GENERALIZED BERNOULLI NUMBERS AND A FORMULA OF LUCAS

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Abstract. An overlooked formula of E. Lucas for the generalized Bernoulli numbers is proved using generating functions. This is then used to provide a new proof and a new form of a sum involving classical Bernoulli numbers studied by K. Dilcher. The value of this sum is then given in terms of the Meixner-Pollaczek polynomials.

1. Introduction

The goal of this paper is to provide a unified approach to two topics that have appeared in the literature. The first one is an expression for the generalized Bernoulli numbers $B_n^{(p)}$ defined by the exponential generating function

$$\sum_{n=0}^{\infty} B_n^{(p)} \frac{z^n}{n!} = \left(\frac{z}{e^z - 1} \right)^p . \tag{1.1}$$

For $n \in \mathbb{N}$, the coefficients $B_n^{(p)}$ are polynomials in p named after Nörlund in [1]. The first few are

$$B_0^{(p)} = 1,\ B_1^{(p)} = -\tfrac{1}{2}p,\ B_2^{(p)} = -\tfrac{1}{12}p + \tfrac{1}{4}p^2,\ B_3^{(p)} = \tfrac{1}{8}p^2(1-p). \tag{1.2}$$

In his 1878 paper E. Lucas [4] gave the formula

$$B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!} \frac{n!}{(n-p)!} \beta^{n-p+1} (1+\beta) \cdots (p-1+\beta) \tag{1.3}$$

for $n \geq p$. This is a symbolic formula: to obtain the value of $B_n^{(p)}$, expand the expression (1.3) and replace β^j by the ratio B_j/j. Here B_j is the classical Bernoulli number $B_n = B_n^{(1)}$ in the notation from (1.1).

The second topic is an expression established by K. Dilcher [2] for the sums of products of Bernoulli numbers

$$S_N(n) := \sum \binom{2n}{2j_1,\, 2j_2,\, \cdots,\, 2j_N} B_{2j_1} B_{2j_2} \cdots B_{2j_N}, \tag{1.4}$$

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where the sum is taken over all nonnegative integers $j_1, \cdots, j_N$ such that $j_1 + \cdots + j_N = n$, and where

$$\binom{2n}{2j_1,\, 2j_2,\, \cdots,\, 2j_N} = \frac{(2n)!}{(2j_1)! \cdots (2j_N)!} \tag{1.5}$$

is the multinomial coefficient and B_{2k} is the classical Bernoulli number. One of the main results of [2] is the evaluation

$$S_N(n) = \frac{(2n)!}{(2n-N)!} \sum_{k=0}^{\lfloor (N-1)/2 \rfloor} b_k^{(N)} \frac{B_{2n-2k}}{2n-2k}, \tag{1.6}$$

where the coefficients $b_k^{(N)}$ are defined by the recurrence

$$b_k^{(N+1)} = -\frac{1}{N} b_k^{(N)} + \frac{1}{4} b_{k-1}^{(N-1)}, \tag{1.7}$$

with $b_0^{(1)} = 1$ and $b_k^{(N)} = 0$ for $k < 0$ and for $k > \lfloor (N-1)/2 \rfloor$.

Lucas's original proof is recalled in Section 2. This section also contains an extension of Lucas's formula for $B_n^{(p)}$ to $0 \leq n \leq p-1$ in terms of the Stirling numbers of the first kind. A unified proof of the two formulas for $B_n^{(p)}$ based on generating functions is given in Section 3. Another proof of Lucas's formula, based on recurrences, is given in Section 4 and Section 5 contains a proof of

$$S_N(n) = \sum_{k=0}^{N} \frac{(2n)!}{(2n-k)!} 2^{-k} \binom{N}{k} B_{2n-k}^{(N-k)} \tag{1.8}$$

that expresses Dilcher's sum (1.4) explicitly in terms of the generalized Bernoulli numbers. Expressing this result in hypergeometric form leads to a formula for $S_N(n)$ in terms of the Meixner-Pollaczek polynomials

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!} e^{\imath n \phi} {}_2F_1 \left(\begin{matrix} -n & \lambda + \imath x \\ & 2\lambda \end{matrix} \middle| 1 - e^{-2\imath\phi} \right). \tag{1.9}$$

It is then established that the recurrence (1.7), provided by Dilcher in [2], is equivalent to the classical three-term relation for this orthogonal family of polynomials.

2. Lucas's theorem

In his paper [4], E. Lucas gave an expression for the generalized Bernoulli numbers $B_n^{(p)}$, for $n \geq p$. This section presents an outline of his proof and an extension of this expression for $B_n^{(p)}$ to the case $0 \leq n \leq p-1$. A proof based on generating functions is given in the next section. Lucas's formula uses the translation

$$\beta^n = \frac{B_n}{n} \tag{2.1}$$

coming from umbral calculus. Observe, for example, that

$$\begin{aligned} B_3^{(2)} &= \frac{(-1)^1}{1!}\frac{3!}{1!}\beta^2(1+\beta) = -6(\beta^2+\beta^3) \\ &= -6\left(\frac{B_2}{2}+\frac{B_3}{3}\right) = -3B_2 = -\frac{1}{2} \end{aligned}$$

Observe also that the symbolic substitution (2.1) should be performed only *after* all the terms have been expanded. For example,

$$\beta^2(1+\beta) = \beta^2+\beta^3 = \frac{B_2}{2}+\frac{B_3}{3} = -\frac{1}{4} \tag{2.2}$$

but

$$\beta^2(1+\beta) \neq \frac{B_2}{2}\left(1+\frac{B_1}{1}\right) = \frac{1}{24}. \tag{2.3}$$

Theorem 2.1 (Lucas)**.** *For $n \geq p$, the generalized Bernoulli numbers $B_n^{(p)}$ are given by*

$$B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!}\frac{n!}{(n-p)!}\beta^{n-p+1}(1+\beta)(2+\beta)\cdots(p-1+\beta) \tag{2.4}$$

where, in symbolic notation,

$$\beta^n = \frac{B_n}{n}. \tag{2.5}$$

Proof. Lucas's argument begins with the identity

$$pB_n^{(p+1)} = (p-n)B_n^{(p)} - pnB_{n-1}^{(p)} \tag{2.6}$$

which follows directly from the identity for generating functions

$$x\frac{d}{dx}\left(\frac{x}{e^x-1}\right)^p = p(1-x)\left(\frac{x}{e^x-1}\right)^p - p\left(\frac{x}{e^x-1}\right)^{p+1}. \tag{2.7}$$

Shifting n to $n-1$ it follows that

$$pB_{n-1}^{(p+1)} = (p-n+1)B_{n-1}^{(p)} - p(n-1)B_{n-2}^{(p)}. \tag{2.8}$$

Now multiplying (2.6) by $n(p+1)$ and (2.8) by $(p-n+1)$ leads to

$$\begin{aligned} p(p+1)B_n^{(p+2)} &= (p-n+1)(p-n)B_n^{(p)} - (p-n+1)(p+p+1)nB_{n-1}^{(p)} \\ &\quad +p(p+1)n(n-1)B_{n-2}^{(p)} \end{aligned}$$

and then, by the same methods, he produces

$$\begin{aligned} (p+2)(p+1)pB_n^{(p+3)} &= (p-n+2)(p-n+1)(p-n)B_n^{(p)} \\ &- (p-n_2)(p-n+1)(p+p+1+p+2)nB_{n-1}^{(p)} \\ &+ (p-n+2)(p(p+1)+p(p+2)+(p+1)(p+2))n(n-1)B_{n-2}^{(p)} \\ &- p(p+1)(p+2)n(n-1)(n-2)B_{n-3}^{(p)} \end{aligned}$$

and then, stating '*and so on*', concludes the proof. □

The following alternate proof of Lucas's theorem using generating functions requires an expression for $B_n^{(p)}$ in the range $0 \leq n \leq p-1$, of the kind given in (2.4). This cannot be obtained directly from (2.4). The Stirling numbers of the first kind $s_k^{(p)}$ are used to produce an equivalent formulation of $B_n^{(p)}$. These numbers are defined by the generating function

$$z(z-1)(z-2)\cdots(z-(p-1)) = \sum_{k=1}^{p} s_k^{(p)} z^k. \tag{2.9}$$

Then (2.4) may be written as

$$\begin{aligned} B_n^{(p)} &= \frac{(-1)^{p-1}}{(p-1)!} n(n-1)\cdots(n-(p-1))\beta^{n-p}(-1)^p \sum_{k=1}^{p} s_k^{(p)}(-\beta)^k \\ &= -\frac{1}{(p-1)!} n(n-1)\cdots(n-(p-1)) \sum_{k=1}^{p} s_k^{(p)}(-1)^k \frac{B_{n-p+k}}{n-p+k}. \end{aligned}$$

Observe that the index n varies in the range $0 \leq n \leq p-1$, therefore the prefactor $n(n-1)\cdots(n-(p-1))$ always vanishes. On the other hand, all the summands are finite, except when $k = p-n$. Performing the translation from $(-\beta)^k$ to B_k/k for this specific index gives

$$-\frac{1}{(p-1)!} n(n-1)\cdots 1 \times (-1)(-2)\cdots(-(p-1-n))) s_{p-n}^{(p)}(-1)^{p-n} = \frac{s_{p-n}^{(p)}}{\binom{p-1}{n}}.$$

This gives:

Theorem 2.2. *The generalized Bernoulli numbers $B_n^{(p)}$, with $0 \leq n \leq p-1$ are given by*

$$B_n^{(p)} = \frac{s_{p-n}^{(p)}}{\binom{p-1}{n}}. \tag{2.10}$$

In fact, this is a classical result. It is, for example, a direct consequence of the identity

$$(z-1)(z-2)\cdots(z-p) = \sum_{\ell=0}^{p} \binom{p}{\ell} z^\ell B_{p-\ell}^{(p+1)} \tag{2.11}$$

which appears (unnumbered) in [5, p.149].

3. The proof via generating function

The expressions for the generalized Bernoulli numbers given in (2.4) and (2.10) are now used to compute the generating function

$$G(z) = \sum_{n=0}^{\infty} B_n^{(p)} \frac{z^n}{n!} \tag{3.1}$$

and to show that it coincides with the generating function of the generalized Bernoulli numbers (1.1).

Split the sum as $G(z) = G_1(z) + G_2(z)$, where

$$G_1(z) = \sum_{n=0}^{p-1} B_n^{(p)} \frac{z^n}{n!} \text{ and } G_2(z) = \sum_{n=p}^{\infty} B_n^{(p)} \frac{z^n}{n!}. \tag{3.2}$$

Observe that

$$\begin{aligned} G_2(z) &= \sum_{n=p}^{\infty} \frac{(-1)^{p-1}}{(p-1)!} \frac{n!}{(n-p)!} \beta^{n-p+1} (1+\beta) \cdots ((p-1)+\beta) \frac{z^n}{n!} \\ &= \frac{(-1)^{p-1}}{(p-1)!} \beta(1+\beta) \cdots (p-1+\beta) \sum_{n=p}^{\infty} \frac{n!}{(n-p)!} \beta^{n-p} \frac{z^n}{n!} \\ &= \frac{(-1)^{p-1}}{(p-1)!} (-1)^p \sum_{k=1}^{p} s_k^{(p)} (-1)^k z^p f_k(z) \end{aligned}$$

with

$$f_k(z) = \sum_{n=p}^{\infty} \frac{B_{n-p+k}}{(n-p)!(n-p+k)} z^{n-p}. \tag{3.3}$$

The $(k-1)$-st antiderivative of $f_k(z)$, denoted by $g_k(z)$, is

$$\begin{aligned} g_k(z) &= \sum_{n=p}^{\infty} \frac{B_{n-p+k}}{(n-p+k)!} z^{n-p+k-1} \\ &= z^{-1} \sum_{\ell=k}^{\infty} \frac{B_\ell}{\ell!} z^\ell \\ &= \frac{1}{z} \left[\frac{z}{e^z - 1} - \sum_{\ell=0}^{k-1} \frac{B_\ell}{\ell!} z^\ell \right], \end{aligned}$$

therefore

$$\begin{aligned} f_k(z) &= \left(\frac{d}{dz} \right)^{k-1} \frac{1}{e^z - 1} - \left(\frac{d}{dz} \right)^{k-1} \frac{1}{z} \\ &= \left(\frac{d}{dz} \right)^{k-1} \frac{1}{e^z - 1} + \frac{(-1)^k (k-1)!}{z^k}. \end{aligned}$$

This gives

$$\begin{aligned} G_2(z) &= -\frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (-1)^k f_k(z) \\ &= -\frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (-1)^k \left(\frac{d}{dz} \right)^{k-1} \left[\frac{1}{e^z - 1} \right] - \frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} \frac{(k-1)!}{z^k}. \end{aligned}$$

On the other hand,

$$\begin{aligned} G_1(z) &= \sum_{n=0}^{p-1} B_n^{(p)} \frac{z^n}{n!} \\ &= \sum_{n=0}^{p-1} \frac{s_{p-n}^{(p)}}{\binom{p-1}{n}} \frac{z^n}{n!} \\ &= \frac{1}{(p-1)!} \sum_{n=0}^{p-1} s_{p-n}^{(p)} (p-1-n)! z^n \\ &= \frac{1}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (k-1)! z^{p-k}. \end{aligned}$$

This sum cancels the second term in the expression for $G_2(z)$. Hence

$$G(z) = G_1(z) + G_2(z) = -\frac{z^p}{(p-1)!} \sum_{k=1}^{p} s_k^{(p)} (-1)^k \left(\frac{d}{dz} \right)^{k-1} \left[\frac{1}{e^z - 1} \right]. \tag{3.4}$$

Using (2.9) this gives

$$G(z) = -\frac{(-z)^p}{(p-1)!} \left((p-1) + \frac{d}{dz} \right) \cdots \left(1 + \frac{d}{dz} \right) \left[\frac{1}{e^z - 1} \right]. \tag{3.5}$$

The next lemma simplifies this expression. Its proof by induction is elementary, so it is omitted.

Lemma 3.1. *For $n \geq 1$, the identity*

$$\frac{(-1)^n}{n!} \left(n + \frac{d}{dz} \right) \left(n - 1 + \frac{d}{dz} \right) \cdots \left(1 + \frac{d}{dz} \right) \frac{1}{e^z - 1} = \frac{1}{(e^z - 1)^{n+1}} \tag{3.6}$$

holds.

Replacing in (3.5) produces

$$G(z) = -\frac{(-z)^p}{(p-1)!} \frac{(p-1)!}{(-1)^{p-1}} \frac{1}{(e^z - 1)^p} = \left(\frac{z}{e^z - 1} \right)^p, \tag{3.7}$$

which is the generating function of the generalized Bernoulli numbers. This proves both Lucas's formula for $B_n^{(p)}$ with $n \geq p$ and the expression (2.10) for $0 \leq p \leq n-1$.

4. Lucas's formula via recurrences

The numbers $B_n^{(p)}$ satisfy the recurrence

$$pB_n^{(p+1)} = (p-n)B_n^{(p)} - pnB_{n-1}^{(p)}. \tag{4.1}$$

Lucas's formula for $B_n^{(p)}$ is now established by showing that the numbers defined by (2.4) satisfy the same recurrence.

Start with

$$(p-n)B_n^{(p)} - pnB_{n-1}^{(p)} = (p-n)\frac{(-1)^{p-1}n!}{(p-1)!(n-p)!}\beta^{n-p}\prod_{k=0}^{p-1}(k+\beta) - pn\frac{(-1)^{p-1}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}\prod_{k=0}^{p-1}(k+\beta),$$

and write it as

$$\begin{aligned}
(p-n)B_n^{(p)} - pnB_{n-1}^{(p)} &= \frac{(-1)^{p-1}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}\left[-\prod_{k=0}^{p-1}(k+\beta) - p\beta\prod_{k=0}^{p-1}(k+\beta)\right] \\
&= \frac{(-1)^{p}n!}{(p-1)!(n-p-1)!}\beta^{n-1-p}(p+\beta)\prod_{k=0}^{p-1}(k+\beta) \\
&= p\frac{(-1)^{p}}{p!}\frac{n!}{(n-p-1)!}\beta^{n-1-p}\prod_{k=0}^{p}(k+\beta) \\
&= pB_n^{(p+1)}.
\end{aligned}$$

To conclude the result, it suffices to check that the initial conditions match. This is clear, since

$$B_n^{(1)} = \frac{n!}{(n-1)!}\beta^n = n\beta^n = n\frac{B_n}{n} = B_n. \tag{4.2}$$

This establishes Lucas's formula for the generalized Bernoulli numbers.

5. A new proof of Dilcher's formula

This section analyzes the sum

$$S_N(n) := \sum \binom{2n}{2j_1,\, 2j_2,\, \cdots,\, 2j_N} B_{2j_1}B_{2j_2}\cdots B_{2j_N}, \tag{5.1}$$

using Lucas's expression for the generalized Bernoulli numbers $B_n^{(p)}$. An alternative formulation is presented.

Proposition 5.1. *The sum $S_N(n)$ is given by*

$$S_N(n) = \sum_{k=0}^{N} \frac{(2n)!}{(2n-k)!}2^{-k}\binom{N}{k}B_{2n-k}^{(N-k)} \tag{5.2}$$

for $2n > N$.

Proof. The umbral method [7] shows that the sum $S_N(n)$ is given by

$$S_N(n) = \frac{1}{2^N}\left(\epsilon_1 B_1 + \cdots + \epsilon_N B_N\right)^{2n} \tag{5.3}$$

with $\epsilon_j = \pm 1$. Introduce the notation

$$Y_{2n}^{(M,N)} = \left(-B_1 - \cdots - B_M + B_{M+1} + \cdots + B_N\right)^{2n} \tag{5.4}$$

where there are M minus signs and $N-M$ plus signs. Thus,

$$S_N(n) = \frac{1}{2^N} \sum_{M=0}^{N} \binom{N}{M} Y_{2n}^{(M,N)}. \tag{5.5}$$

The next step uses the famous umbral identity

$$f(-B) = f(B) + f'(0) \tag{5.6}$$

(see Section 2 of [3] for details) to obtain

$$Y_{2n}^{(M,N)} = Y_{2n}^{(M-1,N)} + 2nY_{2n-1}^{(M-1,N-1)}. \tag{5.7}$$

This may be written as

$$Q_{2n}^{(M)} = Q_{2n}^{(M-1)} + 2nQ_{2n-1}^{(M-1)}, \tag{5.8}$$

where $Q_j^M = Y_j^{(M,P+j)}$ and $P = N - 2n$. Then (5.8) is easily solved to produce

$$Q_{2n}^{(M)} = \sum_{k=0}^{M} \binom{M}{k} \frac{(2n)!}{(2n-k)!} Q_{2n-k}^{(0)}. \tag{5.9}$$

Since the initial condition is

$$Q_{2n-k}^{(0)} = Y_{2n-k}^{(0,N-k)} = B_{2n-k}^{(N-k)}, \tag{5.10}$$

it follows that

$$Y_{2n}^{(M,N)} = \sum_{k=0}^{M} \binom{M}{k} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)}. \tag{5.11}$$

Replacing in (5.5) yields

$$\begin{aligned} S_N(n) &= \frac{1}{2^N} \sum_{M=0}^{N} \binom{N}{M} Y_{2n}^{(M,N)} \\ &= \frac{1}{2^N} \sum_{M=0}^{N} \binom{N}{M} \sum_{k=0}^{M} \binom{M}{k} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)} \\ &= \frac{1}{2^N} \sum_{k=0}^{N} \frac{(2n)!}{(2n-k)!} B_{2n-k}^{(N-k)} \sum_{M=0}^{N} \binom{M}{k} \binom{N}{M}. \end{aligned}$$

Now use the basic identity

$$\sum_{M=0}^{N} \binom{M}{k} \binom{N}{M} = \sum_{M=k}^{N} \binom{M}{k} \binom{N}{M} = 2^{N-k} \binom{N}{k} \tag{5.12}$$

to obtain the result. $\square$

Lucas's identity for generalized Bernoulli numbers is now used to obtain a second expression for the sum $S_N(n)$.

Proposition 5.2. *For $2n > N$, the sum $S_N(n)$ is given by*

$$S_N(n) = \frac{(2n)!}{(2n-N)!}\beta^{2n-N+1}\sum_{\ell=0}^{N-1}\binom{N}{\ell+1}\frac{(-1)^\ell}{2^{N-1-\ell}}\frac{(\beta+1)_\ell}{\ell!}. \tag{5.13}$$

Proof. Using the Pochhammer symbol

$$(\beta+1)_{p-1} = \frac{\Gamma(\beta+p)}{\Gamma(\beta+1)} = (\beta+1)\cdots(\beta+p-1) \tag{5.14}$$

Lucas's formula (2.4) is stated in the form

$$B_n^{(p)} = \frac{(-1)^{p-1}}{(p-1)!}\frac{n!}{(n-p)!}\beta^{n-p+1}(\beta+1)_{p-1}. \tag{5.15}$$

Using Proposition 5.1 and $B_n^{(0)} = \delta_n$ so that $B_{2n-N}^{(0)} = 0$ since $2n > N$, it follows that

$$\begin{aligned} S_N(n) &= \sum_{k=0}^{N-1}\frac{(2n)!}{(2n-k)!}2^{-k}\binom{N}{k}\frac{(-1)^{N-k-1}}{(N-k-1)!}\frac{(2n-k)!}{(2n-N)!}\beta^{2n-N+1}(\beta+1)_{N-k-1} \\ &= \frac{(2n)!}{(2n-N)!}\beta^{2n-N+1}\sum_{k=0}^{N-1}2^{-k}\binom{N}{k}\frac{(-1)^{N-k-1}}{(N-k-1)!}(\beta+1)_{N-k-1} \end{aligned}$$

that reduces to the stated form. □

To obtain a hypergeometric form of the sum $S_N(n)$, observe that

$$N(1-N)_\ell = (-1)^\ell\frac{N!}{(N-\ell-1)!} \tag{5.16}$$

and $(2)_\ell = (\ell+1)!$ give

$$(-1)^\ell\binom{N}{\ell+1} = N\frac{(1-N)_\ell}{(2)_\ell}, \tag{5.17}$$

and the following result follows from Proposition 5.2.

Proposition 5.3. *The hypergeometric form of the sum $S_N(n)$ is given by*

$$S_N(n) = \frac{(2n)!}{(2n-N)!}\beta^{2n-N+1}2^{1-N}N\ {}_2F_1\left(\left.\begin{matrix}1-N, & 1+\beta \\ & 2\end{matrix}\right|2\right). \tag{5.18}$$

The final form of the sum $S_N(n)$ involves the Meixner-Pollaczek polynomials defined by

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!}e^{\imath n\phi}{}_2F_1\left(\left.\begin{matrix}-n, & \lambda+\imath x \\ & 2\lambda\end{matrix}\right|1-e^{-2\imath\phi}\right). \tag{5.19}$$

Choosing $\lambda = 1$ and $\phi = \pi/2$ gives the next result.

Theorem 5.1. *The sum $S_N(n)$ is given by*

$$S_N(n) = \frac{(2n)!}{(2n-N)!}\frac{1}{(2\imath)^{N-1}}\beta^{2n-N+1}P_{N-1}^{(1)}\left(-\imath\beta;\frac{\pi}{2}\right). \tag{5.20}$$

Some examples are presented next.

Example 5.4. *The Meixner-Pollaczek polynomial*

$$P_2^{(1)}\left(x;\frac{\pi}{2}\right) = 2x^2 - 1 \tag{5.21}$$

gives

$$\begin{aligned} S_3(n) &= \frac{(2n)!}{(2n-3)!} \times (-1/4)\beta^{2n-2}(-2\beta^2-1) \\ &= \frac{(2n)(2n-1)(2n-2)}{4}\left[2\frac{B_{2n}}{2n} + \frac{B_{2n-2}}{2n-2}\right] \\ &= (2n-1)(n-1)B_{2n} + \tfrac{1}{2}n(2n-1)B_{2n-2}, \end{aligned}$$

which coincides with [2, eq. (2.6)].

Example 5.5. *The Meixner-Pollaczek of degree* 3 *is*

$$P_3^{(1)}\left(x;\frac{\pi}{2}\right) = \frac{4}{3}(-2x + x^3) \tag{5.22}$$

that produces

$$\begin{aligned} S_3(n) &= \frac{(2n)!}{(2n-4)!}\frac{1}{(2\imath)^3}\beta^{2n-3}\frac{4}{3}(2\imath\beta + \imath\beta^3) \\ &= -\tfrac{1}{3}(2n-1)(n-1)(2n-3)B_{2n} - \tfrac{1}{3}(2n)(2n-1)(2n-3)B_{2n-2}, \end{aligned}$$

which coincides with [2, eq. (2.7)].

The next step is to establish a correspondence between the *Dilcher coefficients* $b_k^{(N)}$ in (1.6) and the coefficients $p_k^{(n)}$ in

$$P_n^{(1)}(x;\pi/2) = \sum_{k=0}^{n} p_k^{(n)} x^k \tag{5.23}$$

the Meixner-Pollaczek polynomials. In particular, it is shown that the recurrence (1.7) is a consequence of the classical three terms recurrence for orthogonal polynomials.

Theorem 5.2. *The coefficients* $b_k^{(N)}$ *defined in* (1.6) *and the coefficients* $p_k^{(n)}$ *are related by*

$$b_k^{(N)} = \frac{(-1)^{N-1-k}}{2^{N-1}} p_{N-1-2k}^{(N-1)}. \tag{5.24}$$

The recurrence relation (1.7) *is equivalent to the three-terms recurrence*

$$(n+1)P_{n+1}^{(1)}\left(x;\frac{\pi}{2}\right) - 2xP_n^{(1)}\left(x;\frac{\pi}{2}\right) + (n+1)P_{n-1}^{(1)}\left(x;\frac{\pi}{2}\right) = 0. \tag{5.25}$$

satisfied by the Meixner-Pollaczek polynomials.

Proof. The Meixner-Pollaczek polynomials are orthogonal, hence they satisfy a three-terms recurrence. The specific form for this family in (5.25) appears in [6, Chapter 18]. In terms of its coefficients $p_k^{(n)}$ this is expressed as

$$(n+1)p_k^{(n+1)} - 2p_{k-1}^{(n)} + (n+1)p_k^{(n-1)} = 0. \tag{5.26}$$

Comparing the two expressions for $S_N(n)$ in (1.6) and (5.20) gives (5.24). This is equivalent to

$$p_\ell^{(N-1)} = 2^{N-1}\imath^{N-1+\ell} b^{(N)}_{\frac{1}{2}(N-1-\ell)}. \tag{5.27}$$

Replacing in (5.26) and simplifying yields (1.7). $\square$

Theorem 2 in [2], stated below, may be proven along the same lines of the proof of Theorem 2.2. Details are omitted.

Theorem 5.3. *If $2n \leq N-1$, then*

$$\begin{aligned} S_N(n) &= (-1)^n \frac{(2n)!(N-2n-1)!}{2^{N-1}} p_{N-2n-1}^{(N-1)} \\ &= (-1)^{N-1}(2n)!(N-2n-1)! b_n^{(N)}. \end{aligned} \tag{5.28}$$

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