}\dot{z}_1 + \frac{1}{\sqrt{1-z_1}}\dot{z}_3 + r_2 c\dot{z}_4\right].$$

Since Assumptions 1 and 2 are invariant under the change of coordinates (23), they are still satisfied. Furthermore,

$$\bar{\gamma}(z) = \frac{1}{\bar{r}}\left(\frac{z_3}{\sqrt{1-z_1}} + r_2 c z_4\right).$$

Notice that it is possible to shape the energy in the first coordinate since the second partial derivative of $\bar{\gamma}$ with respect to z_1 is different to zero.

The set of equilibria in the new coordinates is $z_{2_*} = 0$,

$$z_{3_*} = x_{3_*}\sqrt{1-z_{1_*}}, \qquad z_{4_*} = r_2(1-z_{1_*})\sqrt{\frac{2mg}{n}}.$$

Some simple computations prove that

$$(\nabla^2 H_d)_* = k_e \mathrm{diag}(0, \frac{1}{m}, \frac{1}{m}, c) + \frac{K_I}{\bar{r}^2}\begin{bmatrix} \frac{d_1}{(1-z_{1_*})^2} & 0 & \frac{d_2}{(1-z_{1_*})^{\frac{3}{2}}} & \frac{d_3}{1-z_{1_*}} \\ 0 & 0 & 0 & 0 \\ \frac{d_2}{(1-z_{1_*})^{\frac{3}{2}}} & 0 & \frac{1}{1-z_{1_*}} & \frac{r_2 c}{\sqrt{1-z_{1_*}}} \\ \frac{d_3}{1-z_{1_*}} & 0 & \frac{r_2 c}{\sqrt{1-z_{1_*}}} & r_2^2 c^2 \end{bmatrix}$$

where we defined

$$d_1 := \frac{3}{4}\bar{r}\sqrt{2d_0} + \frac{1}{2}d_0, \quad d_2 := \frac{1}{2}\bar{r} + \sqrt{\frac{d_0}{2}}, \quad d_3 := r_2 c\sqrt{\frac{d_0}{2}}.$$

It is easy to see that, with an appropriate selection of k_e and K_I, condition (11) holds and so, the stabilisation objective is achieved with the proposed controller.

C. LTI systems: Stabilisability is not enough

In the important paper [24] it was shown that IDA–PBC for LTI systems is a *universal stabiliser*, in the sense that it is applicable to all stabilisable systems. On the other hand, it was shown in [20] that stabilisability is *not enough* for IDA–PBC of mechanical system, but a stronger condition must be imposed on the system for stabilisation with IDA–PBC—see Proposition 4.1 in [20].

The difference between these two cases is that, while for general IDA–PBC there is no constraint on the structure of the desired energy function, for mechanical systems a *particular structure* is imposed to it. Since in the methodology proposed in this paper there is also a constraint on the desired energy function, namely (9), it is expected that a condition stronger than stabilisability should be imposed for the method to apply.

Now, recall that for LTI systems the energy function is of the form $H(x) = \frac{1}{2}x^\top Qx$, the matrices F and g are constant and, without loss of generality, we can take $x_* = 0$. Therefore, the control (7) becomes a simple linear, state–feedback of the form $u_{\mathtt{BC}}(x) = Kx$ with

$$K := \left(k_e I - k_a g^\top F^{-\top} g\right)^{-1}(k_a g^\top F^{-\top} FQ + K_I g^\top F^{-\top}), \tag{24}$$

where we have set the vector $C = 0$. To prove the aforementioned conjecture we will construct an LTI, stabilisable pH system for which the controller (24) yields an unstable closed–loop system for all values of the tuning gains k_e, k_a, K_I. It is important to note that the Lyapunov stability test utilised in Proposition 2 is sufficient, but not necessary—even for LTI systems. Therefore, instability must be proved checking directly the closed–loop system matrix. Also, the sign constraints imposed to the tuning gains, which are required to ensure positivity of the shaped energy function, need not be imposed in the LTI case where, as indicated above, a stability analysis—other than Lyapunov—will be carried out.

Consider the following controllable, LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a_1 & 1-a_1 \end{bmatrix}x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u. \tag{25}$$

Some simple calculations show that it admits a pH representation

$$\dot{x} = FQx + gu \tag{26}$$

with $g := \mathrm{col}(0,1)$,

$$F := \begin{bmatrix} -1 & a_1 \\ \frac{1}{2}a_1 & -a_1^2 \end{bmatrix}, \quad Q := -\frac{2}{a_1^2}\begin{bmatrix} a_1^2 & a_1 \\ a_1 & 1-\frac{a_1}{2} \end{bmatrix}, \tag{27}$$

which satisfies $F + F^\top < 0$ and Assumption 1.[3]

Proposition 5: Consider the LTI, pH system (26), (27) in closed–loop with the controller (24). For all values of the controller gains k_e, k_a and K_I the closed–loop system is unstable.

Proof: The closed–loop system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ a_1 - a_1\tilde{k} & 1 - a_1 - \tilde{k} \end{bmatrix}x$$

where

$$\tilde{k} := \frac{2}{a_1^2}\left(k_e + \frac{2k_a}{a_1^2}\right)^{-1}(K_I + k_a).$$

[3] Assumption 2 is always satisfied for single input LTI systems.

Clearly, the closed–loop system matrix is Hurwitz if and only if the following conditions can be satisfied

$$a_1 - a_1\tilde{k} < 0, \quad 1 - a_1 - \tilde{k} < 0.$$

If the system is open–loop stable, that is, $a_1 < 0$, these inequalities are equivalent to

$$\tilde{k} < 1, \quad \tilde{k} > 1 - a_1,$$

Since $1 - a_1 > 1$ the inequalities cannot be simultaneously satisfied. ∎

VII. Concluding Remarks and Future Research

We have presented in this paper a new energy shaping method to stabilize pH systems that, in contrast with the classical PBC methods, does not require the solution of PDEs. The key modification introduced here is to abandon the objective of preservation in closed–loop of the pH structure, which is the condition that gives rise to the PDEs.

The class of systems for which the method is applicable is identified by Assumptions 1 and 2, which can be easily verified from the systems data. The invertibility Assumption 1 is rather weak, and is satisfied in many practical examples. Notice that if it does not hold then there exists equilibria for the open–loop system, which are not *extrema* of the energy function—a situation that its not reasonable in physical systems. The integrability Assumption 2 is a technical condition needed to create the term added to the open–loop energy function (9). As indicated at the end of Section III, this term may be interpreted as an integral term on the power shaping output. Unfortunately, besides this nice interpretation, we don't have at this point any physical, nor practical motivation, for Assumption 2. The controller design parameters are introduced to ensure that $H_d(x)$ is positive definite, hence, it is a Lyapunov function candidate—with the vector C (essentially) needed to make $(\nabla H_d)_* = 0$.

A lines of research that is currently being pursued is to base the construction of the controller on other passive outputs, besides y_{PS}. It was recently revealed in [25], [26] that there exists a large class of passive outputs that can be used for this purpose.

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