Inversion of the Renewal Density with Dead-time

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July 2014

Abstract

Stationary renewal point processes are defined by the probability distribution of the distances between successive points (lifetimes) that are independent and identically distributed random variables. For some applications it is also interesting to define the properties of a renewal process by using the renewal density. There are well-known expressions of this density in terms of the probability density of the lifetimes. It is more difficult to solve the inverse problem consisting in the determination of the density of the lifetimes in terms of the renewal density. Theoretical expressions between their Laplace transforms are available but the inversion of these transforms is often very difficult to obtain in closed form. We show that this is possible for renewal processes presenting a dead-time property characterized by the fact that the renewal density is zero in an interval including the origin. We present the principle of a recursive method allowing the solution of this problem and we apply this method to the case some processes with input dead time. Computer simulations on Poisson and Erlang(2) processes show quite good agreement between theoretical calculations and experimental measurements on simulated data.

Some key words: Point processes, Lifetime, Renewal processes, Poisson and Erlang processes.

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1 Introduction

Renewal point processes are among the most important statistical models of point processes (PP) mainly because of the simplicity of their definition. Indeed in the stationary case they are defined by the fact that the intervals between successive points (lifetimes) are independent and identically distributed (IID) random variables (RV). For simplicity we assume that these RVs are continuous and then defined by their common probability density function (PDF) $p(x)$. This PDF then entirely describes the statistical properties of a renewal PP. For various applications, both in theoretical and practical problems, it is convenient to introduce the so-called *renewal density* (RD). Suppose that the renewal PP begins at a time $x = 0$. The probability $p(x)\mathrm{d}x$ is the probability that the *first point* of the PP after the origin of time appears in the interval $[x, x + \mathrm{d}x]$. On the other hand the renewal density $h(x)$ is such that $h(x)\mathrm{d}x$ is the probability of finding a point in this interval, regardless of whether it is, or not, the first point appearing after the origin.

Since the renewal process is completely defined by $p(x)$, it is clear that the renewal density $h(x)$ can be expressed in terms of $p(x)$ (see p. 53 of Cox and Isham, 1980). The result is

$$h(x) = \sum_{m=1}^{\infty} p_m(x), \tag{1}$$

where $p_1(x) = p(x)$ and $p_m(x)$ is the m-fold convolution of $p(x)$. It is in fact the PDF of the RV $Y_m = X_1 + X_2 + ... + X_m$, where the X_is are the distances between successive points after the origin. The RV Y_m describes the time interval between the origin and the mth point of the PP posterior to this origin.

In various applications it is easier to measure the RD $h(x)$ than the PDF $p(x)$. This is especially the case in statistical physics when PPs are analyzed by the means of coincidence systems in which we measure the probability of finding points in two distinct small intervals regardless of the number or points appearing between these intervals. (Saleh, 1978, Saleh and Teich, 1991, Snyder and Miller, 1991, Picinbono and Bendjaballah, 2010). This leads immediately to the *inverse problem* consisting in deducing $p(x)$ from $h(x)$. For this purpose the form of (1) suggests using the Laplace transforms of the two members of this equation. Let $F(s)$ be the one-sided Laplace transform of $p(x)$ or

$\int_0^\infty p(x)\exp(-sx)dx$, sometimes called the *generating function* of the RV X. It results from the convolution properties that the Laplace transform of $p_m(x)$ is simply $F^m(s)$, and noting that $|F()s| \leq 1$, the Laplace transform $H(s)$ of $h(x)$ becomes

$$H(s) = \sum_{m=1}^{\infty} F^m(s) = \frac{F(s)}{1 - F(s)} \ . \tag{2}$$

This expression can be inverted, which yields

$$F(s) = \frac{H(s)}{1 + H(s)} \ . \tag{3}$$

This equation seems to solve the inverse problem. But it remains a purely algebraic step consisting in the inversion of this Laplace transform in order to obtain the PDF $p(x)$. This step sometimes leads to inextricable calculations and does not produce usable analytical results. Nevertheless it is convenient to note that $F(s)$ itself can yield various interesting characteristics of the RV X. This is especially the case of its moments that can be deduced from a limited expansion of the generating function in the neighborhood of the origin. In various experiments the interest is limited to the measurements of the mean and of the variance. These quantities are easily deduced from $F(s)$, and the Laplace inverse of this function is useless.

The purpose of this paper is to overcome this difficulty of Laplace inversion and to show that in some circumstances an algorithmic recursive procedure can be introduced making it possible to obtain explicitly the PDF $p(x)$ from the renewal density $h(x)$. This is especially the case of renewal processes with dead-time effect (DT). After introducing the principles of this method we analyze its performance by using some examples of renewal PPs simulated by computer. The experimental results indicate quite good agreement with those deduced from calculations.

2 Principles of the inversion procedure in the presence of dead-time

A renewal PP is said to exhibit a *dead-time effect* if there exists a positive value D such that the PDF $p(x)$ satisfies $p(x) = 0$ when $x < D$ and also such that, for any interval $I_\epsilon = [D, D+\epsilon[$, the integral $\int_{I_\epsilon} p(x)dx$ is positive. Physically this means that the first point of the process posterior to the origin is necessarily posterior to D. It is clear that the same property is valid for $h(x)$. Indeed if there exists in interval Δx subset of $[0, D]$ and such that the integral $\int_{\Delta x} h(x)dx$ is positive, then the probability of finding a point of the PP in Δx is not zero. But this is in contradiction with the DT property, or the fact that the first point of the process posterior to the origin is posterior to D. Conversely it results from (1) that since $h(x)$ is a series of non-negative terms $p_n(x)$, the assumption that $h(x) = 0$ implies that all these terms are equal to zero, and in particular $p(x) = 0$.

There are various mechanisms that can generate PPs with DT effect. They are presented in (Cox and Isham, 1980, Picinbono, 2007, Picinbono, 2009) and here we only summarize the basic principles. There are two main kinds of DT effects. In order to explain their structure, let us call $\mathcal{P}$ the input PP assumed here to be a renewal process with points t_i. Some of these points can be erased by the DT effect and the non-erased points, called θ_i, constitute a new PP $\mathcal{P}'$ called output PP deduced from $\mathcal{P}$ by the DT effect.

The first kind of such an effect, called Type-1 or also output DT, is characterized by the fact that only the points θ_i contribute to erase some points of $\mathcal{P}$. More precisely to each point θ_i of $\mathcal{P}'$ is associated an interval $[\theta_i, \theta_i + D]$ such that all the points t_j of $\mathcal{P}$ belonging to this interval are erased. As an important obvious consequence, when the density μ of $\mathcal{P}$ satisfies $\mu >> 1/D$, the output PP $\mathcal{P}'$ becomes a almost periodic sequence of points with a lifetime almost constant and equal to D.

In the Type-2 DT effect, each point t_i of $\mathcal{P}$ generates an erasing interval $[t_i, t_i + D]$ such that all the points t_j of $\mathcal{P}$ belonging to this interval are erased. The DT effect is then due to the input points, which introduces also the term of input DT. In the asymptotic case $\mu >> 1/D$ the situation becomes completely different from that described above for the output DT. Indeed it is then obvious that almost all the points of $\mathcal{P}$ are erased in such a way that $\mathcal{P}'$ contains almost no point. This is why this kind of DT effect is also sometimes called extended, paralyzable or cumulative DT.

These distinctions are however without impact in our following discussion. Indeed the only property of the DT used in our analysis is that the renewal density is zero for $x < D$, and this property appears in the two previous examples of DT effects.

Let us now show that this DT property implies that the PDF $p_m(x)$ is zero for $x < mD$. Let us begin with $p_2(x)$ which is the convolution $[p_1 \star p_1](x)$ defined by

$$p_2(x) = \int p_1(s)u(s-D)p_1(x-s)u(x-s-D)ds, \tag{4}$$

where $u(x)$ is the unit step function equal to 1 for $x > 0$ and to zero otherwise. These two functions $u(.)$ in (4) characterize the DT property.

The product of the two unit functions is 1 if, and only if, $D < s < x - D$, which implies that $x > 2D$. The convolution is then zero if $x < 2D$. The same method is valid for the convolution between $p_m(x)$ and $p_1(x)$ yielding the PDF $p_{m+1}(x)$. In the integral defining the convolution appears the product $u(s - mD)u(x - s - D)$ which yields the condition $mD < s < x - D$, which implies that the convolution is zero if $x < (m + 1)D$. Applying this recursion from $n = 1$ yields $p_m(x) = 0$ if $x < mD$.

This leads us to decompose the renewal density $h(x)$ and all the functions $p_m(x)$ of (1) in the following form

$$h(x) = \sum_{n=1}^{\infty} h_n(x) \ ; \ p_m(x) = \sum_{n=1}^{\infty} a_{mn}(x), \tag{5}$$

in which the functions $h_n(x)$ and $a_{mn}(x)$ are zero outside the intervals $[nD, (n+1)D]$. The DT property on the PDFs $p_m(x)$ indicated above implies that these functions satisfy $a_{mn}(x) = 0$ if $n < m$, which means that the table of these functions is upper triangular, as seen in Table 1.

TABLE 1. TABLE OF FUNCTIONS $h_n(x)$ AND $a_{mn}(x)$.

h	h_1	h_2	h_3	h_4	h_5	$\dots$
p_1	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	$\dots$
p_2		a_{22}	a_{23}	a_{24}	a_{25}	$\dots$
p_3			a_{33}	a_{34}	a_{35}	$\dots$
p_4				a_{44}	a_{45}	$\dots$
p_5					a_{55}	$\dots$

We shall now show that it is possible to calculate recursively all the elements of this table uniquely from those h_i of the first line. Since $p_1(x) = p(x)$, this yields a solution to the problem stated above which is the calculation of $p(x)$ from $h(x)$. This calculation uses a recursive procedure.

It results immediately from the triangular structure presented in Table 1 and from (1) that $a_{11}(x) = h_1(x)$. Let us now calculate a_{12}. The procedure is first to deduce a_{22} from the convolution $p_1 \star p_1$, and then to apply (1) again, which yields $a_{12}(s) = h_2(s) - a_{22}(s)$. In order to calculate a_{22} we use various properties of convolutions of time-limited functions presented in Appendix 1.

It results from (5) that the PDF $p_2(x)$ can be expressed as

$$p_2(x) = \sum_{m=2}^{\infty} a_{2m}(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}[a_{1m} \star a_{1n}](x). \tag{6}$$

The term a_{22} is a function limited to the interval $[2D, 3D]$. On the other hand it is shown in (19) of Appendix 1 that the term $[a_{1m} \star a_{1n}](x)$ is limited to the interval $[MD, (M+2)D]$ with $M = m + n$. This implies that the only contribution to a_{22} of the last term of (6) is obtained for $m = n = 1$, which yields $a_{22} = [a_{11} \star a_{11}]_A(x)$, where the index A means that the convolution is calculated with (21). Once it is calculated, the use of (1) yields

$$a_{12}(x) = h_2(x) - a_{22}(x). \tag{7}$$

This shows that the second column of Table 1 can be deduced from the first two elements of the first line.

Let us show that this property is general. Suppose that all the columns of Table 1 are known till the column $k-1$. By using the previous results and those presented in Appendix 1 we shall see that the elements a_{ik} of the column k for $i > 1$ can be deduced from the elements a_{mn} already known. On the other hand the element a_{1k} is obtained from (1) by a relation similar to (7)

$$a_{1k}(x) = h_k(x) - [a_{2k}(x) + a_{3k}(x) + ... + a_{kk}(x)]. \tag{8}$$

It remains to calculate the terms a_{ik} of this equation. For this we use the fact indicated above that $p_i = p_1 \star p_{i-1}$. Applying (5) to p_1 and p_{i-1} yields a sum of convolutions $a_{1m} \star a_{(i-1)n}$ that can be calculated with the procedure presented in Appendix 1. The first task is to determine the pairs m, n such that the convolution $a_{1m} \star a_{(i-1)n}$ can introduce a non-zero contribution in the interval $kD, (k+1)D$ which is the only set of points where the functions a_{ik} are not equal to zero. It is shown in Appendix 1 that these pairs must satisfy either $m + n = i$, in which case the convolution is given by (21), or $m + n = i + 1$, in which case the convolution is given by (22). The complete calculation can be rather tedious, but introduces no technical difficulties. It solves however the problem stated at the beginning, which is the calculation of the PDF $p_1(x)$ from the renewal density $h(x)$. We shall now evaluate on some particular examples the performance of this recursive procedure of calculation.

3 Computer experiments

The most important renewal PP for physical applications is the *Poisson process*. This comes from the fact that it is the only PP with no memory at all at any scale. This property is introduced in various models of theoretical physics, in such a way that there are numerous examples of experimental measurements in order to verify the Poisson character of a PP. This especially appears in nuclear physics and also in statistical optics.

Various experiments have been introduced to verify whether or not a PP coming from a physical phenomenon is a Poisson process. Among them note especially coincidence or correlation and counting experiments. An overview of such experiments in statistical optics can be found in (Saleh, 1978). When a physical PP does not strictly verify the Poisson structure it is important to detect whether this result comes from a physical fundamental reason or is due to an artifact of the experimental setup such, for example, a DT effect.

Quantum theory of light detection predicts, for example, that the time instants of photons absorption observed at the output of a photodetector are not in general described by a Poisson process and this can be observed by counting or coincidence experiments (Saleh, 1978, Saleh and Teich, 1981, Picinbono and Bendjaballah, 2010). On the other hand number of PPs observed in particle emission in nuclear physics must theoretically constitute Poisson processes. This is why many papers are devoted to the analysis of DT effects in Poisson processes in order to explain the differences that can appear between theoretical prediction and experimental measurements. As a consequence the theoretical description of DT effects on Poisson processes is relatively well known and we shall briefly present the most significant results.

Let $\mathcal{P}$ be a stationary Poisson process of density μ. Consider first the input, or Type-2 DT, introduced above. It is shown that the corresponding output PP $\mathcal{P}'$ is a renewal PP (see p. 102 of Cox and Isham, 1980). Since the condition for a point t_i to be not erased is that there are no points of $\mathcal{P}$

in the interval $[t_i - D, t_i]$, it results from the properties of the Poisson process that the density of $\mathcal{P}'$ is $\lambda = \mu \exp(-\mu D)$. Similarly the renewal density is $h(x) = u(x - D)\lambda$. The fact that λ tends to 0 when $D \to \infty$ illustrates the term "paralyzable" introduced above.

The one-sided Laplace transform of this renewal density is $(\lambda/s) \exp(-sD)$, and applying (3) yields the generating function

$$F(s) = \frac{\lambda \exp(-sD)}{s + \lambda \exp(-sD)} \ . \tag{9}$$

It is possible to deduce from this function the moments of the lifetimes of $\mathcal{P}'$. In particular an expansion of $F(s)$ in terms of s^k limited to s^2 can yield the expected value and the variance of the lifetime and the results of this calculation were presented in (Feller, 1948). The inversion problem was solved several years later (Müller, 1971, Müller, 1973) by using a specific method adapted to the structure of (9). The result remains rather complicated. We have verified that the inversion algorithm presented in the previous section yields the same results.

The case of the output (or Type-1) DT of a Poisson process is quite different. Her also, according to (Cox and Isham, 1980, p. 102), the output PP $\mathcal{P}'$ is a renewal process and it results from the properties of Poisson processes that the PDF of the lifetime of $\mathcal{P}'$ is a displaced exponential or $p(x) = u(x - D) \exp[-\mu(x - D)]$. On the other hand the renewal density is given by a more complicated expression which can be obtained without difficulties.

In order to illustrate the performance of our inversion algorithm it is then appropriate to work with a renewal PP which is not a Poisson process and we have chosen an Erlang 2 PP. It is a renewal PP defined by a PDF of the lifetime given by

$$p(x) = u(x) 4\mu^2 x \exp(-2\mu x), \tag{10}$$

where μ is an arbitrary positive parameter. Its mean value and variance are equal to $1/\mu$ and $1/2\mu^2$ respectively. Consider now the case where this process $\mathcal{P}$ is perturbed by an input DT of duration D. This generates a new renewal process $\mathcal{P}'$ and it is easily found (see Appendix 2) that its renewal density $h(x)$

is

$$h(x) = u(x - D)[\lambda - \lambda' \exp[-4\mu(x - D)], \tag{11}$$

where

$$\lambda = \mu(1 + 2\mu D) \exp(-2\mu D)\ ,\ \lambda' = \mu(1 - 2\mu D) \exp(-2\mu D). \tag{12}$$

The density λ of a renewal PP is the limit of $h(x)$ when $x \to \infty$ (Cox and Isham, 1980, p. 51) and it is equal to the inverse of the mean value m_X of the distance between successive points. The previous equations imply that the density of $\mathcal{P}'$ is λ and then $m_X = 1/\lambda$ given by (12).

It is interesting to note in (12) that $\lambda' = 0$ for $D = 1/(2\mu)$. In this case (11) shows that $h(x) = u(x - D)\lambda$ with $\lambda = 2\mu/\mathrm{e}$. This is the renewal density of a Poisson PP of density 2μ modified by an input DT with the same value $D = 1/(2\mu)$. The inversion problem studied here was solved in this case in (Müller, 1973). Furthermore we note that this value of D introduces two different forms of $h(x)$ because λ' is positive if $D < 1/(2\mu)$ and negative if $D > 1/(2\mu)$. This will clearly appear in the results of computer experiments.

It is now interesting to verify these preliminary results by computer experiments. The starting point for that is to generate a sequence of lifetimes of points t_i considered as a trajectory of a particular Erlang 2 PP. The principle guiding the realization of these simulated data is presented in (Picinbono, 2007). By noting that (10) is the PDF of a sum of two IID positive exponential RVs, we deduce that generating the lifetimes of an Erlang 2 PP is identical to generating two independent sequences of independent exponential RVs. This is an easy task by using a procedure similar to those described in (Devroye, 1986, Ogata, 1981). We then obtain a sequence of N outcomes of IID random variables equal to the distances between successive points of an Erlang 2 process. By a computer procedure we simulate the input DT effect transforming these N outcomes into n other ones corresponding to the distances between successive points *after DT effect*. It is clear that, as indicated above, number of points are erased by this effect which implies that $n < N$, and this reduction can be very important. Since the statistical analysis leading to results of this table are realized with these n samples, the statistical errors due to the finite number of samples analyzed increases with D.

In our experiments $N = 10^7$ in order to obtain a convenient statistical precision of the measurements. In Table 2 we present results of measurements of the mean and variance of the lifetimes with and without DT effect on simulated Erlang 2 PP. In this table m and V are the sample mean and variance of the simulated Erlang 2 PP defined by (10) with $\mu = 1$ and $N = 10^7$, m_T and V_T their corresponding theoretical values, D is the value of the DT, n the number of samples with DT analyzed, m_E is the experimental sample mean value of the lifetime after input DT and m_X and λ are the theoretical values given by (12). We note that the number n can be reduced by approximately 10 when D increases from 0.25 to 2.

Table 2. Theoretical and experimental values of mean and variance of the lifetimes for Erlang 2 process with input DT.

m	m_T	V	V_T	D	n	m_E	m_X	λ
0.999	1	0.4998	0.5	0.25	$9.096\ 10^6$	1.0993	1.0991	0.9096
0.998	1	0.4997	0.5	0.5	$7.359\ 10^6$	1.3591	1.3591	0.7358
0.998	1	0.4997	0.5	0.75	$4.462\ 10^6$	1.7930	1.7927	0.5578
0.999	1	0.4999	0.5	1	$4.060\ 10^6$	2.4626	2.4630	0.4060
0.999	1	0.4999	0.5	2	$0.916\ 10^6$	10.9155	10.9196	0.0916

For $\mu = 1$ used in all our experiments the theoretical values of m and V are 1 and 0.5 respectively, which appears with an excellent precision in the five experiments. The diminution of the number n with D is simply a consequence of the DT effect which appears also on the decreasing of the density λ. This illustrates the term of "paralyzable DT" sometimes used for the input DT. Finally the values of the sample means of the lifetimes correspond with a quite good precision to their theoretical values.

The n values of the lifetimes are now used for the calculation of the renewal density and the results appear in Fig. 1 where $h(x)$ is represented in terms of x for three values of the DT D. The points correspond to the experimental measurements of the density and the continuous curve represents the theoretical value of this density given by (11) and (12). We observe an excellent agreement between experiment and theory.

Furthermore it appears clearly the difference of structures according to the threshold $D = 0.5$. For this precise value of D we find the structure of the renewal density of a Poisson process. Other experiments with $D < 0.5$ exhibit forms of the density very similar to the one obtained for $D = 0.25$. Similarly the densities obtained for $D > 0.5$ are similar to the one obtained with $D = 0.75$. The main difference are that the maximum of $h(x)$ obtained for $x = D$ decreases exponentially with D and the precision of the measurements is also decreasing because the number of points erased by DT effect increases with D. This is clearly shown in Table 2 by the decreasing of the values of n when D increases.

We can now use the same data for the solution of the inversion problem by using the recursive algorithm previously introduced. Let us first describe how the inversion problem can be stated in terms of Laplace transform inversion.

The Laplace transform of $h(x)$ given by (11) and (12) can be expressed in the form

$$H(s) = \frac{As + B}{s(s + 4\mu)} \exp(-Ds), \tag{13}$$

where

$$A = \lambda - \lambda' = 4\mu^2 D \exp(-2\mu D)\ ;\ B = 4\mu\lambda = 4\mu^2(1 + 2\mu D)\exp(-2\mu D). \tag{14}$$

Note that for $2\mu D = 1$ the function $H(s)$ is, as expected, proportional to $\exp(-Ds)/s$, which appears when the input PP is Poisson. By applying (3) we obtain

$$F(s) = \frac{(As + B)\mathrm{e}^{-Ds}}{s(s + 4\mu) + (As + B)\mathrm{e}^{-Ds}}\ . \tag{15}$$

It especially satisfies $F(0) = 1$, a condition necessary for any generating function. This expression is significantly more complicated than (9) obtained for a Poisson process and the method introduced by (Müller, 1971) cannot be directly applied. This leads to the use of the inversion algorithm introduced in Section 2.

The data already used for the results appearing in Table 2 are now analyzed by using normalized histograms in order to obtain values of the PDF of the lifetimes after DT. In all the experiments reported in Fig. 2 the values of the parameters are those used in Table 2, and especially $\mu = 1$. In these figures we present in a continuous line the curves obtained by using the method introduced previously and summarized in Appendix 1. The points correspond to experimental measurements of the PDF of the lifetime deduced from a normalized histogram of the data. The results correspond to the values $D = 0.25$, 0.5 and 0.75 of the DT. These values are chosen in order to illustrate the threshold effect corresponding to $D = 0.5$ and discussed above and also to be sufficiently small to preserve a good statically precision. We note that the curve corresponding to $D = 0.5$ is quite similar to the one published in (Müller, 19731 and 1973) corresponding to a Poisson process.

All these curves show that the experimental measurements are located with a quite good precision on the curves resulting of the application of the recursive algorithm presented in Section 2. This justifies the interest of the method of the PDF calculation introduced and discussed above.

Appendix 1
Convolutions of time-limited functions

Let $r_m(x; D)$ be the rectangular function equal to 1 for $mD \leq x < (m+1)D$ and zero otherwise. A function $f_m(x)$ is said to be time-limited in $[mD, (m+1)D[$ if it satisfies $f_m(x)r_m(x; D) = f_m(x)$. Let us calculate the convolution $c_{mn}(x) = [f_m \star g_n](x)$ between two time-limited functions f_m and g_n. It is defined by

$$c_{mn}(x) = \int f_m(x-s)r_m(x-s; D)g_n(s)r_n(s; D)ds. \tag{16}$$

Since the convolution is commutative, we can assume that $m \leq n$. Because of the functions $r(.; D)$ in (16), obtaining a non-zero integral requires that the integration variable s satisfies the two following equations

$$x - (m+1)D < s < x - mD \tag{17}$$

$$nD < s < (n+1)D. \tag{18}$$

If these equations are not both satisfied, the convolution is zero. This appears if the two intervals $[x-(m+1)D, x-mD]$ and $[nD, (n+1)D]$ do not overlap. This situation depends on the value of the variable x. Note first that if $x \leq 0$ there is no overlapping between these intervals. Indeed this assumption implies that s must be negative, according to (17) while (18) requires that $s > 0$. When x is increasing the overlapping appears as soon as $x - mD > nD$, or $x > MD$, where $M = m + n$. On the other hand the overlapping disappears if $x - (m+1)D > (n+1)D$. This implies that the convolution $c_{mn}(x)$ given by (16) can take nonzero values only if

$$MD < x < (M+2)D \ , \ M = m + n. \tag{19}$$

In this interval the integral appearing in (16) takes two distinct forms. For $MD \leq x \leq (M+1)D$, the integration variable s must belong to the interval $I_A = [nd, x - mD]$. On the other hand, if $(M+1)D \leq x \leq (M+2)D$ this variable s must belong to the interval $[x-(m+1)D, (n+1)D]$. This can be summarized by the relation

$$c_{mn}(x) = J_A(x) r_M(x; D) + J_B(x) r_{M+1}(x; D), \tag{20}$$

with

$$J_A(x) = \int_{nD}^{x-mD} f_m(x-s) r_m(x-s; D) g_n(s) r_n(s; D) ds, \tag{21}$$

$$J_B(x) = \int_{x-(m+1)D}^{(n+1)D} f_m(x-s) r_m(x-s; D) g_n(s) r_n(s; D) ds. \tag{22}$$

Note that, according to the reasoning presented above, $J_A[(m+n)D] = J_B([(m+n+2)D] = 0$.

Appendix 2
Erlang (2) renewal point processes

Erlang PPs appear in the study of congestion problems in processing systems of telephone calls. A good overview of the properties of the Erlang distribution appears in `http://en.wikipedia.org/wiki/Erlangdistribution`. The basic properties of an Erlang (2) RV defined by the PDF $p(x)$ of (10) is that it is a sum of two IID positive exponential RVs with the PDF $f(x) = u(x) 2\mu \exp(-2\mu x)$. It is indeed easy to verify that the convolution $[f \star f](x)$ is $p(x)$. As a consequence an Erlang (2) PP $\mathcal{P}$ is obtained by erasing regularly one point over two in a Poisson process $\mathcal{PP}$ of density 2μ. This yields obviously a renewal PP of density μ defined by the PDF $p(x)$ appearing in (10).

Let us now calculate its renewal density $h(x)$. For this we start of the property indicated in the introduction: suppose that there is a point of $\mathcal{P}$ at the origin O. Then $h(x)\mathrm{d}x$, $x > 0$, is the probability of finding a point of $\mathcal{P}$ in the interval $[x, x + \mathrm{d}x]$, regardless of whether it is, or not, the first point of $\mathcal{P}$ appearing after the origin. Since the number of points of a Poisson process in non-overlapping intervals are independent RVs, it is appropriate to deduce the properties of $\mathcal{P}$ from those of $\mathcal{PP}$. Note first that the fact that there is a point of $\mathcal{P}$ at the origin O means that there a point of $\mathcal{PP}$ at this origin and that this point is not erased. Then the first point of $\mathcal{PP}$ after the origin is erased. Consider the three non-overlapping intervals $I_1 = [0, x - D]$, $I_2 = [x - D, x]$, and $I_3 = [x, x + \mathrm{d}x]$. The event appearing in the definition of $h(x)$ of $\mathcal{P}$ occurs if there is no points of $\mathcal{P}$ in I_2 and at least one point of $\mathcal{P}$ in I_3, regardless the number of points in I_1. This can be decomposed into two events of $\mathcal{PP}$ according to the fact that the number N of points of $\mathcal{PP}$ in I_1 is even or odd. If N is even the first point of $\mathcal{PP}$ posterior to $x - D$ is erased, and then in order to obtain a non-erased point of $\mathcal{PP}$ in I_3 it is necessary that I_2 contains only one point of $\mathcal{PP}$. On the other hand if N is odd the first point of $\mathcal{PP}$ posterior to $x - D$ is not erased and becomes a point of $\mathcal{P}$. Since I_2 must no contain point of $\mathcal{P}$, the same property must be valid for $\mathcal{PP}$. Noting that since $\mathcal{PP}$ is Poisson of density 2μ the probabilities that N is even or odd are $(1/2)[1 + \exp\{-4\mu(x - D)\}]$ and $(1/2)[1 - \exp\{-4\mu(x - D)\}]$ we deduce immediately (11) and (12).

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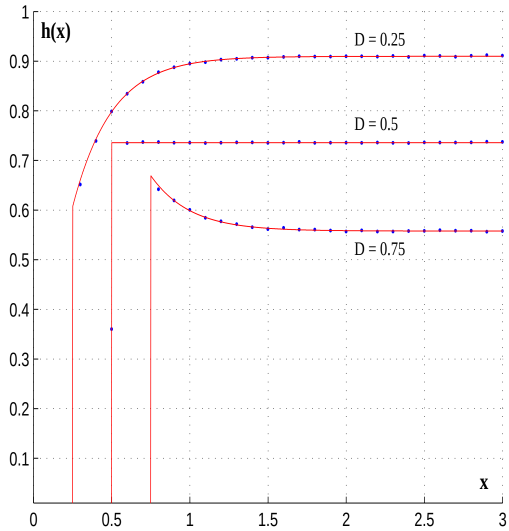
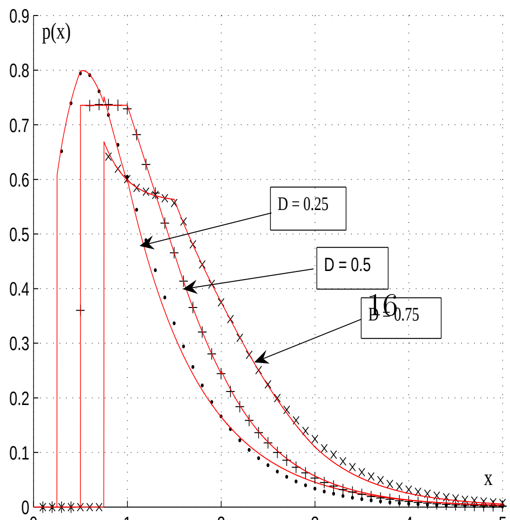
Figures captions

Fig. 1. Renewal density $h(x)$ of Erlang 2 PP after input DT for $D = 0.25, 0.5, 0.75$. Points: experiment, continuous curve: theory.

Fig. 2. PDF of the lifetime of the points of an Erlang 2 PP after input DT and various values of D. Points: experiment, continuous curve: theory.


h(x)
D = 0.25
D = 0.5
D = 0.75
x


p(x)
D = 0.25
D = 0.5
D = 0.75
x