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Tarek Ahmed-Ali, Fouad Giri, Miroslav Krstic, Françoise Lamnabhi-Lagarrigue. Adaptive Observer for a Class of Output-Delayed Systems with Parameter Uncertainty - A PDE Based Approach. 12th IFAC International Workshop on Adaptation and Learning in Control and Signal Processing, IFAC, Jun 2016, Eindhoven, Netherlands. pp.158-163, 10.1016/j.ifacol.2016.07.944 . hal-01313878

HAL Id: hal-01313878

<https://hal-centralesupelec.archives-ouvertes.fr/hal-01313878>

Submitted on 13 Jul 2020

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# Adaptive Observer for a Class of Output-Delayed Systems with Parameter Uncertainty - A PDE Based Approach

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**Abstract:** The problem of state observation is addressed for a class of systems subject to sensor delay and parameter uncertainty. The unknown parameter vector enter a finite-dimensional state equation through a possibly output-dependent regressor. The sensor delay effect is captured by a first-order hyperbolic PDE. Doing so, the system turns out to be an ODE-PDE association with a connection point not accessible to measurements. An adaptive observer is constructed by combining ideas from PDE-based and ODE-based design approaches. The observer provides estimates of the ODE subsystem states and parameters, on the one hand, and of the sensor states, on the other. Observer exponential convergence is established under an ad-hoc persistent excitation condition involving the regressor.

**Keywords:** Delayed systems, Hyperbolic PDE, Adaptive observer, Backstepping design.

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## 1. INTRODUCTION

Time delay is a common property that characterizes several categories of real-life systems. It accounts of physical phenomena such as material transport, traffic flows, networked systems, chemical and biological reactors, and others. From theoretical viewpoints, time delay are infinite dimensional operators and may be source of instability. Therefore, it is natural that an intensive research activity has been devoted to various issues of control system design in presence of time delay, see e.g. (Richard, 2003; Krstic, 2009) and reference lists therein. In this respect, much attention has been paid, over the past three decades, to observability analysis and observer design. Earliest results have mainly concerned linear systems, see e.g. (Richard, 2003; Krstic, 2009; Bhat and Koivo, 1976; Leyva-Ramos and Rearson, 1995; Pearson and Fiagbedzi, 1989; Trinh and Aldeen, 1997). Lately, observer designs for nonlinear delayed systems have been proposed, see e.g. (Watanabe, 1996; Hou and Patton, 2002; Germani et al., 2002; Cacace et al., 2002; Ahmed-Ali et al., 2013).

In this paper we are considering the problem of state observation of delayed systems which are further subject to model parameter uncertainty. We propose an exponentially convergent adaptive observer for a class of output-delayed systems with unknown parameters. The latter enter linearly the state equation and the associated regressor is any nonlinear time function, that is allowed to be output-dependent. Just as in (Krstic and Smyshlyayev, 2008), the time-delay effect is captured through a first-hyperbolic PDE and a backstepping-like design technique is used to design an adaptive observer that estimates the ODE state and parameter vectors as well as the sensor states which, in fact, coincide

with the system future outputs. The observer exponential convergence is established under an ad-hoc persistent excitation condition involving the regressor. Although it does not follow mutatis-mutandis the design approach in (Krstic and Smyshlyayev, 2008), the new observer can be seen as an adaptive extension of the observer proposed there. Compared with classical delay-compensating observers (e.g. Germani et al., 2002; Cacace et al., 2002; Ahmed-Ali et al., 2013), our adaptive observer is full-order because it estimates both the system (finite-dimensional) state and the sensor (infinite-dimensional) state. output that do not estimate the sensor state. A more exhaustive comparison can be found in (Krstic, 2009, ch. 3].

The paper is organised as follows: first, the observation problem under study is formulated in Section 2; then, the observer design and analysis are respectively dealt with in Sections 3 and 4; a conclusion and reference list end the paper. To alleviate the presentation, some technical proofs are appended.

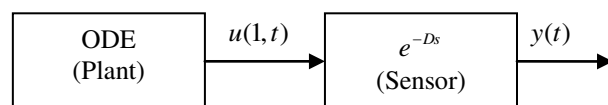


Fig. 1. System structure

## 2. PROBLEM FORMULATION

As this is depicted by Fig. 1, the system under study consists of a finite-dimensional nonlinear subsystem connected in series with a time delay. Analytically, the considered output-delayed system is described as follows:

$$\dot{X}(t) = AX(t) + \phi(t)\theta, \quad t \geq 0, \quad (1a)$$

$$y(t) = CX(t-D) \quad (\text{output}) \quad (1b)$$

where  $A \in \mathbf{R}^{n \times n}$  and  $C \in \mathbf{R}^{1 \times n}$  are known constant matrices and the pair  $(A, C)$  is observable;  $\phi: C([0, \infty): \mathbf{R}^{n \times m})$  is a known bounded continuous function;  $D$  denotes a known time delay which is just supposed to be nonnegative; the output  $y(t)$  is accessible to measurements, but the state vector  $X(t) \in \mathbf{R}^n$  is not.

Following the approach developed in (Ahmed-Ali et al., 2008), the output equation (1b) is represented by a first-order hyperbolic equation. Doing so, the system under study turns out to be modelled by the following state-space representation:

$$\dot{X}(t) = AX(t) + \phi(t)\theta, \quad t \geq 0 \quad (2a)$$

$$u(D, t) = CX(t) \quad (2b)$$

$$u_x(x, t) = u_x(x, t), \quad 0 \leq x \leq D \quad (2c)$$

$$y(t) = u(0, t) \quad (2d)$$

It is well known that the solution of (2b-c) is  $u(x, t) = CX(t+x-D)$ . Therefore, the output equation (2d) gives the delayed output  $y(t) = CX(t-D)$ , which is identical to (1b).

The aim is to design an observer that provides accurate online estimates of the finite-dimensional state  $X(t)$ , the distributed state  $u(x, t)$  ( $0 \leq x \leq D$ ), and the unknown parameter vector  $\theta$ . The observer must only make use of the system output  $y(t)$ .

**Remark 1.** The above observation problem extends a similar problem in (Krstic and Smyshlyaev, 2008) where no uncertain parameters were considered i.e.  $\phi(t)\theta = 0$ . In this regard, note that the vector  $\phi(t)$  in (2a) is allowed to be output-dependent i.e. one can have  $\phi(t) = \psi(t, y(t))$  for some continuous function  $\psi$ . In such a case, the dynamics of the ODE (2a) turns out to be nonlinear. On the other, the present setting is quite different from the one in (Ahmed-Ali et al., 2015) even though an ODE-PDE system structure is considered in both. Indeed, the ODE subsystem in (Ahmed-Ali et al., 2015) is more general than the present one in that it includes a Lipschitz state function. But, it is in the same time less general since it is a triangular structure and involves no parameter uncertainty. Owing to the infinite-dimensional subsystem, it is a parabolic type in (Ahmed-Ali et al., 2015) while it is presently a hyperbolic nature  $\square$

### 3. ADAPTIVE OBSERVER DESIGN

A quite general observer structure is the following:

$$\dot{\hat{X}} = A\hat{X} + \phi(t)\hat{\theta} - K\tilde{u}(0, t) + v_0(t) \quad (3a)$$

$$\hat{u}_t(x, t) = \hat{u}_x(x, t) - k(x)\tilde{u}(0, t) + v_1(x, t) \quad (3b)$$

$$\hat{u}(D, t) = C\hat{X}(t) \quad (3c)$$

for all  $t \geq 0$  and all  $x \in [0, D]$ , where  $\tilde{u}(0, t) = \hat{u}(0, t) - u(0, t) = \hat{u}(0, t) - y(t)$ . The vector and scalar gains,  $K \in \mathbf{R}^n$  and  $k(x) \in \mathbf{R}$ , as well as the additional correction terms,  $v_0(t), v_1(t) \in \mathbf{R}$ , have yet to be defined. To this end, introduce the state and parameter estimation errors:

$$\tilde{X} = \hat{X} - X, \quad \tilde{u} = \hat{u} - u, \quad \tilde{\theta} = \hat{\theta} - \theta \quad (4)$$

From (2a-c) and (3a-c), it is readily seen that these errors undergo the following equations:

$$\dot{\tilde{X}} = A\tilde{X} + \phi(t)\tilde{\theta} - K\tilde{u}(0, t) + v_0(t) \quad (5a)$$

$$\tilde{u}_t(x, t) = \tilde{u}_{xx}(x, t) - k(x)\tilde{u}(0, t) + v_1(x, t) \quad (5b)$$

$$\tilde{u}(D, t) = C\tilde{X}(t) \quad (5c)$$

Consider the following backstepping transformations, partly inspired by (Krstic M. and A. Smyshlyaev, 2008) and (Zhang, 2015):

$$Z(t) = \tilde{X}(t) - \lambda_0(t)\tilde{\theta}(t), \quad (6a)$$

$$\varepsilon(x, t) = \tilde{u}(x, t) - CM(x)\tilde{X}(t) - \lambda_1(x, t)\tilde{\theta}(t) \quad (6b)$$

where  $M(x) \in \mathbf{R}^{n \times n}$ ,  $\lambda_0(t) \in \mathbf{R}^{n \times m}$  and  $\lambda_1(x, t) \in \mathbf{R}^{1 \times m}$  are auxiliary functions yet to be defined. The error system (5a-c) rewrites in terms of the new coordinates  $Z$  and  $\varepsilon$ , as follows (see Appendix A):

$$\begin{aligned} \dot{Z}(t) = & [A - KCM(0)]Z - K\varepsilon(0, t) + v_0(t) - \lambda_0(t)\dot{\tilde{\theta}}(t) \\ & + \left( [A - KCM(0)K]\lambda_0(t) + \phi(t) - K\lambda_1(0, t) - \dot{\lambda}_0(t) \right) \tilde{\theta} \end{aligned} \quad (7a)$$

$$\begin{aligned} \varepsilon_t(x, t) = & \varepsilon_x(x, t) + (CM(x)K - k(x))\tilde{u}(0, t) \\ & + C \left( \frac{d^2 M}{dx^2}(x) - M(x)A \right) \tilde{X}(t) \\ & + (\lambda_{1,x}(x, t) - CM(x)\phi(t) - \lambda_{1,t}(x, t))\tilde{\theta}(t) \\ & + v_1(x, t) - \lambda_1(x, t)\dot{\tilde{\theta}}(t) - CM(x)v_0(t) \end{aligned} \quad (7b)$$

$$\varepsilon(D, t) = C\tilde{X}(t) - \lambda_1(D, t)\tilde{\theta}(t) - CM(D)\tilde{X}(t) \quad (7c)$$

We seek functions  $k(x)$ ,  $v_0(t)$ ,  $v_1(x, t)$ ,  $M(x)$ ,  $\lambda_0(t)$  and  $\lambda_1(x, t)$  that make the error system (7a-c) coincide with the following target system:

$$\dot{Z}(t) = [A - KCM(0)]Z - K\varepsilon(0, t) \quad (8a)$$

$$\varepsilon_t(x, t) = \varepsilon_x(x, t) \quad (8b)$$

$$\varepsilon(D, t) = 0 \quad (8c)$$

for all  $t \geq 0$  and all  $x \in [0, D]$ . The target system (8a-c) is motivated by the fact that, the subsystem (8b) (which represents a time delay) has the solution

$\varepsilon(x,t) = \varepsilon(D,t+x-D)$  for all  $t \geq 0$ . Due to (8c), this entails  $\varepsilon(x,t) = 0$  for  $t \geq D$ . Then, it follows from (8a) that the (finite-dimensional) state  $Z(t)$  is exponentially vanishing provided that the matrix  $A - KCM(0)$  is Hurwitz which will prove not to be an issue. Bearing in mind these observations, it follows by comparing (7a-c) and (8a-c) that the various auxiliary functions and constants introduced so far must meet the following requirements:

$$A - KCM(0) \text{ is Hurwitz} \quad (9a)$$

$$\frac{dM}{dx}(x) = M(x)A, \quad M(D) = I \quad (9b)$$

$$k(x) = CM(x)K \quad (9c)$$

$$\dot{\lambda}_0(t) = [A - KCM(0)K]\lambda_0(t) + \phi(t) - K\lambda_1(0,t) \quad (9d)$$

$$\lambda_{1,t}(x,t) = \lambda_{1,x}(x,t) - CM(x)\phi(t) \quad (9e)$$

$$\lambda_1(D,t) = 0 \quad (9f)$$

$$v_0(t) = \lambda_0(t)\dot{\theta}(t) \quad (9g)$$

$$v_1(x,t) = \lambda_1(x,t)\dot{\theta}(t) + CM(x)v_0(t) \quad (9h)$$

where the initial values of the auxiliary states  $\lambda_0(0) \in \mathbf{R}$  and  $\lambda_1(x,0) \in \mathbf{R}^{1 \times m}$  are arbitrary. In the sequel, we simply let them to be zero i.e.  $\lambda_0(0) = 0$  and  $\lambda_1(x,0) = 0, \forall x \in [0, D]$ . The solution of (9b) is:

$$M(x) = e^{(x-D)A} \in \mathbf{R}^{n \times n} \quad (10)$$

which immediately implies that  $M(x)$  is invertible and commutating with  $A$ . That is,

$$AM(x) = M(x)A \text{ and } M^{-1}(x)A = AM^{-1}(x) \quad (11)$$

These properties prove to be useful for meeting the requirement (9a), See Remark 2 (Part a).

Finally, writing equations (6a-b) at  $t = 0$  suggest the following least-squares parameter adaptive law:

$$\dot{\hat{\theta}}(t) = -\rho R(t)(CM(0)\lambda_0(t) + \lambda_1(0,t))^T \tilde{u}(0,t) \quad (12a)$$

$$\begin{aligned} \dot{R}(t) = R(t) - R(t)(CM(0)\lambda_0(t) + \lambda_1(0,t))^T \\ \times (CM(0)\lambda_0(t) + \lambda_1(0,t))R \end{aligned} \quad (12b)$$

where  $\hat{\theta}(0)$  and  $R(0) = R^T(0) > 0$  are arbitrarily chosen. The adaptive observer thus designed is constituted of equations (3a-c), (9b-h) and (12a-b). For convenience, the observer is summarized in Table I.

Table I. Adaptive Observer

State observer:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + \phi(t)\hat{\theta} - K\tilde{u}(0,t) + \lambda_0(t)\dot{\theta}(t) \quad (13a)$$

$$\begin{aligned} \hat{u}_t(x,t) = \hat{u}_x(x,t) - CM(x)K\tilde{u}(0,t) \\ + (\lambda_1(x,t) + CM(x)\lambda_0(t))\dot{\theta}(t) \end{aligned} \quad (13b)$$

$$\hat{u}(D,t) = C\hat{X}(t) \quad (13c)$$

for all  $t \geq 0$  and all  $x \in [0, 1]$ , where  $\hat{X}(0) \in \mathbf{R}^n$  is arbitrary and  $K \in \mathbf{R}^n$  is such that  $A - KCM(0)$  is Hurwitz.

Parameter adaptive law

$$\dot{\hat{\theta}}(t) = -\rho R(t)\Lambda(t)\tilde{u}(0,t) \quad (13d)$$

$$\dot{R}(t) = R(t) - R(t)\Lambda(t)\Lambda(t)^T R \quad (13e)$$

$$\Lambda(t) = (CM(0)\lambda_0(t) + \lambda_1(0,t))^T \in \mathbf{R}^m \quad (13f)$$

where  $\hat{\theta}(0) \in \mathbf{R}^m$  and  $R(0) \in \mathbf{R}^{m \times m}$  are arbitrarily chosen but  $R(0) = R^T(0) > 0$ . An insight on how to choose the parameter  $\rho > 0$  will be given latter (see Theorem 1).

Auxiliary states and functions

$$\dot{\lambda}_0(t) = [A - KCM(0)K]\lambda_0(t) + \phi(t) - K\lambda_1(0,t) \quad (13g)$$

$$\lambda_{1,t}(x,t) = \lambda_{1,x}(x,t) - CM(x)\phi(t) \quad (13h)$$

$$\lambda_0(0) = 0, \quad \lambda_1(D,t) = 0 \in \mathbf{R}^{1 \times m}, \quad \lambda_1(x,0) = 0 \quad (13i)$$

$$M(x) = e^{(x-D)A}, \quad \forall x \in [0, D] \quad (13j)$$

**Remark 2.** a) Using (11) one has:

$$\begin{aligned} A - KCM(0) &= M^{-1}(0)AM(0) - KCM(0) \\ &= M^{-1}(0)[A - M(0)KC]M(0) \end{aligned} \quad (14)$$

which shows that there is similarity between the  $A - KCM(0)$  and  $A - M(0)KC$  and, accordingly, their eigenvalues are identical. On the other hand, since  $(A, C)$  is observable, there exists a gain  $L$  such that  $A - LC$  is Hurwitz. Letting  $K = M^{-1}(0)L$ , it follows that  $A - M(0)KC$  is Hurwitz which, in view of (14), implies that so is  $A - KCM(0)$ . It is thus demonstrated that the requirement (9a) is not an issue.

b) Using (13j), one has  $K = e^{AD}L$  and  $M(x)K = e^{Ax}L$ , with  $L$  as in Part a. Then, the adaptive observer equations (13a-c) rewrites in term of  $L$  as follows:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + \phi(t)\hat{\theta} - e^{AD}L\tilde{u}(0,t) + \lambda_0(t)\dot{\theta}(t) \quad (15)$$

$$\begin{aligned} \hat{u}_t(x,t) = \hat{u}_x(x,t) - Ce^{Ax}L\tilde{u}(0,t) \\ + (\lambda_1(x,t) + CM(x)\lambda_0(t))\dot{\theta}(t) \end{aligned} \quad (16a)$$

$$\hat{u}(D,t) = C\hat{X}(t) \quad (16b)$$

Clearly, these equations are an adaptive version of the observer (88)-(90) in (Krstic and Smyshlyaev, 2008). Indeed, if  $\phi(t)$  and  $\dot{\theta}(t)$  are set to zero then, (15)-(16) boil down to (88)-(90). On the other hand, the present adaptive observer design clearly applies mutatis-mutandis to the case where the function  $\phi(t) = \psi(t, y(t))$  is a function of the output. That is, the present observer design and analysis are not limited to linear systems, unlike (Krstic and Smyshlyaev, 2008).

c) In (Ahmed-Ali et al., 2015), a nonadaptive observer has been proposed for a different class of ODE-PDE systems (see Remark 1). The proposed observer is a high-gain type while the observer of Table I is a Kalman-like  $\square$

#### 4. ADAPTIVE OBSERVER ANALYSIS

The next result, proved in Appendix B, is on the boundedness of the auxiliary states  $\lambda_0(t)$  and  $\lambda_1(x,t)$ .

**Proposition 1.** The auxiliary state vectors  $\lambda_0(t)$  and  $\lambda_1(x,t)$ , generated by (13g-i) are uniformly bounded  $\square$

In addition to Propositions 1, the following persistent excitation (PE) assumption is needed to establish the observer exponential convergence:

**PE Assumption.** The vector signal  $\lambda_1(0,t)$  is persistently exciting (PE), in the sense that,

$$\exists \delta, \varepsilon_0 > 0, \forall t > 0: \int_t^{t+\delta} \Lambda(s)\Lambda^T(s)ds > \varepsilon_0 I \quad (17)$$

where  $I \in \mathbf{R}^{m \times m}$  denotes the identity matrix.

Intuitively, the PE assumption means that the family of vectors  $\{\Lambda(s); t \leq s \leq t + \delta\}$  spans the parameter vector space  $\mathbf{R}^m$ . It is readily seen from (13f-i) that the signal  $\Lambda(t)$  only depends on the well known signal  $\phi(t)$  (but not on the state estimates). Therefore, it is quite possible to check whether the PE condition is satisfied or not. Given the linearity of (18g-h), the PE requirement is (likely to be) met if  $\phi(t)$  has a sufficiently wide power spectrum. Now, it is shown in many places that, if (17) holds then the time-varying matrix gain inverse  $R^{-1}(t)$  exists, is positive definite, and stays bounded away from 0. Specifically,  $R^{-1}(t)$  undergoes the following equation:

$$\frac{dR^{-1}}{dt} = -R^{-1} + \Lambda(t)\Lambda(t)^T \quad (18)$$

and satisfies,

$$r_0 \leq R^{-1}(t) \leq r_1, \text{ for all } t \geq 0 \quad (19)$$

for some couple  $(r_0, r_1)$  of positive real numbers. In the sequel, condition (17) will be supposed to be true, so that one can make use of (18)-(19). The exponential convergence of the adaptive observer of Table I is established in Theorem 1.

**Theorem 1** Consider the adaptive observer of Table I and let there the gain  $\rho$  of the parameter adaptive law be such that  $\rho > 1/2$ . Then, when applied to the system (2a-d), the observer is globally exponentially convergent in the sense that the observation errors  $\tilde{X}(t)$ ,  $\tilde{\theta}(t)$  and the norm

$\int_0^D \tilde{u}^2(x,t)dx$  are all exponentially vanishing (as  $t \rightarrow \infty$ ),

whatever the initial conditions  $\hat{X}(0)$ ,  $\hat{u}(x,0)$ ,  $\hat{\theta}(0)$   $\square$

**Proof.** First, let us analyze the error system consisting of the target system (8a-b) and the parameter adaptive law (13d) and (18), rewritten in terms of the errors  $\varepsilon(0,t)$ ,  $Z(t)$  and  $\tilde{\theta}(t)$ . For convenience, the error system is rewritten:

$$\dot{Z}(t) = (A - KCM(0))Z - K\varepsilon(0,t) \quad (20a)$$

$$\varepsilon_t(x,t) = \varepsilon_x(x,t) \quad (20b)$$

$$\varepsilon(D,t) = 0 \quad (20c)$$

$$\begin{aligned} \dot{\tilde{\theta}}(t) = & -\rho R(t)\Lambda(t)\Lambda^T(t)\tilde{\theta}(t) - \rho R(t)\Lambda(t)\varepsilon(0,t) \\ & - \rho R(t)\Lambda(t)CM(0)Z(t) \end{aligned} \quad (20d)$$

$$\dot{R}^{-1} = -R^{-1} + \Lambda(t)\Lambda(t)^T \quad (20e)$$

where (6a-b) has been used to get (20d). Inspired by (Krstic and Smyshlyaev, 2008), we consider the Lyapunov function,

$$V = \tilde{Z}^T P \tilde{Z} + a \int_0^D (1+x)\varepsilon^2(x,t)dx + \tilde{\theta}^T R^{-1} \tilde{\theta} \quad (21)$$

with  $P$  any symmetric positive definite matrix satisfying the algebraic equation,

$$P(A - KCM(0)) + (A - KCM(0))^T P \leq -\mu I, \quad (22)$$

for some scalars  $a > 0, \mu > 0$ . Differentiating (21) yields, using (20a-e) and (22):

$$\begin{aligned} \dot{V} = & \tilde{Z}^T P \dot{\tilde{Z}} + \tilde{Z}^T P \dot{\tilde{Z}} + 2a \int_0^D (1+x)\varepsilon(x,t)\varepsilon_t(x,t)dx \\ & + b\tilde{\theta}^T \dot{R}^{-1} \tilde{\theta} + 2b\tilde{\theta}^T R^{-1} \dot{\tilde{\theta}} \\ = & ((A - KCM(0))Z - K\varepsilon(0,t))^T P \tilde{Z} \\ & + \tilde{Z}^T P((A - KCM(0))Z - K\varepsilon(0,t)) \\ & + \tilde{\theta}^T (-R^{-1} + \Lambda\Lambda^T) \tilde{\theta} \\ & + 2\tilde{\theta}^T R^{-1} (-\rho R\Lambda\Lambda^T \tilde{\theta} - \rho R\Lambda\varepsilon(0,t) - \rho R\Lambda CM(0)Z) \\ \leq & -\mu \|\tilde{Z}\|^2 - 2\tilde{Z}^T PK \varepsilon(0,t) \\ & - a\varepsilon^2(0,t) - a \int_0^D \varepsilon^2(x,t)dx \\ & - \tilde{\theta}^T R^{-1} \tilde{\theta} + |\Lambda^T \tilde{\theta}|^2 - 2\rho |\Lambda \tilde{\theta}|^2 \\ & - 2\rho \tilde{\theta}^T \Lambda \varepsilon(0,t) - 2\rho \tilde{\theta}^T \Lambda CM(0)Z \end{aligned} \quad (23)$$

where the last inequality is obtained using an integration by part. Applying Young's inequality to cross terms, equality (23) develops as follows:

$$\begin{aligned} \dot{V} \leq & -\mu \|\tilde{Z}\|^2 + \|\tilde{Z}\|^2 + \|PK\|^2 \varepsilon^2(0,t) \\ & - a\varepsilon^2(0,t) - a \int_0^D \varepsilon^2(x,t)dx \\ & - \tilde{\theta}^T R^{-1} \tilde{\theta} - (2\rho - 1) |\Lambda \tilde{\theta}|^2 \\ & + \frac{\rho \zeta}{2} |\tilde{\theta}^T \Lambda|^2 + \frac{\rho}{2\zeta} \varepsilon^2(0,t) + \frac{\rho \zeta}{2} |\tilde{\theta}^T \Lambda|^2 \end{aligned}$$

$$\begin{aligned}
& + \frac{\rho}{2\zeta} \|CM(0)\|^2 \|Z\|^2 \\
\leq & - \left( \mu - 1 - \frac{\rho}{2\zeta} \|CM(0)\|^2 \right) \|\tilde{Z}\|^2 \\
& - \frac{a}{1+D} \int_0^D (1+x) \varepsilon^2(x,t) dx - \tilde{\theta}^T R^{-1} \tilde{\theta} \\
& - \left( a - \|PK\|^2 - \frac{\rho}{2\zeta} \right) \varepsilon^2(0,t) \\
& - \left| \Lambda^T \tilde{\theta} \right|^2 (2\rho - 1 - \rho\zeta) \tag{24}
\end{aligned}$$

whatever  $\zeta > 0$ . Let the free parameters  $a$ ,  $\mu$  and  $\zeta$  be set so that the following conditions hold:

$$\mu - 1 - \frac{\rho}{2\zeta} \|CM(0)\|^2 > 0 \tag{25}$$

$$a - \|PK\|^2 - \frac{\rho}{2\zeta} > 0 \tag{26}$$

$$2\rho - 1 - \rho\zeta > 0 \tag{27}$$

To meet the last inequality, set  $\zeta < (2\rho - 1) / \rho$  which is not an issue since  $\rho > 1/2$ . Inequalities (25)-(26) are also feasible because  $a$  and  $\mu$  are free and so can be chosen arbitrarily large. In view of (25)-(27), it follows from (24) and (21) that:

$$\dot{V} \leq -\alpha V \tag{25}$$

using (21), with  $\alpha = \min\left(\frac{1}{1+D}, \mu - 1 - \frac{\rho}{2\zeta} \|CM(0)\|^2\right)$ .

Clearly, this implies that  $V$  is exponentially vanishing (as  $t \rightarrow 0$ ). Due to (21), so are  $\tilde{Z}(t)$ ,  $\tilde{\theta}(t)$  and  $\int_0^D \varepsilon^2(x,t) dx$ .

Then, it follows from (6a-b) that in turn  $\tilde{X}(t)$  and  $\int_0^D \tilde{u}^2(x,t) dx$  are exponentially vanishing, using the fact that  $M(x)$ ,  $\lambda_0(t)$  and  $\lambda_1(x,t)$  are bounded. The Proof of Theorem 1 is completed ■

## 5. CONCLUSION

The problem of state observation is addressed for the class of nonlinear systems, represented by the ODE-PDE association of Fig. 1, analytically modelled by equations (1a-e). The aim is to get online estimates of both the finite-dimensional state  $X(t)$  and the infinite-dimensional state  $u(x,t)$  over the  $x$ -domain  $(0,D)$ , for some  $D > 0$ . A major difficulty is that the connexion point (between the ODE and the PDE subsystems), is not accessible to measurements making useless existing observers developed separately for ODE and PDE systems. The problem is dealt with using the high-gain type observer defined by equations (11a-e) which is a generalization of (Krstic, 2009) to the case where the ODE subsystem is nonlinear with triangular structure. The matrix function  $M(x)$  emphasizes the difference with standard high-gain observers and plays an instrumental role in making

(11a-e) an exponential convergence (Theorem 1). The present study can be pursued in several directions including: (i) re-designing the observer so that to make its convergence rate dependent on the the design parameters  $\mu$  and  $\theta$ ; (ii) the design of an adaptive version of the observer and the generalisation to other ODE and PDE subsystems.

## APPENDICES

### APPENDIX A. PROOF OF (7a-b).

Differentiating  $Z(t) = \tilde{X}(t) - \lambda_0(t)\tilde{\theta}(t)$ , with respect to time, and using (5a) and (6a-b), one successively gets the following equalities (where the argument 't' is omitted when it comes alone):

$$\begin{aligned}
\dot{Z} &= \left( A\tilde{X} + \phi\tilde{\theta} - K\tilde{u}(0,t) + v_0 \right) - \dot{\lambda}_0\tilde{\theta} - \lambda_0\dot{\tilde{\theta}} \\
&= A\left( Z + \lambda_0\tilde{\theta} \right) - K\tilde{u}(0,t) + \left( \phi - \dot{\lambda}_0 \right) \tilde{\theta} - \lambda_0\dot{\tilde{\theta}} + v_0 \\
&= AZ - K\varepsilon(0,t) - KCM(0)\tilde{X} \\
&\quad + \left( A\lambda_0 + \phi - K\lambda_1(0,t) - \dot{\lambda}_0(t) \right) \tilde{\theta} - \lambda_0\dot{\tilde{\theta}} + v_0 \\
&= (A - KCM(0))Z - K\varepsilon(0,t) \\
&\quad + \left( [A - KCM(0)]\lambda_0 + \phi - K\lambda_1(0,t) - \dot{\lambda}_0(t) \right) \tilde{\theta} \\
&\quad - \lambda_0\dot{\tilde{\theta}} + v_0 \tag{A1}
\end{aligned}$$

which establishes (7a). To prove (7b), differentiate both sides of (6b) (with respect to time) and use (5a-b) and (6a). Doing so, one successively gets:

$$\begin{aligned}
\varepsilon_t(x,t) &= \tilde{u}_t(x,t) - CM(x)\dot{\tilde{X}} - \lambda_{1,t}(x,t)\tilde{\theta} - \lambda_1(x,t)\dot{\tilde{\theta}} \\
&= \tilde{u}_{xx}(x,t) - k(x)\tilde{u}(0,t) + v_1 \\
&\quad - CM(x)[A\tilde{X} + \phi\tilde{\theta} - K\tilde{u}(0,t) + v_0] \\
&\quad - \lambda_{1,t}(x,t)\tilde{\theta} - \lambda_1(x,t)\dot{\tilde{\theta}} \\
&= \varepsilon_{xx}(x,t) + C \frac{d^2 M}{dx^2}(x)\tilde{X} + \lambda_{1,xx}(x,t)\tilde{\theta} \\
&\quad - k(x)\tilde{u}(0,t) \\
&\quad [ -CM(x)A\tilde{X} - CM(x)\phi\tilde{\theta} + CM(x)K\tilde{u}(0,t) \\
&\quad - \lambda_{1,t}(x,t)\tilde{\theta} + v_1 - \lambda_1(x,t)\dot{\tilde{\theta}} - CM(x)v_0 \\
&= \varepsilon_{xx}(x,t) + (CM(x)K - k(x))\tilde{u}(0,t) \\
&\quad + C \left( \frac{d^2 M}{dx^2}(x)\tilde{X} - M(x)A \right) \tilde{X} \\
&\quad + \left( \lambda_{1,xx}(x,t) - CM(x)\phi\tilde{\theta} - \lambda_{1,t}(x,t) \right) \tilde{\theta} \\
&\quad + v_1 - \lambda_1(x,t)\dot{\tilde{\theta}} - CM(x)v_0 \tag{A2}
\end{aligned}$$

This proves (7b). Equation (7c) is readily obtained by writing (6b) for  $x=1$  and using (5c) and (6a). To get (7d), differentiate (6b) (with respect to  $x$ ), let  $x=0$  in the

obtained equality, and use (5d). This completes the proof that the system (7a-d) holds ■

#### APPENDIX B. PROOF OF PROPOSITION 1.

Proof that  $\lambda_0(t)$  and  $\lambda_1(x, t)$  are bounded.

Recall that the vector signal  $\phi(t)$  is bounded by assumption. Then, as  $A - KCM(0)K$  is Hurwitz it follows from (13g) that  $\lambda_0(t)$  is bounded provided  $\lambda_1(0, t)$  is so. That is, it only remains to show that  $\lambda_1(x, t)$  is bounded. One possibility is to solve equation (13h) using the Laplace transform. Indeed, the it follows transforming (13h):

$$s\Lambda_1(x, s) - \lambda_1(x, 0) = \Lambda_{1,x}(x, s) - CM(x)\Phi(s)$$

which rewrites as follows, for all  $\forall x \in [0, D]$ :

$$\Lambda_{1,x}(x, s) = s\Lambda_1(x, s) + Ce^{(x-D)A}\Phi(s) \quad (B1)$$

where we have used (13j) and the fact that  $\lambda_1(x, 0) = 0$ . Equation (B1) has the following solution:

$$\begin{aligned} \Lambda_1(x, s) &= e^{s(x-D)}\Lambda_1(D, s) + \int_D^x e^{s(x-v)}Ce^{(v-D)A}\Phi(s)dv \\ &= -\int_x^D e^{s(x-v)}Ce^{(v-D)A}\Phi(s)dv \end{aligned} \quad (B2)$$

with  $0 \leq x \leq D$ , using the fact  $\lambda_1(D, t) = 0$  ( $\forall t \geq 0$ ) which entails  $\Lambda_1(D, s) = L(\lambda_1(D, t)) = 0$ . Taking the Laplace Transform inverse of both sides of (B2), one gets for all  $\forall x \in [0, D]$ :

$$\lambda_1(x, t) = -\int_x^D Ce^{(v-D)A}\phi(t+x-v)dv \quad (B2)$$

which shows that  $\lambda_1(x, t)$  is bounded, since  $\phi(t)$  is so. This establishes Proposition 1 ■

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