Strong iISS for neutrally stable systems by saturated linear state feedback

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Abstract: In this note, we propose a linear state-feedback that ensures Strong iISS to any neutrally stable system affected by actuator saturation. This robustness property was recently proposed as a compromise between the strength of input-to-state stability (ISS) and the generality of integral input-to-state stability (iISS). It ensures not only that the system is globally asymptotically stable in the absence of disturbances, but also that trajectories are bounded in response to any perturbation whose amplitude is below a certain threshold. It also guarantees that the state converges to the origin in response to any vanishing disturbance. Here Strong iISS is proven with respect to any additive disturbance acting outside the saturation, without any matching requirement.

Key Words: Saturated feedback, LTI systems, robustness

1 Introduction

It is well known that a necessary and sufficient condition for the stabilizability of a linear time-invariant (LTI) plant by saturated feedback is that its internal dynamics has no pole with positive real part [17]. Several works in the literature have proposed bounded stabilizing feedback for particular classes of systems whose internal dynamics exhibits no exponential instability. For neutrally stable systems (meaning LTI systems whose internal dynamics exhibits no unbounded solutions), it is also known that stabilization can be achieved using a saturated linear static feedback. Nonetheless, some classes of systems, although having no poles with positive real parts, cannot be stabilized by saturated linear static state-feedback; this class includes chains of three or more integrators [6, 18]. Several control strategies, by state or output feedback, have been proposed to stabilize such systems, including nested saturations [19] and linear combinations of saturated linear functions [16]. For LTI systems having no eigenvalues with positive real parts, it has been shown in [15] that global stabilization by bounded output feedback can be achieved if and only if the system is both detectable and stabilizable.

Beyond stabilization, it is often desirable to ensure some robustness properties in order to cope, for instance, with parameter uncertainty, measurement noise or exogenous disturbances. L_p-stabilization with respect to disturbances acting inside the saturation was achieved in [11] based on the low-and-high gain control law introduced in [10]. This robust satbilization has been extended to disturbances acting outside the saturation in [20] for chains of integrators under matching conditions. Also, explicit estimates of L_p input/output gains have been obtained for neutrally stable systems based on a saturated linear static feedback in [9].

Another natural candidate for the evaluation of robustness to exogenous inputs is the framework of input-to-state stability (ISS, [12, 14]) and its weaker variant integral ISS (iISS, [13]). In [1], a saturated linear state-feedback was proposed to ensure ISS with respect to sufficiently small disturbances despite parameter uncertainty for systems of dimension smaller than or equal to three, as well as feedforward systems. ISS of neutrally stable systems with respect to disturbances acting outside the saturation have been proposed in [2] under matching conditions. Other approaches guarantee ISS and iISS with bounded control to nonlinear systems based on Arstein's "universal constructions" [3, 8].

Among other robustness features, ISS ensures a bounded response to any bounded disturbance. Intuitively, one may expect that bounded controls fail in general at guaranteeing the solutions' boundedness if the disturbance acts outside the saturation with a too large amplitude (unless matching conditions are met between the saturated actuator and the disturbance: see *e.g.* [2, 20]). At first sight, for these systems, nothing more than ISS with respect to small inputs can be established, thus providing no information on the system's behavior for larger inputs. In this note, we provide sufficient conditions under which a more interesting property, namely Strong iISS, can be achieved by saturated feedback. This property, introduced in [4], not only guarantees ISS with respect to small inputs but also iISS. In particular, it ensures global asymptotic stability in the absence of disturbances, a bounded response to any disturbance whose amplitude is below a given threshold, but also the existence of solutions at all times even for disturbances above that threshold. It also guarantees that the state converges to zero in response to any vanishing disturbance, and it is known to be preserved under cascade interconnection. More details on this property can be found in [4, 5].

In this paper, we show that any LTI system with neutrally stable internal dynamics can be made Strongly iISS by a simple saturated linear state feedback. Strong iISS is established with respect to disturbances acting outside the saturation, and no matching condition between the actuation and the disturbances is assumed. We provide an explicit estimate of the maximum disturbance amplitude that can be tolerated without compromising solutions' boundedness. This estimate is confronted to numerical observations in the example of an harmonic oscillator robust stabilization.

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2 Notation and problem statement

2.1 Notation

A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{PD}$ if it is continuous and positive definite. It is of class $\mathcal{K}$ if, in addition, it is increasing. It is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$. $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if, given any fixed $t \geq 0$, $\beta(\cdot, t) \in \mathcal{K}$ and, given any fixed $s \geq 0$, $\beta(s, \cdot)$ is continuous, nonincreasing and tends to zero as its argument tends to infinity. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a positive integer p, $\mathcal{U}^p$ denotes the set of all measurable locally essentially bounded functions $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$. For a given $d \in \mathcal{U}^p$, $\|d\| := \text{ess}\sup_{t\geq 0} |d(t)|$. Given a constant $R > 0$, we let $\mathcal{U}^p_{<R}$ denote the set $\{d \in \mathcal{U}^p : \|d\| < R\}$.

2.2 Considered class of systems

We recall that a neutrally stable matrix is any matrix A such that the solutions of $\dot{x} = Ax$ are bounded. This is equivalent to requiring that there exists a symmetric positive definite matrix P such that $A^T P + PA$ is a negative semidefinite matrix. This paper focuses on the class of neutrally stable systems affected by actuator saturation and additive exogenous disturbances:

$$\dot{x} = Ax - B\sigma(u) + d, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and $d \in \mathbb{R}^n$ is the disturbance. No matching condition on the disturbance d is assumed: it can affect all directions of $\mathbb{R}^n$, not only those defined by the B matrix. The function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is assumed to belong to the class $\mathcal{S}^m$, as defined below.

Definition 1 ($\mathcal{S}$ and $\mathcal{S}^m$ functions, [9]) $\mathcal{S}$ denotes the set of all locally Lipschitz bounded function $S : \mathbb{R} \to \mathbb{R}$ satisfying: $S(0) = 0$, $S(s)s > 0$ for all $s \neq 0$, $\liminf_{s\to 0} S(s)/s > 0$, and $\liminf_{|s|\to\infty} |S(s)| > 0$. A vector function $\sigma : \mathbb{R}^m \to \mathbb{R}^m$ is said to be in the class $\mathcal{S}^m$ if it reads $\sigma(u) = (S_1(u_1), \ldots, S_m(u_m))^T$, for all $u \in \mathbb{R}^m$, where $S_i \in \mathcal{S}$ for each $i \in \{1, \ldots, m\}$.

We stress that the class $\mathcal{S}$ includes, but is not limited to, "usual" saturation functions such as sigmoids, arctan, tanh or $\text{sat}_\ell : s \mapsto \text{sign}(s)\min\{\ell|s|, 1\}$ with $\ell > 0$. For more details about the functions included in this class, the reader is invited to refer to [9].

We state the following useful property of $\mathcal{S}^m$-functions, which is a straightforward m-dimensional extension of [9, Remark 2].

Fact 1 *If $\sigma \in \mathcal{S}^m$, then there exist constants $a, b, K > 0$ and a measurable diagonal-matrix valued function $\tau : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ such that, for all $u \in \mathbb{R}^m$,*

$$aI \leq \tau(u) \leq bI, \quad |\sigma(u) - \tau(u)u| \leq Ku\sigma(u). \tag{2}$$

2.3 The Strong iISS property

As already stressed, the purpose of this paper is to robustly stabilize the system (1) by linear state feedback. We rely on the formalism of ISS, introduced by Sontag in [12]. We recall that a system is ISS if there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that its solutions satisfy $|x(t)| \leq \beta(|x_0|, t) + \gamma(\|d\|)$ at all times. In particular, ISS ensures boundedness of solutions for any bounded disturbance, and a vanishing state in response to any vanishing perturbation; see [14] for a survey.

Without matching requirements between the disturbance and the actuation, it is hopeless to try to make (1) ISS. Indeed, due to the bounded nature of σ, we can always pick a sufficiently large disturbance d that makes solutions diverge.

An alternative robustness property could be the integral input-to-state stability (iISS, [13]). Instead of considering the impact of the *amplitude* of the disturbance d, iISS evaluates the effect of the input *energy* on the solutions behavior. More precisely, a system is iISS if there exist $\beta \in \mathcal{KL}$ and $\mu_1, \mu_2, \mu \in \mathcal{K}_\infty$ such that its solutions satisfy $|x(t)| \leq \beta(|x_0|, t) + \mu_1\left(\int_0^t \mu_2(|d(s)|)ds\right)$ at all times. Like ISS, iISS ensures global asymptotic stability (GAS) of the disturbance-free system. However, it is a much weaker robustness property than ISS, as some iISS systems can be destabilized by arbitrarily small, and even vanishing, disturbances. In the present context, it seems that better robustness properties can be achieved. We believe that a better candidate to evaluate the robustness of (1) to exogenous disturbances is the Strong iISS, recently introduced in [4].

Definition 2 (Strong iISS, [4]) The system $\dot{x} = f(x, d)$ is said to be *Strongly iISS* if it is both ISS wrt small inputs and iISS. In other words, there exist $R > 0$, $\beta \in \mathcal{KL}$ and $\mu_1, \mu_2, \mu \in \mathcal{K}_\infty$ such that, for all $d \in \mathcal{U}^p$, all $x_0 \in \mathbb{R}^n$ and all $t \geq 0$, its solution satisfies the two properties:

$$|x(t)| \leq \beta(|x_0|, t) + \mu_1\left(\int_0^t \mu_2(|d(s)|)ds\right) \tag{3}$$

$$\|d\| < R \Rightarrow |x(t)| \leq \beta(|x_0|, t) + \mu(\|d\|). \tag{4}$$

The constant R is then called an *input threshold.*

It can easily be seen from this definition that Strong iISS ensures GAS in the absence of disturbances, just like iISS and ISS. Moreover, the state remains bounded if the disturbance amplitude is below the input threshold R. It was also shown in [4] that, if the disturbance d tends to zero, then the solution of a Strongly iISS system will also eventually tend to zero. Finally, like ISS, Strong iISS is well behaved under cascade interconnection [5].

2.4 From neutral stability to skew-symmetry

Reasoning as in [9, Section 3.2] the question of robustly stabilizing (1) when A is neutrally stable boils down to the case when A is skew-symmetric. This comes from the fact that any neutrally stable matrix is similar to $\text{diag}(A_H, A_S)$ where A_H is a Hurwitz matrix and A_S is skew-symmetric. This observation lets us consider that A is skew-symmetric without loss of generality. This will be assumed in the remainder of the paper.

3 Main result

3.1 Strong iISS by saturated linear feedback

Beyond L_p-stability results for disturbances entering the function σ, it was shown in [9] that the linear state feedback $u = B^T x$ globally asymptotically stabilizes the saturated system (1) in the absence of exogenous disturbances. Theorem 2 in [9] also shows that solutions are bounded if the disturbance d is of sufficiently low amplitude. The following result unifies and goes further these observations, by establishing that the same feedback law ensures Strong iISS to the system (1).

Theorem 1 *Let $\sigma \in \mathcal{S}^m$ and assume that $A \in \mathbb{R}^{n\times n}$ is skew-symmetric and that the pair (A, B) is controllable. Then the linear static state feedback $u = B^T x$ makes the saturated system (1) Strongly iISS.*

The proof of this result, provided in Section 5.1, relies on the following technical result.

Lemma 1 *Let $A \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix and $B \in \mathbb{R}^{n\times m}$ be such that the pair (A, B) is controllable. Let $D : \mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times m}$ be any bounded measurable matrix-valued function satisfying $D(t) + D(t)^T \geq \varepsilon I$ for almost all $t \in \mathbb{R}_{\geq 0}$, where ε denotes a positive constant. Then, there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that, for almost all $t \in \mathbb{R}_{\geq 0}$,*

$$(A - BD(t)B^T)^T P + P(A - BD(t)B^T) \leq -I.$$

Moreover, this matrix P can be picked as $P = P_0 + \chi I$, where $P_0 \in \mathbb{R}^{n\times n}$ denotes any symmetric positive definite matrix satisfying

$$(A - \varepsilon BB^T)^T P_0 + P_0(A - \varepsilon BB^T) \leq -2I, \qquad (5)$$

as ensured by the controllability of (A, B), and χ denotes any constant satisfying

$$\chi \geq \frac{1}{\varepsilon} \sup_{t\geq 0} |P_0 B(\varepsilon I - D(t))| \,. \qquad (6)$$

The proof of Lemma 1 can be found along the lines of [9, Lemma 3.2 and Corollary 1] and is therefore omitted.

3.2 Estimate of the input threshold

The proof of Theorem 1 is constructive: once a matrix P such as the one generated by Lemma 1 is known, we can estimate the resulting input threshold for (1), meaning the maximum disturbance amplitude that does not compromise solutions' boundedness. These findings are summarized by the following result.

Corollary 1 *Let $\sigma \in \mathcal{S}^m$ and let $A \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix such that the pair (A, B) is controllable. Let a, b and K be any positive constants such that (2) holds for all $u \in \mathbb{R}^m$, where $\tau : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is a measurable diagonal-matrix valued function. Let $P_0 \in \mathbb{R}^{n\times n}$ be any symmetric positive definite matrix such that*

$$(A - 2aBB^T)^T P_0 + P_0(A - 2aBB^T) \leq -2I. \qquad (7)$$

Finally, let χ be the positive constant defined as

$$\chi = \frac{|2a - b|}{2a} |P_0 B| \,, \qquad (8)$$

and let $P := P_0 + \chi I$. Then (1) in closed loop with the control law $u = B^T x$ is Strongly iISS with input threshold

$$R = \frac{1}{2K\,|PB| + |P|\,(4K\,|PB|\,/3)^{1/3}}. \qquad (9)$$

Note that the existence of such constants a, b, K and such a function τ is ensured by Fact 1. The existence of a matrix P_0 satisfying (7) is guaranteed by the controllability of the pair (A, B). The proof of this result follows straightforwardly from that of Theorem 1. We provide its main steps in Section 5.2 for the sake of completeness.

For some particular saturations, the function τ of Fact 1 can be picked as a constant matrix. This includes for instance saturation functions of the form $\sigma(s) = (S(s), \ldots, S(s))^T$ where

$$S(s) = \text{sat}_\ell(s) := \text{sign}(s) \min\{\ell\,|s|\,; 1\}, \quad \forall s \in \mathbb{R}, \quad (10)$$

where $\ell > 0$ denotes its linear slope. For such functions, the expression of the input threshold estimate is slightly simpler. We summarize this in the following corollary.

Corollary 2 *Let $S \in \mathcal{S}^m$ and let $A \in \mathbb{R}^{n\times n}$ be a skew-symmetric matrix such that the pair (A, B) is controllable. Assume that there exist $\bar{\tau}, K > 0$ such that*

$$|\sigma(u) - \bar{\tau}u| \leq K u\sigma(u), \quad \forall u \in \mathbb{R}^m. \qquad (11)$$

Let $P_0 \in \mathbb{R}^{n\times n}$ be any symmetric positive definite matrix such that

$$(A - 2\bar{\tau}BB^T)^T P_0 + P_0(A - 2\bar{\tau}BB^T) \leq -2I, \qquad (12)$$

as ensured by the controllability of the pair (A, B). Finally, let $P := P_0 + \frac{1}{2}|P_0 B|\, I$. Then the system (1) in closed loop with the control law $u = B^T x$ is Strongly iISS with the input threshold given in (9).

4 Example: the harmonic oscillator

We finally illustrate the above results with the following planar system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{sat}_\ell(u) + d, \qquad (13)$$

where $\ell > 0$ and sat_ℓ was introduced in (10). This system is in the form of (1) with

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma(s) = \text{sat}_\ell(s).$$

Note that the perturbation is not matching, as it affects the dynamics of x_1 although no control is available in that direction. The pair (A, B) being controllable, we conclude with Theorem 1 that the state feedback $u = B^T x = x_2$ makes the system (13) Strong iISS.

In addition, we can estimate its input threshold by invoking Corollary 2. To that end, notice that $|\text{sat}_\ell(s) - \ell s| \leq \ell\, s\, \text{sat}_\ell(s)$ for all $s \in \mathbb{R}$, meaning that (11) is satisfied with $\bar{\tau} = K = \ell$. Furthermore, the Lyapunov equation (12) is satisfied with the following symmetric positive definite matrix:

$$P_0 = \frac{1}{2} \begin{pmatrix} 2\ell + 1/\ell & 1 \\ 1 & 1/\ell \end{pmatrix}.$$

Noticing that $|P_0 B| = \frac{1}{2}\sqrt{1 + \ell^2}$, the input threshold of (13) in closed loop with $u = B^T x = x_2$ can be computed as a function of the slope ℓ of the saturation according to the estimate (9). We obtain the curve reported in Figure 1.

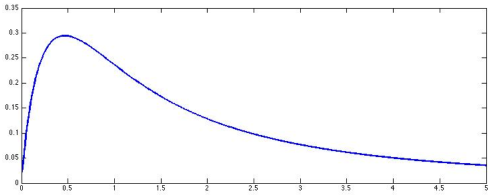

Fig. 1: Input threshold estimate of the planar system (13) as a function of the slope ℓ of the saturation.

This plot shows that the estimate of R provided by Corollary 2 is not optimally accurate for this specific example, as it tends to zero for large values of the slope ℓ (meaning when sat_ℓ tends to the sign function). Simulations suggest that the actual input threshold rather tends to 1. Although providing an explicit estimate of the input threshold, the tools employed in this paper fail at giving a tighter bound. The authors believe that a better choice of the feedback law than merely $u = B^T x$ would allow a better estimate of the guaranteed input threshold, but this goes beyond the scope of the paper.

5 Proofs

5.1 Proof of Theorem 1

In order to establish Theorem 1, we make use of the Lyapunov function proposed in [9]. More precisely, letting

$$V(x) := \frac{c}{3}|x|^3 + x^T P x, \quad \forall x \in \mathbb{R}^n,$$

where $P \in \mathbb{R}^{n\times n}$ is a convenient symmetric positive definite matrix and $c > 0$ denotes some well chosen constant, we claim that the proper C^1 Lyapunov function candidate

$$\tilde{V}(x) := \frac{1}{3}\left[(1+V(x))^{1/3} - 1\right] \tag{14}$$

satisfies, along the solutions of (1) in closed loop with $u = B^T x$,

$$\dot{\tilde{V}}(x) \leq -W(x) + \gamma(|d|), \tag{15}$$

where γ denotes a class $\mathcal{K}_\infty$ function and W is a continuous positive definite function satisfying $\liminf_{x\to\infty} W(x) > 0$. According to [4, Theorem 1], this establishes Strong iISS, as claimed in the statement.

To that aim, we start by decomposing the function V as $V(x) = cV_1(x) + V_2(x)$, where $V_1(x) := |x|^3/3$ and $V_2(x) := x^T P x$. We will study the time derivative of V_1 and V_2 along the solutions of (1) separately. We start by the function V_1: recalling that $x^T A x = 0$ for all $x \in \mathbb{R}^n$ (as A is skew-symmetric), direct computations show that

$$\begin{aligned}\dot{V}_1 &= |x|\, x^T \dot{x} \\ &= -|x|\, x^T B\sigma(B^T x) + |x|\, x^T d.\end{aligned} \tag{16}$$

In order to study the derivative of V_2, consider any initial state $x_0 \in \mathbb{R}^n$ and any input signal $d \in \mathcal{U}^n$. Let τ be the function generated by Fact 1 and let

$$\tilde{A}(t) := A - B\tau(B^T x(t))B^T, \quad \forall t \geq 0,$$

where $x(\cdot)$ denotes the solution of (1) starting from x_0. Note that the saturation function σ being locally Lipschitz and bounded, the existence of the solutions of (1) is ensured at all forward time, which makes the above matrix $\tilde{A}(t)$ well defined at all times $t \geq 0$. With this notation, the system (1) in closed loop with $u = B^T x$ can be rewritten as

$$\dot{x} = \tilde{A}(t)x + B\left[\tau(B^T x)B^T x - \sigma(B^T x)\right] + d. \tag{17}$$

Now, in view of Fact 1 and recalling that $\tau(u)$ is a diagonal matrix for each $u \in \mathbb{R}^m$, it holds that

$$\tau(B^T x(t)) + \tau(B^T x(t))^T \geq 2aI, \quad \forall t \geq 0,$$

for some constant $a > 0$ independent of x_0. It follows that all the conditions of Lemma 1 are fulfilled with $D(t) = \tau(B^T x(t))$ and $\varepsilon = 2a$. Consequently, there exists a symmetric positive definite matrix $P = P^T$ such that

$$\tilde{A}(t)^T P + P\tilde{A}(t) \leq -I, \quad \forall t \geq 0.$$

The derivative of the function $V_2(x) = x^T P x$ with this particular matrix P along the solutions of (17) reads

$$\begin{aligned}\dot{V}_2 = {} & x^T\left(\tilde{A}(t)^T P + P\tilde{A}(t)\right)x \\ & + 2x^T PB\left[\tau(B^T x)B^T x - \sigma(B^T x)\right] + 2x^T P d \\ \leq {} & -|x|^2 + 2|x|\,|PB|\left|\tau(B^T x)B^T x - \sigma(B^T x)\right| + 2x^T P d.\end{aligned}$$

Recalling that, in view of Fact 1, $\left|\tau(B^T x)B^T x - \sigma(B^T x)\right| \leq K x^T B\sigma(B^T x)$, it follows that

$$\dot{V}_2 \leq -|x|^2 + 2K|x|\,|PB|\, x^T B\sigma(B^T x) + 2x^T P d. \tag{18}$$

Combining (16) and (18) and picking $c = 2K|PB|$, we obtain that the derivative of $V(x) = cV_1(x) + V_2(x)$ along the solutions of (1) satisfies

$$\dot{V} \leq -|x|^2 + c|x|^2|d| + 2|P|\,|x|\,|d|.$$

Thus the function $\tilde{V}$, defined in (14), satisfies

$$\begin{aligned}\dot{\tilde{V}} = {} & \frac{\dot{V}(x)}{[1+V(x)]^{2/3}} \\ \leq {} & -\frac{|x|^2}{[1+V(x)]^{2/3}} + c\frac{|x|^2}{[1+V(x)]^{2/3}}|d| \\ & + 2|P|\frac{|x|}{[1+V(x)]^{2/3}}|d|.\end{aligned} \tag{19}$$

We now analyze the two nonnegative term of this upper bound separately. First, it holds that

$$\begin{aligned}\frac{|x|^2}{[1+V(x)]^{2/3}} &= \frac{|x|^2}{\left[1 + x^T P x + \frac{c}{3}|x|^3\right]^{2/3}} \\ &\leq \frac{|x|^2}{\left[1 + \frac{c}{3}|x|^3\right]^{2/3}} \\ &\leq \left(\frac{|x|^3}{1+\frac{c}{3}|x|^3}\right)^{2/3} \\ &\leq \left(\frac{3}{c}\right)^{2/3}.\end{aligned} \tag{20}$$

In the same way, we have that

$$\begin{aligned}\frac{|x|}{[1+V(x)]^{2/3}} &= \frac{|x|}{\left[1+x^TPx+\frac{c}{3}|x|^3\right]^{2/3}}\\ &\leq \frac{|x|}{\left[1+\frac{c}{3}|x|^3\right]^{2/3}}\\ &\leq \left(\frac{|x|^{3/2}}{1+\frac{c}{3}|x|^3}\right)^{2/3}\\ &\leq \left(\frac{3}{4c}\right)^{1/3}, \qquad (21)\end{aligned}$$

where the last inequality follows form the fact that the function $s \mapsto s/(1+cs^2/3)$ reaches its maximum $\sqrt{3/4c}$ at $s = \sqrt{3/c}$. Plugging (20)-(21) into (19) leads, as claimed, to

$$\dot{V}(x) \leq -W(x) + \gamma(|d|),$$

where γ denotes the following $\mathcal{K}_\infty$ function:

$$\gamma(s) := \left[(9c)^{1/3} + |P|\left(\frac{6}{c}\right)^{1/3}\right] s, \quad \forall s \geq 0.$$

and $W : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is the function defined as

$$W(x) := \frac{|x|^2}{\left[1+x^TPx+\frac{c}{3}|x|^3\right]^{2/3}}, \quad \forall x \in \mathbb{R}^n,$$

where we recall that $c = 2K|PB|$. Notice that W is continuous and positive definite and satisfies

$$\liminf_{|x|\to\infty} W(x) = \liminf_{|x|\to\infty} \frac{|x|^2}{(c/3)^{2/3}|x|^2} = \left(\frac{3}{c}\right)^{2/3}.$$

Thus, we can apply [4, Theorem 1] to conclude that (1) is Strongly iISS with input threshold

$$\begin{aligned}\gamma^{-1}\left((3/c)^{2/3}\right) &= \frac{1}{c+|P|(2c/3)^{1/3}}\\ &= \frac{1}{2K|PB|+|P|(4K|PB|/3)^{1/3}}.\end{aligned}$$

5.2 Proof of Corollary 1

Since $aI \leq \tau(u) \leq bI$, the constant χ introduced in (8) satisfies (6). By Lemma 1, the proof of Theorem 1 can be repeated with $P = P_0 + \chi I$. The rest of the proof is identical.

6 Conclusion

Focusing on the specific class of LTI systems whose internal dynamics is neutrally stable, we have shown that a simple linear static state-feedback ensures Strong iISS to both matching and non-matching additive disturbances acting outside the saturation. This result was illustrated on an academic example, that underlined the limits of the input threshold estimate and suggests that further work is needed to choose the feedback gains in order to obtain a tighter input threshold estimate. Finally, further research can be envisioned to check the robustness properties of saturated stabilization of chains of integrators; a first step in that direction will be presented in [7].

References

[1] D. Angeli, Y. Chitour, and L. Marconi. Robust stabilization via saturated feedback. *IEEE Trans. Autom. Control*, 50(12):1997–2014, January 2005.

[2] M. Arcak and A.R. Teel. Input-to-state stability for a class of Lurie systems. *Automatica*, 38(11):1945–1949, 2002.

[3] R. Azouit, A. Chaillet, and L. Greco. Robustness under saturated feedback: Strong iISS for a class of nonlinear systems. In *European Control Conference*, Strasbourg, France, 2014.

[4] A. Chaillet, D. Angeli, and H. Ito. Combining iISS and ISS with respect to small inputs: the Strong iISS property. *IEEE Trans. on Automat. Contr.*, 59(9):2518–2524, Sept 2014.

[5] A. Chaillet, D. Angeli, and H. Ito. Strong iISS is preserved under cascade interconnection. *Automatica*, 50(9):2424–2427, Sept. 2014.

[6] A.T. Fuller. In-the-large stability of relay and saturating control systems with linear controllers. *International Journal of Control*, 10(4):457–480, 1969.

[7] J. Laporte, A. Chaillet, and Y. Chitour. Global stabilization of multiple integrators by a bounded feedback with constraints on its successive derivatives. In *Submitted to IEEE Conf. on Decision and Control*, Japan, Decemeber 2015.

[8] D. Liberzon. ISS and integral-ISS disturbance attenuation with bounded controls. *Proc. 41st. IEEE Conf. Decision Contr.*, 3:2501–2506, 2002.

[9] W. Liu, Y. Chitour, and E.D. Sontag. On finite-gain stabilizability of linear systems subject to input saturation. *SIAM Journal on Control and Optimization*, 34(4):1190–1219, 1996.

[10] A. Megretski. l_2 bibo output feedback stabilization with saturated control. In *In IFAC World Congress*, pages 435–440, 1996.

[11] A. Saberi, P. Hou, and A.A. Stoorvogel. On simultaneous global external and global internal stabilization of critically unstable linear systems with saturating actuators. *IEEE Trans. Autom. Control*, 45(6):368–378, 2000.

[12] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Autom. Control*, 34(4):435–443, 1989.

[13] E.D. Sontag. Comments on integral variants of ISS. *Systems & Control Letters*, 34:93–100, 1998.

[14] E.D. Sontag. *Input to state stability: Basic concepts and results*, chapter in Nonlinear and Optimal Control Theory, pages 163–220. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2006. P. Nistri and G. Stefani eds.

[15] E.D. Sontag and H.J. Sussmann. Nonlinear output feedback design for linear systems with saturating controls. In *Proc. 29th. IEEE Conf. Decision Contr.*, pages 3414–3416, Honolulu, HI, 1990.

[16] H.J. Sussmann, E.D. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. *Proc. of the 32nd IEEE Conference on Decision and Control*, pages 1802–1807, 1993.

[17] H.J. Sussmann, E.D. Sontag, and Y. Yang. A general result on the stabilization of linear systems using bounded controls. *IEEE Trans. Autom. Control*, 39(12):2411–2425, 1994.

[18] H.J. Sussmann and Y. Yang. On the stabilizability of multiple integrators by means of bounded feedback controls. In *Proc. 30th. IEEE Conf. Decision Contr.*, Brighton, UK, Dec. 1991.

[19] A.R. Teel. Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, 18(3):165–171, 1992.

[20] X. Wang, A. Saberi, A.A. Stoorvogel, and H.F. Grip. Control of a chain of integrators subject to actuator saturation and disturbances. *International Journal of Robust and Nonlinear Control*, 22(14):1562–1570, 2012.