Modelling and structural properties of distributed parameter wind power systems

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***Abstract*—We examine two distributed parameter models for strings of generators connected to a wind farm. We show that these models boil down to delay systems either with or without continuous dynamics, depending on the type of the chosen boundary conditions. We then investigate the differential flatness of the systems, giving some solution to the actuation placement problem (i.e. where to place the farms along the generators string).**

***Index Terms*—Infinite dimensional systems, rings, modules, controllability, differential flatness, wave equation, wind power systems.**

I. Introduction

Inter area oscillations is a known and annoying phenomenon in power systems. A simple yet representative modelisation of this phenomena can be done through a distributed parameter system of hyperbolic type (namely a wave equation) by assuming a distributed placement of strings of generators ([3], [22], [28], [37], [41]). The involved wave equations are amenable to delay systems, the study of which has a rich literature. Quite a few authors have used algebraic techniques ([1], [7], [16]). We here envision the problem using module theoretic techniques, which have been used for delay systems ([8], [12], [14] as well as for distributed parameter systems [15], [26], [33], [40]). The main advantages of this approach are threefold: notions are intrinsic to the system, many controllability notions can be recast in this setting using extension of scalars, and a complete system parametrization is obtained through the freeness property (the linear analogue to differential flatness of lumped nonlinear systems).

The module properties of torsion freeness, projectivity and freeness, as well as the change of base ring through tensor product (*i.e.* extension of scalars) give rise to a huge number of possible controllability notions. One can then combine the choice of the base ring (the simplest, *i.e.* the nearest to a principal ideal domain, the better) and the module property (the strongest, *i.e.* the nearest to freeness, the better) to obtain a basis which can generate all the distributed system (such a system can be viewed as a collection of input/output systems parametrized by the spatial variable).

The paper is organized as follows: in a first Section, the module theoretic setting is recalled, including the module associated to a distributed parameter system and its controllability properties. In a second one, two modelizations are reviewed, one stemming from the literature, and another new one. The control appearing at the boundary is supposed to come from a wind farm. We then show that the first input/output model is a purely discrete one (*i.e.* it has no continuous dynamics, from a control perpective) and that the second is a neutral distributed delay one. In a third Section, the controllability properties of the two systems are given, wherefrom a placement of the wind farm can be deduced.

II. Module theoretic setting

A. R-linear systems

We shall consider in this section quite general definitions for linear systems viewed as modules over a ring. In the next section, we shall be more specific in order to describe boundary value problems as modules over a ring parametrized by space.

Definition 1: An *R-system* Λ, or a *system over R*, is an R-module. A *presentation matrix* of a finitely presented R-system Σ is a matrix P such that $\Sigma \cong [v]/[Pv]$ where $[v]$ is free with basis v. An *output $\boldsymbol{y}$* is a subset, which may be empty, of Λ. An *input-output R-system*, or an *input-output system over R*, is an R-dynamics equipped with an output.
The next definition allows, by extension of scalars, to obtain much nicer algebraic properties when needed.

Definition 2: Let A be an R-algebra and Λ be an R-system. The A-module $A \otimes_R \Lambda$ is an A-system, which *extends* Λ.

B. Boundary value problems as systems parametrized by space

We shall here consider boundary value PDE systems as modules over rings. A space parametrization is embedded in the chosen rings.

1) Model class: Models are here considered as space dynamics with time differential operator coefficients.

Distributed equations: The envisioned model equations are based on a Cauchy-Kowalevski form:

$$\partial_x \boldsymbol{w}_i = A_i \boldsymbol{w}_i + B_i \boldsymbol{u}, \quad \boldsymbol{w}_i : \Omega_i \to (\mathcal{D}^{'*})^p, \quad \boldsymbol{u} \in (\mathcal{D}^{'*})^m,$$
$$A_i \in (\mathbb{R}[\partial_t])^{p_i \times p_i},\ B_i \in (\mathbb{R}[\partial_t])^{p_i \times m},\ i \in \{1, \ldots, l\} \quad \text{(1a)}$$

where $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_l$ are the distributed variables, the lumped variables are $\boldsymbol{u} = (u_1, \ldots, u_m)$, and $\mathcal{D}^{'*}$ denotes a space of (ultra -) distributions.

Assumptions: We shall make two assumptions:

- The intervals $\Omega_1, \dots, \Omega_l$ are given by an open neighborhood of
$$\tilde{\Omega}_i = [x_{i,0}, x_{i,1}], \quad \ell_i = x_{i,1} - x_{i,0} = q_i \ell \\ q_i \in \mathbb{Q}, \ell \in \mathbb{R} \tag{1b}$$
Without loss of generality, assume $x_{i,0} = 0$.
- The characteristic polynomials of the matrices $A_1, \dots, A_l$ can be written
$$P_i(\lambda) := \det(\lambda I - A_i) = \sum_{\nu=0}^{p_i} a_{i,\nu} \lambda^\nu, \tag{1c}$$
$$a_{i,\nu} = \sum_{\nu+\mu \le p_i} a_{i,\nu,\mu} \partial_t^\mu \tag{1d}$$
with $a_{i,j,k} \in \mathbb{R}$, $a_{i,p_i,0} = 1$. Moreover, their principal parts $\sum_{\mu+\nu=p_i} a_{i,\mu,\nu} \partial_t^\mu \lambda^\nu$ are hyperbolic w.r.t. the time t, i.e. the roots of $\sum_{\mu+\nu=p_i} a_{i,\mu,\nu} \lambda^j$ are real.

Remark 1: Note that the above assumptions apply to most spatially one-dimensional boundary controlled evolution equations including Euler-Bernoulli or Timoshenko beam equations, more general parabolic diffusion-reaction-convection equations, damped and undamped wave-equations, etc. The only notable exception is the case where the maximal order derivative is a mixed one, such as, e.g. models of structural damping $(\alpha\partial_t + 1)(\partial_x^2 - \partial_t^2)\boldsymbol{w} = 0$

Boundary conditions: The models are completed by the following boundary conditions
$$\sum_{i=1}^{l} L_i \boldsymbol{w}_i(0) + R_i \boldsymbol{w}_i(\ell_i) + D\boldsymbol{u} = 0 \tag{1e}$$
with $D \in (\mathbb{R}[\partial_t])^{q \times m}$ and $L_i, R_i \in (\mathbb{R}[\partial_t])^{q \times p_i}$.

2) Solution of the Cauchy Problem: Some properties of the solution of the Cauchy problem (1a) with initial conditions given by $x = \xi$, i.e.
$$\partial_x \boldsymbol{w} = A\boldsymbol{w} + B\boldsymbol{u}, \quad \boldsymbol{w}(\xi) = \boldsymbol{w}_\xi \tag{2}$$
with $A \in (\mathbb{R}[\partial_t])^{p \times p}$, $B \in (\mathbb{R}[\partial_t])^{p \times q}$ as assumed in the previous section for A_i, B_i, will be used. The notation of the previous section is used in what follows, dropping the index $i \in \{1, \dots, l\}$.

Consider the initial value problem
$$P(\partial_x)v(x) = 0, \quad (\partial_x^j v)(0) = v_j \in \mathcal{E}^*(\mathbb{R}), \quad j = 0, \dots, p-1 \tag{3}$$
associated with the characteristic equation
$$P(\lambda) := \det(\lambda I - A) = \sum_{j=0}^{p} a_j \lambda^j, \quad a_j = \sum_{j+\mu \le p} a_{j,\mu} \partial^\mu$$
According to [18, Thrm. 12.5.6] or [30, Thm 2.5.2, and Prop. 2.5.6] the initial value problem (3) has a unique solution which may be written as
$$v(x) = \sum_{j=0}^{p-1} C_j(x) v_j$$
where the juxtaposition of symbols means convolution and $C_0, \dots, C_{p-1}$ are smooth functions mapping Ω to the space of compactly supported Beurling ultradistributions $\mathcal{E}'^*(\mathbb{R}) := \mathcal{E}'^{(p/(p-1))}(\mathbb{R})$ of Gevrey order $p/(p-1)$. The functions $C_0, \dots, C_{p-1}$ satisfy ($k, j \in \{0, \dots, p-1\}$)
$$\partial_x^k C_j(0) = \begin{cases} 1, & k = j \\ 0, & k \ne j \end{cases} \tag{4}$$
and
$$\partial_x C_j = C_{j-1} - a_j C_{p-1}, \quad j = 1, \dots, p-1, \tag{5}$$
$$\partial_x C_0 = -a_0 C_{p-1}. \tag{6}$$

With this preparatory steps, the unique solution $x \mapsto \Phi(x, \xi)$ of the initial value problem (2) can be expressed as
$$\boldsymbol{w}(x) = \Phi(x, \xi)\boldsymbol{w}_\xi + \Psi(x, \xi)\boldsymbol{u}. \tag{7}$$
Therein, $\Phi(x, \xi) \in \mathcal{E}'^*(\mathbb{R})^{p \times p}$ is the fundamental matrix of the initial value problem
$$\boldsymbol{w}(x) = \Phi(x, \xi)\boldsymbol{w}(x), \quad \boldsymbol{w}(\xi) = \boldsymbol{w}_\xi$$
and $\Psi(x, \xi) \in \mathcal{E}'^*(\mathbb{R})^{p \times m}$ corresponds to the particular solution of (2) with vanishing data $w_\xi = 0$.

Explicit expressions for Φ and Ψ can be given using the ultradistribution-valued functions $C_0, \dots, C_{p-1}$
$$\Phi(x, \xi) = \sum_{j=0}^{p-1} A^j C_j(x - \xi), \quad \Psi(x, \xi) = \int_\xi^x \Phi(x, \zeta) d\zeta B. \tag{8}$$

Substituting the general solutions of the initial value problems into the boundary conditions, one obtains the following linear system of equations:
$$\boldsymbol{w}(x) = W_{\boldsymbol{\xi}}(x)\boldsymbol{c}_{\boldsymbol{\xi}}, \quad P_{\boldsymbol{\xi}}\boldsymbol{c}_{\boldsymbol{\xi}} = 0. \tag{9}$$
Here, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$, $\boldsymbol{c}_{\boldsymbol{\xi}}^T = (\boldsymbol{w}_1^T(\xi_1) \cdots \boldsymbol{w}_l^T(\xi_l), \boldsymbol{u}^T)$,
$$W_{\boldsymbol{\xi}} = \begin{pmatrix} \Phi_1(x, \xi_1) & 0 & 0 & \Psi_1(x, \xi_1) \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & \Phi_l(x, \xi_l) & \Psi_l(x, \xi_l) \end{pmatrix},$$
$$P_{\boldsymbol{\xi}} = \left(P_{\boldsymbol{\xi},1} \cdots P_{\boldsymbol{\xi},l+1}\right)$$
with
$$P_{\boldsymbol{\xi},i} = L_i \Phi_i(0, \xi_i) + R_i \Phi_i(\ell_i, \xi_i), \quad i = 1, \dots, l$$
$$P_{\boldsymbol{\xi},l+1} = D + \sum_{i=1}^{l} L_i \Psi_i(0, \xi_i) + R_i \Psi_i(\ell_i, \xi_i).$$

A possible choice for the coefficient ring is the ring $R_{\mathbb{X}}^I = \mathbb{C}[\partial_t, \mathfrak{S}_{\mathbb{X}}, \mathfrak{S}_{\mathbb{X}}^I]$, with $\mathbb{X} \subseteq \mathbb{R}$ and
$$\mathfrak{S}_{\mathbb{X}} = \{C(z\ell), S(z\ell) | z \in \mathbb{X}\},$$
$$\mathfrak{S}_{\mathbb{X}}^I = \{C^I(z\ell), S^I(z\ell) | z \in \mathbb{X}\},$$

ℓ defined as in (1b), and

$$S^I(x) = \int_0^x S(\zeta) d\zeta, \quad C^I(x) = \int_0^x C(\zeta) d\zeta.$$

Inspired by the results given in [23], [1], [16], and in view of the simplification of the analysis of the module properties, instead of the ring $R^I_{\mathbb{R}}$, we shall use a slightly larger ring, given by $\mathcal{R}_{\mathbb{R}} = \mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{R}}] \cap \mathcal{E}'^*$.

Definition 3: The *convolutional system* Σ associated with the boundary value problem (1) is the $R^I_{\mathbb{R}}$-module generated by the elements of $\boldsymbol{c_\xi}$ with presentation matrix $P_{\boldsymbol{\xi}}$. By $\Sigma_{\mathbb{R}}$ (resp. $\Sigma_{\mathbb{Q}}$) we denote $\Sigma_{\mathbb{R}} = \mathcal{R}_{\mathbb{R}} \otimes_{R^I_{\mathbb{R}}} \Sigma$ (resp. $\Sigma_{\mathbb{Q}} = \mathcal{R}_{\mathbb{Q}} \otimes_{R^I_{\mathbb{R}}} \Sigma$).

C. System controllabilities

1) General controllabilities: In this section we emphasize several controllability notions which are defined directly without referring to a solution space. Let us start with some purely algebraic definitions:

*Definition 4 (*see, e.g. *, [12, Def. 2.4.]):* Let A be an R algebra. An R-system Λ is said to be *A-torsion free controllable* (resp. *A-projective controllable*, *A-free controllable*) if the A-module $A \otimes_R \Lambda$ is torsion free (resp. projective, free). An *R-torsion free* (resp. *R-projective*, *R-free*) *controllable* R-system is simply called *torsion free* (resp. *projective*, *free*) *controllable*.

Elementary homological algebra (*see, e.g.,* [31]) yields

Proposition 1: A-free (resp. A-projective) controllability implies A-projective (resp. A-torsion free) controllability.

Proposition 2: R-free controllability implies A-free controllability for any R-algebra A. More generally, given any R-system Σ that is a direct sum of a torsion module $\mathrm{t}\Sigma$ and a free module Λ, the extended system $A \otimes_R \Sigma$ is a direct sum of the torsion module $A \otimes_R \mathrm{t}\Sigma$ and the free module $A \otimes_R \Lambda$.

Definition 5: Take an A-free controllable R-system Λ with a finite output $\boldsymbol{y}$. This output is said to be *A-flat*, or *A-basic*, if $\boldsymbol{y}$ is a basis of $A \otimes_R \Lambda$. If $A \cong R$ then $\boldsymbol{y}$ is simply called *flat*, or *basic*.

2) Bézout character of $k[\mathfrak{S}_{\mathbb{Q}}]$ and $\mathcal{R}_{\mathbb{Q}}$:

Proposition 3 ([40, Cor. 3.2., 3.7.]): Let k be a field. The ring $k[\mathfrak{S}_{\mathbb{Q}}]$ is a Bézout domain, *i.e.*, any finitely generated ideal is principal.

Remark 2: Note that $k[\mathfrak{S}_{\mathbb{Q}}]$ is not Noetherian. Indeed, the following ideal is not finitely generated: $(\{S_{1/2^n} \mid n \in \mathbb{N}\})$.

Theorem 1 ([40, Thm. 3.9.]): The ring $\mathcal{R}_{\mathbb{Q}}$ is a Bézout domain.

Corollary 1: For any $\mathbb{X} \supset \mathbb{Q}$, a finitely presented $\mathcal{R}_{\mathbb{X}}$-system $\Sigma_{\mathbb{X}}$ is free, if and only if, it is torsion free. More generally $\Sigma_{\mathbb{X}} = \mathrm{t}\Sigma_{\mathbb{X}} \oplus \Sigma_{\mathbb{X}}/\mathrm{t}\Sigma_{\mathbb{X}}$ where $\mathrm{t}\Sigma_{\mathbb{X}}$ is torsion and $\Sigma_{\mathbb{X}}/\mathrm{t}\Sigma_{\mathbb{X}}$ is free.

3) Spectral controllability: In finite dimensional linear systems theory, the so called Hautus criterion is a quite popular tool for checking controllability. This criterion has been generalized to delay systems (*see, e.g.,* [23, Def. 5.1]) and to more general convolutional systems [38, Def. 10] and [40]. All those rings may be embedded into the ring of entire functions via the Laplace transform. This motivates the following quite general definition.

Definition and proposition 1 (see [40]): Let R be any ring that is isomorphic to a subring of the ring $\mathcal{O}$ of entire functions with pointwise defined multiplication. Denote the embedding $R \to \mathcal{O}$ by $\mathscr{L}$. A finitely presented R-system with presentation matrix P is said to be R-spectrally controllable if one of the following equivalent conditions holds:

(i)) The $\mathcal{O}$-matrix $\hat{P} = \mathscr{L}(P)$ satisfies

$$\exists k \in \mathbb{N}, \forall \sigma \in \mathbb{C}, \operatorname{rank}_R \hat{P}(\sigma) = k$$

(ii)) The module $\Sigma_{\mathcal{O}} = \mathcal{O} \otimes_R \Sigma$ is torsion free.

In the case of the existence of a presentation matrix of full generic row-rank equivalence of spectral controllability and torsion free controllability has been shown for delay systems in [23, Thrm. 5.1]) and for $\mathcal{M}_0$-systems in [39, Satz 4.4]. As [38, Example. 6] shows, such an equivalence does not hold for more general presentation matrices. However, for Bézout domains we have following.

Proposition 4 (see [40]): Let R be any Bézout domain that is isomorphic to a subring of $\mathcal{O}$ with the embedding $R \to \mathcal{O}$ denoted by $\mathscr{L}$. Then the notions of spectral controllability and R-torsion free controllability are equivalent if and only if $\mathscr{L}$ maps non-units in R to non-units in $\mathcal{O}$.

We are now able to state the main result of this section:

Theorem 2 (see [40]): A finitely presented $\mathcal{R}_{\mathbb{Q}}$-system $\Sigma_{\mathbb{Q}}$ is spectrally controllable, if and only if it is torsion free.

III. MODELLING

A. Distributed Parameter System models

Consider a transmission line with series of generators. The generation G_i and power angle change δ_i are supposed to be continuously distributed over the spatial dimension z. The Rotor dynamics of the i^{th} generator is taken to be (see, e.g. [])

$$\left(\frac{2H_i}{\Omega_s}\right) G_i \ddot{\delta}_i + \xi \dot{\delta}_i = P_i \tag{10}$$

with the following

H_i	The inertia constant
Ω_s	The electrical frequency with 60Hz base
P_i	The real power flowing out the i^{th} machine
ξ	A damping coefficient

Then, the real power flow from node i to node $i+1$ over a lossless line is

$$P_{i,i+1} = \frac{E_i E_{i+1} \sin(\delta_i - \delta_{i+1})}{x_i}$$

with E_i the voltage magnitude at bus i.

We then make the following two common assumptions: the change angle δ_i is small and $E_i = 1$. With these, we get

$$P_i = P_{i+1,i} - P_{i,i+1} \frac{(\delta_{i-1} - \delta_i)(\delta_i - \delta_{i+1})}{x_i}$$

By substitution, one obtains

$$\frac{2}{\Omega_i} \frac{H_i}{\Delta L} \ddot{\delta}_i + \frac{\xi}{\Delta l} \dot{\delta}_i = \frac{\Delta L}{x_i} \frac{\delta_i - \delta_{i-1}}{(\Delta L)^2} - \frac{\Delta L}{x_i} \frac{\delta_i - \delta_{i+1}}{(\Delta L)^2}$$

Then, taking the limit $\Delta L \to 0$, and setting

$$H_T = \frac{1}{L}\int_0^L dH(z) = \frac{H(L)}{L}, \quad \gamma = \frac{x(L)}{L}, \quad \eta = \frac{\xi(L)}{L}$$

yields, with $\nu = \sqrt{377/2H_T G_T \gamma}$

$$\partial_t^2 \delta(z,t) + \eta\, \partial_t \delta(z,t) = \nu^2\, \partial_z^2 \delta(z,t) \tag{11}$$

The corresponding power flow is

$$P(z,t) = -\frac{1}{\gamma}\, \partial_z \delta(z,t)$$

This type of model has been used to take into account inter area oscillation phenomena.

Adding power injection to the previous model leads to

$$\partial_t^2 \delta(z,t) + \eta\, \partial_t \delta(z,t) - \nu^2\, \partial_z^2 \delta(z,t) = W(z,t) \tag{12}$$

wiht boundary conditions

$$P(0,t) = P(1,t) = 0, \quad \text{or} \quad \partial_z \delta(0,t) = \partial_z \delta(1,t) = 0$$

A first model, used in [22], is a point source injection

$$W(u,t) = \rho P_g(t) \bar{\delta}(z-\alpha)$$

where $\bar{\delta}$ denotes the delta Dirac distribution, and P_g the net power injected. Another possible model, which we introduce here, is a power flow injection

$$W(u,t) = -\gamma P_g(t) \bar{\delta}'(z-\alpha)$$

with $\bar{\delta}'$ is the Dirac's derivative, in the distributional sense. The previous model (12) with point source injection

$$\partial_t^2 \delta_p(z,t) + \eta \partial_t \delta_p(z,t) - \nu^2 \partial_z^2 \delta_p(z,t) = \rho P_g(t) \bar{\delta}_p(z-\alpha)$$

is equivalent to the following model

$$\begin{aligned}
&\forall z \in [0,\alpha], \\
&\quad \partial_t^2 \delta_p^-(z,t) + \eta\, \partial_t \delta_p^-(z,t) - \nu^2\, \partial_z^2 \delta_p^-(z,t) = 0 && \text{(13a)} \\
&\quad \partial_z \delta_p^-(0,t) = 0 && \text{(13b)} \\
&\quad \delta_p^-(\alpha,t) = \rho P_g(t) && \text{(13c)} \\
&\forall z \in [\alpha, L], \\
&\quad \partial_t^2 \delta_p^+(z,t) + \eta\, \partial_t \delta_p^+(z,t) - \nu^2\, \partial_z^2 \delta_p^+(z,t) = 0 && \text{(13d)} \\
&\quad \delta_p^+(\alpha,t) = \rho P_g(t) && \text{(13e)} \\
&\quad \partial_z \delta_p^+(L,t) = 0 && \text{(13f)}
\end{aligned}$$

The other model we introduce, corresponding to the model (12) with power flow injection:

$$\partial_t^2 \delta_f(z,t) + \eta \partial_t \delta_f(z,t) - \nu^2 \partial_z^2 \delta_f(z,t) = -\gamma P_g(t) \bar{\delta}_f'(z-\alpha)$$

is equivalent to the following model

$$\begin{aligned}
&\forall z \in [0,\alpha], \\
&\quad \partial_t^2 \delta_f^-(z,t) + \eta\, \partial_t \delta_f^-(z,t) - \nu^2\, \partial_z^2 \delta_f^-(z,t) = 0 && \text{(14a)} \\
&\quad \partial_z \delta_f^-(0,t) = 0 && \text{(14b)} \\
&\quad \partial_z \delta_f^-(\alpha,t) = -\gamma P_g(t) && \text{(14c)} \\
&\forall z \in [\alpha, L], \\
&\quad \partial_t^2 \delta_f^+(z,t) + \eta\, \partial_t \delta_f^+(z,t) - \nu^2\, \partial_z^2 \delta_f^+(z,t) = 0 && \text{(14d)} \\
&\quad \partial_z \delta_f^+(\alpha,t) = -\gamma P_g(t) && \text{(14e)} \\
&\quad \partial_z \delta_f^+(L,t) = 0 && \text{(14f)}
\end{aligned}$$

B. *Point source model solution*

General solution: Let us consider the first half point source model for $z \in [0,\alpha]$ (equations (13a)–(13c)):

$$\begin{aligned}
&\partial_t^2 \delta_p^-(z,t) + \eta\, \partial_t \delta_p^-(z,t) - \nu^2\, \partial_z^2 \delta_p^-(z,t) = 0 && \text{(15a)} \\
&\partial_z \delta_p^-(0,t) = 0 && \text{(15b)} \\
&\delta_p^-(\alpha,t) = \rho P_g(t) && \text{(15c)}
\end{aligned}$$

The temporal Laplace transform of (15) yields

$$\begin{aligned}
&s^2 \hat{\delta}_p^-(z,s) + \eta s \hat{\delta}_p^-(z,s) - \nu^2 \partial_z^2 \hat{\delta}_p^-(z,s) = 0 \\
&\partial_z \hat{\delta}_p^-(0,s) = 0 \\
&\hat{\delta}_p^-(\alpha,s) = \rho \hat{P}_g(s)
\end{aligned}$$

Freezing s leads to an ODE in space:

$$s^2 \hat{\delta}_p^-(z) + \eta s \hat{\delta}_p^-(z) - \nu^2 \frac{d\hat{\delta}_p^-}{dz^2}(z) = 0 \tag{16}$$

$$\frac{d\hat{\delta}_p^-}{dz}(0) = 0, \qquad \hat{\delta}_p^-(\alpha) = \rho \hat{P}_g(s) \tag{17}$$

where we have kept the symbol δ_p^- by abuse of notation.

The general solution of the previous ODE is investigated through the characteristic equation in ξ:

$$s^2 + \eta s - \nu^2 \xi^2 = 0$$

yielding

$$\xi = \pm\, \varsigma \sqrt{s^2 + \eta s} = \pm\, \sigma(s), \quad \text{with} \quad \varsigma = 1/\nu$$

Thus, the general solution of (16) is

$$\hat{\delta}_p^-(z) = e^{\varsigma z \sqrt{s^2+\eta s}}\, \hat{\lambda}_1 + e^{-\varsigma z \sqrt{s^2+\eta s}}\, \hat{\lambda}_2 = e^{\sigma z} \hat{\lambda}_1 + e^{-\sigma z} \hat{\lambda}_2$$

Boundary value problem solution: Following Subsection II-B2 and equations (7) to (9), the general solution of (15) is rewritten as

$$\hat{\delta}_p^-(z,s) = \hat{C}_z(s) \hat{\mu}_{p1}^-(s) + \hat{S}_z(s) \hat{\mu}_{p2}^-(s), \qquad \text{where}$$

$$\hat{C}_z(s) = \cosh(\sigma z), \ \hat{S}_z(s) = \frac{\sinh(\sigma z)}{\sigma}$$

The sole advantage of using these operators is that:

$$\hat{C}_0(s) = 1, \qquad \hat{S}_0(s) = 0$$

which simplifies the boundary conditions expressions. The spatial derivatives of $\hat{C}_z$ and $\hat{S}_z$ are:

$$\partial_z \hat{C}_z = \sigma^2 \hat{S}_z, \qquad \partial_z \hat{S}_z = \hat{C}_z$$

And the spatial derivative of $\hat{\delta}_p^-$ is

$$\partial_z \hat{\delta}_p^-(z,s) = \sigma^2 \hat{S}_z \hat{\mu}_{p1}^- + \hat{C}_z \hat{\mu}_{p2}^-$$

Similarily, the general solution of the second half point source model for $z \in [\alpha, L]$ (equations (13d)–(13f)) and its spatial derivatives are

$$\begin{aligned} \hat{\delta}_p^+(z,s) &= \hat{C}_z \hat{\mu}_{p1}^+ + \hat{S}_z \hat{\mu}_{p2}^+ \\ \partial_z \hat{\delta}_p^+(z,s) &= \sigma^2 \hat{S}_z \hat{\mu}_{p1}^+ + \hat{C}_z \hat{\mu}_{p2}^+ \end{aligned}$$

The boundary conditions of the point source model (13)

$$\begin{aligned} \partial_z \delta_p^-(0,t) &= 0 \\ \delta_p^-(\alpha,t) &= \rho P_g(t) \\ \delta_p^+(\alpha,t) &= \rho P_g(t) \\ \partial_z \delta_p^+(L,t) &= 0 \end{aligned}$$

are then expressed as

$$\hat{\mu}_{p2}^- = 0 \tag{18a}$$
$$\hat{C}_\alpha \hat{\mu}_{p1}^- + \hat{S}_\alpha \mu_{p2}^- = \rho \hat{P}_g(s) \tag{18b}$$
$$\hat{C}_\alpha \hat{\mu}_{p1}^+ + \hat{S}_\alpha \hat{\mu}_{p2}^+ = \rho \hat{P}_g(s) \tag{18c}$$
$$\sigma^2 \hat{S}_L \hat{\mu}_{p1}^+ + \hat{C}_L \hat{\mu}_{p2}^+ = 0 \tag{18d}$$

Recalling the general solutions and/or spatial derivatives:

$$\begin{aligned} \hat{\delta}_p^-(z,s) &= \hat{C}_z \hat{\mu}_{p1}^- \\ \hat{\delta}_p^+(z,s) &= \hat{C}_z \hat{\mu}_{p1}^+ + \hat{S}_z \hat{\mu}_{p2}^+ \\ \partial_z \hat{\delta}_p^+(z,s) &= \sigma^2 \hat{S}_z \hat{\mu}_{p1}^+ + \hat{C}_z \hat{\mu}_{p2}^+ \end{aligned}$$

we get the following expressions for $\hat{\mu}_{p1}^-$, $\hat{\mu}_{p1}^+$ and $\hat{\mu}_{p2}^+$:

$$\hat{\mu}_{p1}^- = \hat{\delta}_p^-(0,s) = \hat{\delta}_{p0}^- \tag{19a}$$
$$\hat{\mu}_{p1}^+ = \hat{\delta}_p^+(0,s) = \hat{\delta}_{p0}^+ \tag{19b}$$
$$\hat{\mu}_{p2}^+ = \partial_z \hat{\delta}_p^+(0,s) = \hat{\delta}_{p0}^{+'} \tag{19c}$$

Thus, the preceding equations (18) become

$$\hat{C}_\alpha \hat{\delta}_{p0}^+ + \hat{S}_\alpha \hat{\delta}_{p0}^{+'} = \hat{C}_\alpha \hat{\delta}_{p0}^- \tag{20a}$$
$$\sigma^2 \hat{S}_L \hat{\delta}_{p0}^+ + \hat{C}_L \hat{\delta}_{p0}^{+'} = 0 \tag{20b}$$

which form the relations of the $\mathcal{R}_\mathbb{Q}$-system $\Lambda_\mathbb{Q}^p = [\hat{\delta}_{p0}^-, \hat{\delta}_{p0}^+, \hat{\delta}_{p0}^{+'}]_{\mathcal{R}_\mathbb{Q}}$. The presentation of $\Lambda_\mathbb{Q}^p$ is then

$$\begin{pmatrix} -\hat{C}_\alpha & \hat{C}_\alpha & \hat{S}_\alpha \\ 0 & \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{p0}^- \\ \hat{\delta}_{p0}^+ \\ \hat{\delta}_{p0}^{+'} \end{pmatrix} = 0 \tag{21}$$

To be more specific from a physical viewpoint, the model can be written as

$$\hat{C}_\alpha \hat{\delta}_{p0}^+ + \hat{S}_\alpha \hat{\delta}_{p0}^{+'} = \hat{C}_\alpha \hat{\delta}_{p0}^- \tag{22a}$$
$$\sigma^2 \hat{S}_L \hat{\delta}_{p0}^+ + \hat{C}_L \hat{\delta}_{p0}^{+'} = 0 \tag{22b}$$
$$\rho \hat{P}_g = \hat{C}_\alpha \hat{\delta}_{p0}^- \tag{22c}$$
$$\hat{\delta}_{pz}^- = \hat{C}_z \hat{\delta}_{p0}^- \tag{22d}$$
$$\hat{\delta}_{pz}^+ = \hat{C}_z \hat{\delta}_{p0}^+ + \hat{S}_z \hat{\delta}_{p0}^{+'} \tag{22e}$$

with $\delta_{pz}^- = \delta_p^-(z,s)$, $\delta_{pz}^+ = \delta_p^+(z,s)$.

C. Power flow model solution

General and boundary value problem solution: The general solution of power flow model (14) and its spatial derivatives are

$$\begin{aligned} \hat{\delta}_f^-(z,s) &= \hat{C}_z \hat{\mu}_{f1}^- + \hat{S}_z \hat{\mu}_{f2}^- \quad \partial_z \hat{\delta}_f^-(z,s) = \sigma^2 \hat{S}_z \hat{\mu}_{f1}^- + \hat{C}_z \mu_{f2}^- \\ \hat{\delta}_f^+(z,s) &= \hat{C}_z \hat{\mu}_{f1}^+ + \hat{S}_z \hat{\mu}_{f2}^+ \quad \partial_z \hat{\delta}_f^+(z,s) = \sigma^2 \hat{S}_z \hat{\mu}_{f1}^+ + \hat{C}_z \hat{\mu}_{f2}^+ \end{aligned}$$

The boundary conditions of the point source model (14)

$$\begin{aligned} \partial_z \delta_f^-(0,t) &= 0 \\ \partial_z \delta_f^-(\alpha,t) &= -\gamma P_g(t) \\ \partial_z \delta_f^+(\alpha,t) &= -\gamma P_g(t) \\ \partial_z \delta_f^+(L,t) &= 0 \end{aligned}$$

are then expressed as

$$\hat{\mu}_{f2}^- = 0 \tag{23a}$$
$$\sigma^2 \hat{S}_\alpha \hat{\mu}_{f1}^- + \hat{C}_\alpha \mu_{f2}^- = -\gamma \hat{P}_g(s) \tag{23b}$$
$$\sigma^2 \hat{S}_\alpha \hat{\mu}_{f1}^+ + \hat{C}_\alpha \mu_{f2}^+ = -\gamma \hat{P}_g(s) \tag{23c}$$
$$\sigma^2 \hat{S}_L \hat{\mu}_{f1}^+ + \hat{C}_L \hat{\mu}_{f2}^+ = 0 \tag{23d}$$

The following expressions are obtained for $\hat{\mu}_{f1}^-$, $\hat{\mu}_{f1}^+$ and $\hat{\mu}_{f2}^+$:

$$\hat{\mu}_{f1}^- = \hat{\delta}_f^-(0,s) = \hat{\delta}_{f0}^- \tag{24}$$
$$\hat{\mu}_{f1}^+ = \hat{\delta}_f^+(0,s) = \hat{\delta}_{f0}^+ \tag{25}$$
$$\hat{\mu}_{f2}^+ = \partial_z \hat{\delta}_f^+(0,s) = \hat{\delta}_{f0}^{+'} \tag{26}$$

We then get the following equations

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{f\,0}^{\;+} + \hat{C}_\alpha \delta_{f0}^{+'} = \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^- \tag{27a}$$
$$\sigma^2 \hat{S}_L \hat{\delta}_{f0}^+ + \hat{C}_L \delta_{f0}^{+'} = 0 \tag{27b}$$

which form the relations of the $\mathcal{R}_\mathbb{Q}$-system $\Lambda_\mathbb{Q}^f = [\hat{\delta}_{f0}^-, \hat{\delta}_{f0}^+, \hat{\delta}_{f0}^{+'}]_{\mathcal{R}_\mathbb{Q}}$. The presentation of $\Lambda_\mathbb{Q}^f$ is then

$$\begin{pmatrix} -\sigma^2 \hat{S}_\alpha & \sigma^2 \hat{S}_\alpha & \hat{C}_\alpha \\ 0 & \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{f0}^- \\ \hat{\delta}_{f0}^+ \\ \hat{\delta}_{f0}^{+'} \end{pmatrix} = 0 \tag{28}$$

To be more specific from a physical viewpoint, the model can be written as

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{\,+} - \hat{C}_\alpha \delta_{f0}^{+'} = \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{-} \tag{29a}$$
$$\sigma^2 \hat{S}_L \hat{\delta}_{f0}^{+} - \hat{C}_L \delta_{f0}^{+'} = 0 \tag{29b}$$
$$-\gamma \hat{P}_g = \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{-} \tag{29c}$$
$$\hat{\delta}_{fz}^{-} = \hat{C}_z \hat{\delta}_{f0}^{-} \tag{29d}$$
$$\hat{\delta}_{fz}^{+} = \hat{C}_z \hat{\delta}_{f0}^{+} + \hat{S}_z \hat{\delta}_{f0}^{+'} \tag{29e}$$

with $\delta_{fz}^{-} = \delta_f^{-}(z,s)$, $\delta_{fz}^{+} = \delta_f^{+}(z,s)$.

D. Associated I/O systems

Point source I/O system: To obtain the input/output system, we should extract from (22) the relation between δ_{pz}^{-} and $\hat{P}_g$ on the one hand and between δ_{pz}^{+} and $\hat{P}_g$ on the other hand. From (22c) and (22c), we get

$$\hat{C}_\alpha \hat{\delta}_{pz}^{-} = \rho\, \hat{C}_z \hat{P}_g \tag{30}$$

Then, from (22a) and (22b):

$$\begin{pmatrix} \hat{C}_\alpha & \hat{S}_\alpha \\ \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{p0}^{+} \\ \hat{\delta}_{p0}^{+'} \end{pmatrix} = \begin{pmatrix} \hat{C}_\alpha \hat{\delta}_{p0}^{-} \\ 0 \end{pmatrix}$$

Hence, we get

$$\hat{C}_{L-\alpha} \begin{pmatrix} \hat{\delta}_{p0}^{+} \\ \hat{\delta}_{p0}^{+'} \end{pmatrix} = \begin{pmatrix} \hat{C}_L & -\hat{S}_\alpha \\ -\sigma^2 \hat{S}_L & \hat{C}_\alpha \end{pmatrix} \begin{pmatrix} \hat{C}_\alpha \hat{\delta}_{p0}^{-} \\ 0 \end{pmatrix}$$

wherefrom the expression involving $\hat{\delta}_{pz}^{+}$:

$$\begin{aligned} \hat{C}_{L-\alpha} \hat{\delta}_{pz}^{+} &= \hat{C}_z \hat{C}_{L-\alpha} \hat{\delta}_{p0}^{+} + \hat{S}_z \hat{C}_{L-\alpha} \hat{\delta}_{p0}^{+'} \\ &= \big(\hat{C}_z \hat{C}_L - \sigma^2 \hat{S}_z \hat{S}_L\big) \hat{C}_\alpha \hat{\delta}_{p0}^{-} \\ &= \rho\, \hat{C}_{L-z} \hat{P}_g \end{aligned} \tag{31}$$

And, gathering (30) and (31), we get the point source injection input/output model:

$$\hat{C}_\alpha \hat{\delta}_{pz}^{-} = \rho\, \hat{C}_z \hat{P}_g \tag{32}$$
$$\hat{C}_{L-\alpha} \hat{\delta}_{pz}^{+} = \rho\, \hat{C}_{L-z} \hat{P}_g \tag{33}$$

on which we can see that this model is a purely discrete one, i.e. from a system dynamics viewpoint, it has no dynamics.

Power flow I/O system: Similarily to obtain the input/output system, we should extract from (29) the relation between δ_{fz}^{-} and $\hat{P}_g$ on the one hand and between δ_{fz}^{+} and $\hat{P}_g$ on the other hand.

From (29c) and (29c), we get

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{fz}^{-} = -\gamma\, \hat{C}_z \hat{P}_g \tag{34}$$

Then, from (29a) and (29b):

$$\begin{pmatrix} \sigma^2 \hat{S}_\alpha & -\hat{C}_\alpha \\ \sigma^2 \hat{S}_L & -\hat{C}_L \end{pmatrix} \begin{pmatrix} \hat{\delta}_{f0}^{+} \\ \hat{\delta}_{f0}^{+'} \end{pmatrix} = \begin{pmatrix} \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{-} \\ 0 \end{pmatrix}$$

Hence, we get

$$-\hat{S}_{L-\alpha} \begin{pmatrix} \hat{\delta}_{f0}^{+} \\ \hat{\delta}_{f0}^{+'} \end{pmatrix} = \begin{pmatrix} -\hat{C}_L & \hat{C}_\alpha \\ -\sigma^2 \hat{S}_L & \sigma^2 \hat{S}_\alpha \end{pmatrix} \begin{pmatrix} \sigma^2 \hat{S}_\alpha \hat{\delta}_{f0}^{-} \\ 0 \end{pmatrix}$$

wherefrom the expression involving $\hat{\delta}_{fz}^{+}$:

$$\begin{aligned} \hat{S}_{L-\alpha} \hat{\delta}_{fz}^{+} &= \hat{C}_z \hat{S}_{L-\alpha} \hat{\delta}_{f0}^{+} + \hat{S}_z \hat{S}_{L-\alpha} \hat{\delta}_{f0}^{+'} \\ &= \big(\hat{C}_z \hat{C}_L - \sigma^2 \hat{S}_z \hat{S}_L\big) \sigma^2 S_\alpha \hat{\delta}_{f0}^{-} \\ &= -\gamma\, \hat{C}_{L-z} \hat{P}_g \end{aligned} \tag{35}$$

And, gathering (34) and (35), we get the power flow injection input/output model:

$$\sigma^2 \hat{S}_\alpha \hat{\delta}_{fz}^{-} = -\gamma\, \hat{C}_z \hat{P}_g \tag{36}$$
$$\hat{S}_{L-\alpha} \hat{\delta}_{fz}^{+} = -\gamma\, \hat{C}_{L-z} \hat{P}_g \tag{37}$$

on which we can see that this model is a second order neutral distributed delay system.

IV. Structural properties

A. Point source system

The presentation matrix associated to $\Lambda_{\mathbb{Q}}^{p}$ is

$$P_{\Lambda_{\mathbb{Q}}^{p}} = \begin{pmatrix} -\hat{C}_\alpha & \hat{C}_\alpha & \hat{S}_\alpha \\ 0 & \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \tag{38}$$

and the associated minors are:

$$m_1(s) = -\sigma^2 \hat{C}_\alpha \hat{S}_L \tag{39a}$$
$$m_2(s) = -\hat{C}_\alpha \hat{C}_L \tag{39b}$$
$$m_3(s) = \hat{C}_\alpha \hat{C}_L - \sigma^2 \hat{S}_L \hat{S}_\alpha = \hat{C}_{L-\alpha} \tag{39c}$$

We then have the following proposition

Theorem 3: The $\mathcal{R}_{\mathbb{Q}}$-system $\Lambda_{\mathbb{Q}}^{p}$ is $\mathcal{R}_{\mathbb{Q}}$-free controllable if, and only if, $\hat{C}_{L-\alpha}$ and $\hat{C}_\alpha$ have no common zeros in $\mathbb{C}$.

Proof: Common zeros between the minors m_1 and m_2 can only be the ones of their common factor $\hat{C}_\alpha$. Thus, if $\hat{C}_{L-\alpha}$ and $\hat{C}_\alpha$ have no common zeros, applying Proposition 1 and Theorem 2, $\Lambda_{\mathbb{Q}}^{p}$ is $\mathcal{R}_{\mathbb{Q}}$-spectrally controllable. ■

When $\Lambda_{\mathbb{Q}}^{p}$ is free, a basis may be obtained as follows. Consider (p_1, p_2, p_3) a solution to the Bézout equation in $\mathcal{R}_{\mathbb{Q}}$:

$$p\, m_1 + q\, m_2 + r\, m_3 = 1$$

Then, a basis is given by (*see, e.g.,* [24, Rmk II.5, p. 30])

$$\omega = p_1 \hat{\delta}_{p0}^{-} - p_2 \hat{\delta}_{p0}^{+} + p_3 \hat{\delta}_{p0}^{+'}$$

B. Power flow system

The presentation matrix associated to $\Lambda_{\mathbb{Q}}^{p}$ is

$$P_{\Lambda_{\mathbb{Q}}^{p}} = \begin{pmatrix} -\sigma^2 \hat{S}_\alpha & \sigma^2 \hat{S}_\alpha & \hat{C}_\alpha \\ 0 & \sigma^2 \hat{S}_L & \hat{C}_L \end{pmatrix} \tag{40}$$

and the associated minors are:

$$m_1(s) = -\sigma^4 \hat{S}_\alpha \hat{S}_L \tag{41a}$$
$$m_2(s) = -\sigma^2 \hat{S}_\alpha \hat{C}_L \tag{41b}$$
$$m_3(s) = \sigma^2 \hat{S}_\alpha \hat{C}_L - \sigma^2 \hat{S}_L \hat{C}_\alpha = \sigma^2 \hat{S}_{\alpha - L} \tag{41c}$$

We then have the following proposition

Proposition 5: The $\mathcal{R}_{\mathbb{Q}}$-system $\Lambda_{\mathbb{Q}}^{f}$ is not $\mathcal{R}_{\mathbb{Q}}$-free controllable.

Proof: The minors m_1 to m_3 have $\sigma = 0$ (corresponding to $s = 0$) as a common zero. Hence, $\Lambda_{\mathbb{Q}}^{f}$ is not $\mathcal{R}_{\mathbb{Q}}$-spectrally controllable, and hence not $\mathcal{R}_{\mathbb{Q}}$-torsion free. ■

Let $\mathcal{R}_{\mathbb{Q}}^{\sigma}$ be the ring $\mathcal{R}_{\mathbb{Q}}^{\sigma} = \mathbb{C}(\partial_t)[\sigma^{-1}, \mathfrak{S}_{\mathbb{Q}}] \cap \mathcal{E}'^{*}$ Then we have:

Theorem 4: The system $\mathcal{R}_{\mathbb{Q}}^{\sigma} \otimes_{\mathcal{R}_{\mathbb{Q}}} \Lambda_{\mathbb{Q}}^{f}$ is $\mathcal{R}_{\mathbb{Q}}^{\sigma}$-free controllable if, and only if, $\hat{S}_{\alpha}$ and $\hat{S}_{L-\alpha}$ have no common zeros in $\mathbb{C}$.

Proof: Common zeros between the minors m_1 and m_2 can only be the ones of their common factor $\hat{S}_{\alpha}$. Thus, if $\hat{S}_{L-\alpha}$ and $\hat{S}_{\alpha}$ have no common zeros, applying Proposition 1 and Theorem 2, $\Lambda_{\mathbb{Q}}^{p}$ is $\mathcal{R}_{\mathbb{Q}}$-spectrally controllable. ■

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