Quantum Observables for Binary, Multi-Valued and Fuzzy Logic : Eigenlogic

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We propose a linear algebraic method, named Eigenlogic [1], for two-, many-valued and fuzzy logic using observables in Hilbert space. All logical connectives are represented by observables where the truth values correspond to eigenvalues and the atomic input propositional cases, i.e. the "models" of a propositional system, to the respective eigenvectors. In this way propositional logic can be formalized by using combinations of tensored elementary quantum observables. The outcome of a "measurement" of a logical observable will give the truth value of the associated logical proposition, and becomes "interpretable" when applied to vectors of its eigenspace, leading to an original insight into the quantum measurement postulate. Recently the concept of "quantum predicate" has been proposed in [2] leading to similar concepts.

We develop logical observables for binary logic and extend them to many-valued logic.

For binary logic and truth values $\{0, 1\}$ logical observables are commuting projector operators [1].

For truth values $\{+1, -1\}$ the logical observables are isometries formally equivalent to the ones of a composite quantum spin $\frac{1}{2}$ system, these observables are reversible quantum logic gates.

The analogy of many-valued logic with quantum angular momentum is then established using a general algebraic method, based on classical interpolation framework suggested by the finite-element method. Logical observables for three-valued logic are formulated using the orbital angular momentum observable L_z with $\ell = 1$. The representative 3-valued 2-argument logical observables for the ternary threshold logical connectives Min and Max are then explicitly obtained [3].

Also in this approach fuzzy logic arises naturally when considering vectors outside the eigensystem. The fuzzy membership function [4] is obtained by the quantum mean value (Born rule) of the logical projector observable and turns out to be a probability measure. Fuzziness arises because of the quantum superposition of atomic propositional cases, the truth of a proposition being in this case a probabilistic value ranging from completely false to completely true.

This method could be employed for developing algorithms in high-dimensional vector spaces for example in modern semantic theories, such as distributional semantics or in connectionist models of cognition [5]. For practical implementation, due to the exponential growth of the vector space dimension, adapted logical reduction methods must then be used. LSA (Latent Semantic Analysis) algorithms are often used in quantum-like approaches, this was done in [6] using the HAL (Hyperspace Analogue to Language) algorithm.

Our approach is also of interest for quantum computation because several of the observables in Eigenlogic are well-known quantum gates (for example CONTROL-Z) and other ones can be derived by unitary transformations. Ternary-logic quantum gates using qu-trits lead to less complex circuits, our formulation of multi-valued logical observables could help the development of new multi-valued quantum gate architectures.

[1] Toffano, Z. : "Eigenlogic in the spirit of George Boole", ArXiv :1512.06632 (2015)
[2] Ying, M.S. : "Foundations of Quantum Programming", 4.1 "Quantum Predicates", p.103, Morgan Kaufmann, (2016).
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Eigenlogic [1]

We propose a linear algebraic model for two- and many-valued logic using families of observables in Hilbert space, the **eigenvalues give the truth values** of logical propositions where the input propositional cases (models for the logical proposition) are represented by the respective eigenvectors.

quantum logical observables ⟶ logical connectives

eigenvectors of the observables ⟶ models of the propositional system

eigenvalues of the observables ⟶ truth values

The **outcome of a measurement on a logical observable** will give the truth value of the associated logical proposition.

In classical logic, predicates are used to describe properties of individuals or systems. A natural idea is that a **quantum predicate** [2] should be a physical observable.

Projector Binary Logical Observables

J. von Neumann considered **projectors as propositions** (**quantum logic**).

Two-dimensional rank-1 projector (qubit) $\boldsymbol{\Pi} = |1\rangle\langle 1|$

$\boldsymbol{\Pi}|1\rangle = 1|1\rangle$; eigenvalue 1. $\boldsymbol{\Pi}|0\rangle = 0|0\rangle$; eigenvalue 0.

Logical observable as a development in $\boldsymbol{\Pi}$

Here the elements of the development are the **rank-1 projectors** and the **truth values** $f(0)$ and $f(1)$:

$$\boldsymbol{F} = f(0)\boldsymbol{\Pi}_0 + f(1)\boldsymbol{\Pi}_1 = f(0)(\boldsymbol{I_d} - \boldsymbol{\Pi}) + f(1)\boldsymbol{\Pi} = \begin{pmatrix} f(0) & 0 \\ 0 & f(1) \end{pmatrix}$$

The two-argument logical observables can be developed on the corresponding four rank-1 projectors:

$$\boldsymbol{F} = \begin{pmatrix} f(0,0) & 0 & 0 & 0 \\ 0 & f(0,1) & 0 & 0 \\ 0 & 0 & f(1,0) & 0 \\ 0 & 0 & 0 & f(1,1) \end{pmatrix}$$

The conjunction is here: $\boldsymbol{F}_{AND} = \boldsymbol{\Pi} \otimes \boldsymbol{\Pi}$

All 16 two-argument connectives are derived.

connective for Boolean A, B	truth table {F, T}: {0, 1} ; { + 1, − 1}	{0, 1} projective logical observable	{ + 1, − 1} isometric logical observable
False F	F F F F	*0*	*+ I*
NOR	F F F T	*I − A − B + AB*	*(1/2) (+ I − U − V − UV)*
A ⇍ B	F F T F	*B − AB*	*(1/2) (+ I − U + V+UV)*
¬A	F F T T	*I − A*	*− U*
A ⇏ B	F T F F	*A − AB*	*(1/2) (+ I + U − V + UV)*
¬B	F T F T	*I − B*	*− V*
XOR ; A⊕B	F T T F	*A + B − 2 AB*	*U·V = Z⊗Z*
NAND	F T T T	*I − AB*	*(1/2) (− I − U − V + UV)*
AND ; A∧B	T F F F	*AB = Π⊗Π*	*(1/2) (+ I + U + V − UV)*
A ≡ B	T F F T	*I − A − B + 2 AB*	*− UV*
B	T F T F	*B = I⊗Π*	*V = I⊗Z*
A ⇒ B	T F T T	*I − A + AB*	*(1/2) (− I − U + V − UV)*
A	T T F F	*A = Π⊗I*	*U = Z⊗I*
A ⇐ B	T T F T	*I − B + AB*	*(1/2) (− I + U − V − UV)*
OR ; A∨B	T T T F	*A + B − AB*	*(1/2) (− I + U + V + UV)*
True T	T T T T	*I*	*− I*

The 16 - Binary Logical Connectives - Truth Tables - Eigenlogic Projective Observables - Eigenlogic Isometric Observables

Isometric Binary Logical Observables

There is a linear bijection (isomorphism) from the projector $\boldsymbol{F}$ to a reversible observable $\boldsymbol{G}$ (isometry) using the **Householder transform**:

$$\boldsymbol{G} = \boldsymbol{I_d} - 2\boldsymbol{F}$$

both observables commute and have the same system of eigenvectors.

The eigenvalues that correspond to the truth values, respectively "false" and "true", are now $+1$ and -1. The seed observable is **Pauli-Z**:

$$\boldsymbol{Z} = \boldsymbol{I_d} - 2\boldsymbol{\Pi} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$$

The **logical dictators** $\boldsymbol{U}$ and $\boldsymbol{V}$ are expressed as a function of the seed observable $\boldsymbol{Z}$: $\boldsymbol{U} = \boldsymbol{Z} \otimes \boldsymbol{I_d}$ $\boldsymbol{V} = \boldsymbol{I_d} \otimes \boldsymbol{Z}$

For the isometric observables negation is obtained by multiplying by -1.

The exclusive disjunction (XOR) : $\boldsymbol{G}_{XOR} = \boldsymbol{Z} \otimes \boldsymbol{Z}$

The logical conjunction (AND) :

$$G_{AND} = \frac{1}{2}(I + Z \otimes I + I \otimes Z - Z \otimes Z) = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = C^Z$$

which is the quantum **Control-Z** gate

Multivalued Eigenlogic

Identification with **quantum orbital angular moment (OAM)**, The z component of the OAM gives the observable $\boldsymbol{L_z}$:

$$L_z = \hbar \Lambda = \hbar \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigenvalues, , $\{+1, 0, -1\}$, are the logical values:

"false" $F \equiv +1$ "neutral" $N \equiv 0$ "true" $T \equiv -1$

The logical observables can be expressed as spectral decompositions on the rank-1 projectors of the three-dimensional vector space:

$$\boldsymbol{\Pi}_{+1} = \frac{1}{2}\Lambda(\Lambda + I) \qquad \boldsymbol{\Pi}_0 = I - \Lambda^2 \qquad \boldsymbol{\Pi}_{-1} = \frac{1}{2}\Lambda(\Lambda - I)$$

We can define the dictators $\boldsymbol{U}$ and $\boldsymbol{V}$ simply by the rule of composition:

$$U = \Lambda \otimes I \qquad V = I \otimes \Lambda \qquad U \cdot V = \Lambda \otimes \Lambda$$

The logical observables are in this case represented by 9x9 matrices.

Tri-Valued Logical Observables : Min and Max

In tri-valued logic, popular connectives are **Min** and **Max**, defined by their tables:

Min U \\ V	F	N	T
F ≡ +1	+1	+1	+1
N ≡ 0	+1	0	0
T ≡ − 1	+1	0	− 1

Max U \\ V	F	N	T
F ≡ +1	+1	0	− 1
N ≡ 0	0	0	− 1
T ≡ − 1	− 1	− 1	− 1

The corresponding logical observables are given function of the dictators:

$$\begin{cases} \mathrm{Min}_{m=3}(U,V) = \frac{1}{2}(U + V + U^2 + V^2 - U \cdot V - U^2 \cdot V^2) \\ \mathrm{Max}_{m=3}(U,V) = \frac{1}{2}(U + V - U^2 - V^2 + U \cdot V + U^2 \cdot V^2) \end{cases}$$

The **total number of logical connectives** for a system of m values and n arguments is the combinatorial number $\boldsymbol{m}^{\boldsymbol{m}^{\boldsymbol{n}}}$. For a two-argument system taking 3 values the total number of connectives will be 19683.

Fuzzy Logic Interpretation

Fuzzy logic deals with propositions that can lie between "completely true" and "completely false".

When the **quantum state is not an eigenstate of the logical system** one can always express a state vector as a superposition, for example:

$$|\psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle$$

when more than one coefficient is non-zero we are in the case of a **fuzzy proposition**, that can be interpreted as a quantum superposition of propositional models. Mean values are **fuzzy membership functions** μ [3].

Using the Born rule for the state: $|\varphi> = \sin\alpha\,|0> + e^{i\beta}\cos\alpha\,|1>$

$$\mu(a) = <\varphi|\boldsymbol{\Pi}|\varphi> = \cos\alpha e^{-i\beta} < 1\,(||1><1|)\cos\alpha e^{i\beta}|1> = \cos^2\alpha$$

For a compound quantum state: $|\psi> = |\varphi_p> \otimes |\varphi_q>$

$$\mu(a) = <\psi|\boldsymbol{\Pi} \otimes \boldsymbol{I}|\psi> = p(1-q) + p \cdot q = p, \quad \mu(b) = <\psi|\boldsymbol{I} \otimes \boldsymbol{\Pi}|\psi> = q$$

We can give the "measure" of the conjunction, $A \wedge B$, and the disjunction, $A \vee B$ using the seed projector observable:

$$\begin{cases} \mu(a \wedge b) = <\psi|\boldsymbol{\Pi} \otimes \boldsymbol{\Pi}|\psi> = p \cdot q = \mu(a) \cdot \mu(b), \\ \mu(a \vee b) = p + q - p \cdot q = \mu(a) + \mu(b) - \mu(a) \cdot \mu(b) \end{cases}$$

Discussion and Potential Applications

- Several observables in Eigenlogic are well-known quantum gates. *e.g.* $\boldsymbol{C^Z}$ (**Control-Z**) corresponds here to the conjunction logical observable $\boldsymbol{G}_{AND}$.
- Operational correspondence between **control logic** (Deutsch's paradigm) and **propositional logic**. A goal could be to express the diagonal forms of observables in Eigenlogic as products of known quantum gates.
- Questions of **how could Eigenlogic be used for quantum computing?**: measurements give truth values, but how to reinject them in a circuit?
- Ternary-logic quantum gates using **qu-trits** lead to less complex circuits (*e.g.* **Adder**). Our formulation of multi-valued logical observables could help the development of new multi-valued quantum gate architectures.
- The **fuzzy membership function** is obtained by the quantum mean value (Born rule) of the logical projector observable.
- This method could be employed for developing **algorithms** in high-dimensional **vector spaces** for example in **modern semantic theories**, such as distributional semantics or in connectionist models of cognition [4].

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