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Explicit robustness and fragility margins for linear discrete systems with piecewise affine control law

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Abstract

In this paper, we focus on the robustness and fragility problem for piecewise affine (PWA) control laws for discrete-time linear system dynamics in the presence of parametric uncertainty of the state space model. A generic geometrical approach will be used to obtain robustness/fragility margins with respect to the positive invariance properties. For PWA control laws defined over a bounded region in the state space, it is shown that these margins can be described in terms of polyhedral sets in parameter space. The methodology is further extended to the fragility problem with respect to the partition defining the controller. Finally, several computational aspects are presented to transform the theoretical formulations to *explicit* representations (vertex/halfspace representation of polytopes) of these sets.

Key words: PWA control, explicit robustness margin, fragility analysis

1 Introduction

When analyzing a control law, both practitioner and theoretician take into account the capacity to cope with disturbances and model uncertainties. This characteristic is classically denoted in control theory as *robustness*. The presence of additive disturbances in the control system structure is due to measurement noises and external perturbation sources. Otherwise, the uncertainty stems from model reduction, linearization of nonlinear elements, imperfect mathematical model or partial information on the parameters. These elements are unavoidable in the control design by the essence of their causes and the practical need of complexity reduction in model-based design, and consequently the robustness consideration of the closed-loop is necessary.

The present study concentrates on the robustness problem in the presence of model uncertainty for PWA feedback control laws. This class of controllers is known to lead in closed loop to a hybrid system formulation [13]. Another motivation for the study of the PWA controllers and their robustness is the recent interest in the optimization-based design via parametric convex programming [3,28,25,22,20] or the approximate explicit solutions in Model Predictive Control (MPC) [16,19]. Recall that explicit MPC aims to minimize a cost function subject to a set of constraints wherein the sequence of control variables over the prediction horizon stands for the decision variables and the current state represents the parameter.

There are various types of uncertainties in the robust control literature according to [9,18,30]. In this paper, our interest is in parametric uncertainties, understood as variations of coefficients of a model with a pre-imposed structure. This is mainly due to the fact that the unstructured uncertainty will generally lead to an augmented state space and the extension of a predefined controller leads to nonuniqueness and related well-posedness problems which are beyond the scope of the current study of PWA dynamics over a given state space partition.

At the same time, from the practical point of view, the implementation of control laws in general leads to numerical round-offs. This may affect closed-loop stability. The maxi-

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mal admissible set of numerical errors, for which the implemented control law still guarantees the stability, is denoted as the *fragility margin*. This problem has already been investigated in literature [10,17], but these studies neither provide a constructive procedure to compute such a margin, nor cover our interests in the class of PWA control laws. As far as it concerns the fragility margin of PWA control laws, we will refer to the possible inaccuracy in the coefficients of the PWA controllers without assuming any uncertainty on the state space partition. Perturbations in the region description will lead to overlapping regions or topological changes in the partition with implications on non-uniqueness of the trajectories. All these aspects are addressed for the first time in the literature to our best knowledge.

Based on the preliminary results in [23,21], this paper provides a theoretical framework and mathematical computation for the explicit robustness/fragility margin of a nominal discrete-time linear system, controlled by a given PWA control law. The methodology is centered around the *robust positive invariance* properties which have been studied since the late '80s [6,29,7,8]. Note that the robust positive invariance is associated with robust stability by the fact that the closed-loop is designed to keep the trajectory inside of a subset of the state space, namely a positively invariant set. Guaranteeing robust asymptotic stability is beyond the scope of the present paper. Based on the same constructive principle, the problem of finding the biggest set of errors in the description of the regions of the given state space polyhedral partition is also tackled in this study.

Notation and basic definitions

Throughout the paper, $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{N}$ and $\mathbb{N}_+$ denote the field of real numbers, the set of nonnegative real numbers, the set of integers, the set of non-negative integers, the set of positive integer numbers, respectively. For two column vectors: $x \in \mathbb{R}^n$ $x = [x_1\ x_2,\ \dots\ x_n]^T, y \in \mathbb{R}^n$ $y = [y_1\ y_2,\ \dots\ y_n]^T$ and two $n \times m$ matrices $A \in \mathbb{R}^{n\times m}$ $A = [a_{ij}], B \in \mathbb{R}^{n\times m}$ $B = [b_{ij}]$, then the partial order relation $x \le y$ and $A \le B$ are equivalent to $x_i \le y_i,\ \forall i = 1, \dots, n$ and $a_{ij} \le b_{ij},\ \forall i = 1, \dots, n$ and $\forall j = 1, \dots, m$ respectively.

A vector with its elements equal to one (zero) is denoted by $\mathbf{1}$ ($\mathbf{0}$) or by 1_n (0_n) in the case when the dimension n must be explicitly stated. Similarly, $\mathbf{I}$ denotes an identity matrix of appropriate dimension, with a subscript when the dimension of this matrix needs to be specified i.e. $\mathbf{I}_n$ means that $\mathbf{I} \in \mathbb{R}^{n\times n}$.

For a matrix A with m rows and n columns, then $\mathrm{vec}(A)$ represents the vector composed by the columns of the matrix A as follows: $\mathrm{vec}(A) := \begin{bmatrix} A(\cdot,1)^T & \dots & A(\cdot,n)^T \end{bmatrix}^T$, where $A(\cdot, i)$ denotes the i^{th} column of matrix A.

Given two matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$, their Kronecker tensor product is denoted as $A \otimes B \in \mathbb{R}^{mp\times nq}$ and is defined as:

$$A \otimes B := \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}.$$

For an arbitrary set $\mathcal{S} \subseteq \mathbb{R}^n$, $\mathrm{int}(\mathcal{S})$ denotes the interior of $\mathcal{S}$. By $\dim(\mathcal{S})$, we denote the dimension of its affine hull. $\mathcal{V}(\mathcal{S})$ describes the set of vertices whenever $\mathcal{S}$ is a polytope (bounded polyhedral set). If $\mathcal{S} \subset \mathbb{R}^n$ is composed by a finite number of vectors $\mathcal{S} = \{s_1, s_2, \dots, s_m\}$, then $[\mathcal{S}]$ denotes a matrix for which the columns are the elements of $\mathcal{S}$ in an arbitrary order: $[\mathcal{S}] = \begin{bmatrix} s_1 & s_2 & \dots & s_m \end{bmatrix}$. Moreover, by $\mathrm{conv}(\mathcal{S})$, we denote the convex hull of the points in $\mathcal{S}$. Given a map $f : \mathbb{R}^m \to \mathbb{R}^n$, a set $\mathcal{S} \subset \mathbb{R}^m$,

$$f(\mathcal{S}) = \{y \in \mathbb{R}^n \mid \exists x \in \mathcal{S} \text{ such that } y = f(x)\}$$

denotes the image of the set $\mathcal{S}$ via the mapping f. For a linear map $f(x) = Ax$ with $A \in \mathbb{R}^{n\times m}$, the image of a set $S \subset \mathbb{R}^m$ is briefly rewritten as $f(\mathcal{S}) = A\mathcal{S}$. The Minkowski sum of two sets P_1 and P_2 is denoted as $P_1 \oplus P_2$ and is defined by the relation:

$$P_1 \oplus P_2 := \{y \mid \exists x_1 \in P_1, x_2 \in P_2 \text{ such that } y = x_1 + x_2\}.$$

The unit simplex in $\mathbb{R}^L$ is defined as

$$\mathcal{S}_L = \left\{x \in \mathbb{R}^L_+ \mid \mathbf{1}^T x = 1\right\}. \tag{1}$$

Finally, for an $N \in \mathbb{N}_+$, $\mathcal{I}_N$ denotes the set of integers: $\mathcal{I}_N := \{i \in \mathbb{N}_+ \mid i \le N\}$.

2 Preliminaries

In this section, some basic notions related to the piecewise affine control functions and the discrete dynamics will be introduced to facilitate the problem formulation and the presentation of the main results of the paper.

Definition 2.1 A set of $N \in \mathbb{N}_+$ full-dimensional polyhedra $\mathcal{X}_i \subset \mathbb{R}^n$, i.e. $\mathcal{P}_N(\mathcal{X}) = \{\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_N\}$ is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^n$ if:

(1) $\bigcup_{i\in\mathcal{I}_N} \mathcal{X}_i = \mathcal{X}$.
(2) $\mathrm{int}(\mathcal{X}_i) \bigcap \mathrm{int}(\mathcal{X}_j) = \emptyset$ with $i \ne j,\ (i,j) \in \mathcal{I}_N^2$,

Also, $(\mathcal{X}_i, \mathcal{X}_j)$ are called neighbours if $(i,j) \in \mathcal{I}_N^2$, $i \ne j$ and $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = n - 1$. If $\mathcal{X}$ is a polytope, we call $\mathcal{P}_N(\mathcal{X})$ a *polytopic partition*.

Definition 2.2 A function $f_{pwa} : \mathcal{X} \to \mathbb{R}^m$ defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of the polyhedron $\mathcal{X}$ by the relation $f_{pwa}(x) = A_i x + a_i$ for $x \in \mathcal{X}_i$, $i \in \mathcal{I}_N$, with $A_i \in \mathbb{R}^{m\times n}, a_i \in \mathbb{R}^m$, is said to be a *piecewise affine function* over $\mathcal{P}_N(\mathcal{X})$.

In the present paper, we consider discrete linear time–invariant (LTI) systems described by state equations:

$$x_{k+1} = Ax_k + Bu_k, \tag{2}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ denotes the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

If the control action is synthesized in terms of a piecewise affine state feedback defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of a polyhedral set $\mathcal{X} \subset \mathbb{R}^n$ then it will be described by

$$u(x_k) = f_{pwa}(x_k) = G_i x_k + g_i \text{ for } x_k \in \mathcal{X}_i,\ i \in \mathcal{I}_N, \tag{3}$$

with $G_i \in \mathbb{R}^{m\times n}$ and $g_i \in \mathbb{R}^m$. With this control law, the resulting closed-loop system (2)-(3) is a piecewise affine system described by the state equation:

$$x_{k+1} = (A + BG_i)x_k + Bg_i \text{ for } x_k \in \mathcal{X}_i. \tag{4}$$

Definition 2.3 A set $\mathcal{X} \subset \mathbb{R}^n$ is positively invariant with respect to the system $x_{k+1} = f(x_k)$ if $x \in \mathcal{X}$ implies $f(x) \in \mathcal{X}$.

In the context of robustness analysis for the closed loop PWA dynamics, the introduction of discrete time-varying uncertainty on $[A\ B]$ in the dynamical model (2) is of use. We assume that the matrix $[A\ B]$ belongs to a polytopic set Ω:

$$\Omega = \text{conv}\{[A_1\ B_1], \ldots, [A_L\ B_L]\} \tag{5}$$

where *conv* denotes the convex hull. Thus, if $[A\ B] \in \Omega$, then there exists non negative scalars $\alpha_1, \ldots, \alpha_L, \sum_{i=1}^{L} \alpha_i = 1$ satisfying the relation $[A\ B] = \sum_{i=1}^{L} \alpha_i [A_i\ B_i]$. It is clear that a polytope can be described by the convex hull of its vertices, given as vectors in an Euclidean space. Therefore, for the convex hull of matrices, one can exploit the isomorphism between $\mathbb{R}^{m\times n}$ and $\mathbb{R}^{mn}$. With a slight abuse of notation, we call Ω a *parametric uncertainty polytope*, for ease of presentation. Also, a subset of Ω is called *polytope* if its associated set of coefficients $\alpha = [\alpha_1 \ldots \alpha_L]^T$ is a polytope.

The development of the results in this paper is based on a set of hypotheses and working assumptions. Part of them will represent instrumental conditions for the establishment of constructive procedures for our results:
Assumption: Given a nominal LTI system (2) and a piecewise affine state feedback function $u(x)$ (3), defined over a polyhedral partition $\mathcal{P}_N(\mathcal{X})$ of the set $\mathcal{X} \subset \mathbb{R}^n$, it is assumed that

(1) The set $\mathcal{X}$ is a polytope.
(2) The set $\mathcal{X}$ is positively invariant with respect to the PWA dynamics generated by the nominal model (2) in closed loop with the PWA controller (3).
(3) The control function $f_{pwa} : \mathcal{X} \to \mathbb{R}^m$ is continuous.
(4) $0 \in \text{int}(\mathcal{X})$ [1].

There are several control design methodologies based on a PWA formulation. We mention here only MPC or the interpolation-based control [20] which are based on a polyhedral partition of the state space inherited from optimal solution to a linear/quadratic parametric programming problem. In the most general case, the partition is not convex as for example in the case when the state/input constraints are not convex. In this context, assumption 1 implies that we restrict our attention to bounded convex domains and particularly to polytopes. Assumption 2 implies that with the given PWA control law $u(x)$, the trajectories of the nominal linear system (2) are confined in $\mathcal{X}$.

From assumption 1, and the polytopic partition characteristic of $\mathcal{X}$, it follows that its components $\mathcal{X}_i$, $\forall i \in \mathcal{I}_N$ are also polytopes. Therefore, these sets can be defined either via the vertex or via the halfspace representation. The problem of obtaining the vertices of a given polytope from its halfspace representation, is called *vertex enumeration*. Many studies have been dedicated to this problem. A prominent solution is reported in [2] with a computation time in $O(ndv)$, where n denotes the number of halfspaces, d denotes the dimension of this polytope, v is its number of vertices. The halfspace representation of the polytopes of interest can be defined as follows for every $i \in \mathcal{I}_N$:

$$\begin{aligned} \mathcal{X} &= \{x : Fx \le h\}, \quad \text{with } F \in \mathbb{R}^{r\times n}, h \in \mathbb{R}^r \\ \mathcal{X}_i &= \{x : F_i x \le h_i\}, \text{ with } F_i \in \mathbb{R}^{r_i\times n}, h_i \in \mathbb{R}^{r_i}. \end{aligned} \tag{6}$$

The vertex representation of polytopic sets $\mathcal{X}$ and $\mathcal{X}_i$ with corresponding sets of vertices $\mathcal{V}(\mathcal{X}) = \{v_1, v_2, \ldots, v_q\}$, and $\mathcal{V}(\mathcal{X}_i) = \{w_{i1}, w_{i2}, \ldots, w_{iq_i}\}$ are defined as:

$$\begin{aligned} \mathcal{X} &= \text{conv}\{v_1, v_2, \ldots, v_q\}, \\ \mathcal{X}_i &= \text{conv}\{w_{i1}, w_{i2}, \ldots, w_{iq_i}\}. \end{aligned} \tag{7}$$

For ease of presentation, define the following sets of vertices:

$$\mathcal{W}_i = \mathcal{V}(\mathcal{X}_i),\ \mathcal{W} = \bigcup_{i\in\mathcal{I}_N} \mathcal{V}(\mathcal{X}_i). \tag{8}$$

With respect to an arbitrary order, the following matrices can be defined such that their columns are the elements of their associated sets:

$$\begin{aligned} &V = [\mathcal{V}(\mathcal{X})] \in \mathbb{R}^{n\times q},\ U = [f_{pwa}(\mathcal{W})] \in \mathbb{R}^{m\times p}, \\ &V_i = [\mathcal{W}_i] \in \mathbb{R}^{n\times q_i}, U_i = [f_{pwa}(\mathcal{W}_i)] \in \mathbb{R}^{m\times q_i}, \\ &W = [\mathcal{W}] \in \mathbb{R}^{n\times p}. \end{aligned} \tag{9}$$

[1] In the context of control design, the origin is usually supposed to be asymptotically stable and the set $\mathcal{X}$ is understood as basin of attraction. However, our results presented next still hold true without this assumption.

3 Explicit robustness margin for PWA control laws

The present study aims to find the set of model uncertainties, while closed-loop stability is ensured by the given PWA control law.

3.1 Problem formulation and structure of the solution

Given a continuous PWA state feedback control law (3) and $[A(k)\ B(k)]$ belongs to a polytopic uncertainty set Ω defined by (5), the *robustness problem* aims to find the set of coefficients, denoted by $\Omega^{\alpha}_{rob} \subseteq \mathcal{S}_L$, associated with $\Omega_{rob} \subseteq \Omega$ such that the polytope $\mathcal{X}$ is positively invariant with respect to the closed loop system:

$$x_{k+1} = (A(k) + B(k)G_i)x_k + Bg_i \quad \text{for } x \in \mathcal{X}_i, \tag{10}$$

$\forall\, [A(k)\ B(k)] \in \Omega_{rob}$. The set Ω_{rob} can be alternatively called the *robustness margin*.

The set Ω_{rob} can be characterized based on the local structure of the dynamics. The next result shows a strong property that can be obtained despite the global nonlinearity (PWA formulation) of the dynamics.

Theorem 3.1 *The set Ω_{rob} for the PWA dynamics robustness problem is convex.*

PROOF. Let $[A(k_1)\ B(k_1)]$ and $[A(k_2)\ B(k_2)] \in \Omega_{rob}$. The invariance property of the set $\mathcal{X}$ with respect to (10) implies the following set inclusions:

$$\begin{aligned}&(A(k_1) + B(k_1)G_i)\mathcal{X}_i \oplus B(k_1)g_i \subseteq \mathcal{X}, \forall i \in \mathcal{I}_N,\\ &(A(k_2) + B(k_2)G_i)\mathcal{X}_i \oplus B(k_2)g_i \subseteq \mathcal{X}, \forall i \in \mathcal{I}_N.\end{aligned}$$

respectively. Since, by Assumption 1, the set $\mathcal{X}$ is convex, one has:

$$\begin{aligned}&(1-\mu)\left((A(k_2) + B(k_2)G_i)\mathcal{X}_i \oplus B(k_2)g_i\right) \oplus\\ &\mu\left((A(k_1) + B(k_1)G_i)\mathcal{X}_i \oplus B(k_1)g_i\right) \subseteq \mathcal{X},\end{aligned} \tag{11}$$

$\forall i \in \mathcal{I}_N$ and $0 \le \mu \le 1$. Inclusion (11) proves $\mu\,[A(k_1)\ B(k_1)] + (1-\mu)\,[A(k_2)\ B(k_2)] \in \Omega_{rob}$ and consequently the convexity of the set Ω_{rob}. $\square$

As a consequence of the convexity of both Ω_{rob} and Ω, the robustness margin can be expressed as an equivalent set:

$$\begin{aligned}\Omega^{\alpha}_{rob} = \Big\{&\alpha \in \mathbb{R}^L_+ \mid \forall i \in \mathcal{I}_N,\ \mathbf{1}^T\alpha = 1,\\ &\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)\mathcal{X}_i \oplus \alpha_j B_j g_i \subseteq \mathcal{X}\Big\}.\end{aligned}$$

The isomorphic relationship between Ω_{rob} and Ω^{α}_{rob} follows directly from the one-to-one correspondence between the elements of these sets. Consequently, the constructive procedures for the characterization of robustness margins will be expressed in terms of $\Omega^{\alpha}_{rob} \subset \mathbb{R}^L$. If $L < n(m+n)$, this expression is more effective than the one via the elements of $[A\ B]$. However, the paper still handles the latter case.

3.2 Construction based on the vertex representation

With respect to definitions (7)–(9), the first result can be stated as follows:

Theorem 3.2 *Consider the system* (10) *subject to a parametric uncertainty* (5). *For a given piecewise affine control law* (3) *satisfying Assumptions 1-3, the robustness margin is obtained as the projection*

$$\Omega^{\alpha}_{rob} = \mathrm{Proj}_{\mathbb{R}^L}\mathcal{R} \tag{12}$$

where $\mathcal{R}$ represents the polyhedral set:

$$\begin{aligned}\mathcal{R} = \Big\{&(\alpha, \Gamma) \in \mathcal{S}_L \times \mathbb{R}^{q\times p}_+ \,|\mathbf{1}^T\Gamma = \mathbf{1}^T,\\ &\sum_{j=1}^{L} \alpha_j(A_jW + B_jU) = V\Gamma\Big\},\end{aligned} \tag{13}$$

with W, U defined in (9), *$\mathcal{S}_L$ defined in* (1), *$p = \mathrm{Card}(\mathcal{W}), q = \mathrm{Card}(\mathcal{V}(\mathcal{X}))$ and Γ represents any matrix with the non-negative elements, satisfying* (13).

PROOF. If Ω_{rob} describes the robustness margin, then for all $[A\ B] \in \Omega_{rob}$ and $\forall x \in \mathcal{X}_i$, $\forall i \in \mathcal{I}_N$:

$$(A + BG_i)x + Bg_i \in \mathcal{X}. \tag{14}$$

First of all, we remark that the calculation of the robustness margin corresponds to the search for the subset of polytopic uncertainty $\Omega_{rob} \subseteq \Omega$ and equivalently to the description of the set Ω^{α}_{rob}. Clearly, (14) can be written by:

$$\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)x + \alpha_j B_j g_i \in \mathcal{X},\ \forall x \in \mathcal{X}_i \tag{15}$$

with α_j as the elements of a vector $\alpha \in \mathcal{S}_L$. On the other hand, by expressing the state $x \in \mathcal{X}_i$ as a convex combination of the vertices $x = \sum_{l=1}^{q_i} \beta_l w_{il}$ for $\beta_l \in \mathbb{R}_+$ and $\sum_{l=1}^{q_i} \beta_l = 1$, it follows that (15) is equivalent to:

$$\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)w_{il} + \alpha_j B_j g_i \in \mathcal{X},\ \forall i \in \mathcal{I}_N,\ \forall l \in \mathcal{I}_{q_i}.$$

Further, the inclusion can be explicitly described by the existence of $y_{il} \in \mathcal{X}$ such that:

$$\sum_{j=1}^{L} \alpha_j(A_j + B_jG_i)w_{il} + \alpha_j B_j g_i = y_{il}. \tag{16}$$

y_{il} can be expressed as: $y_{il} = [\mathcal{V}(\mathcal{X})]\gamma_{il}$ for $\gamma_{il} \in \mathcal{S}_q$. By replacing this inclusion in (16) with notation (9), we obtain:

$$\sum_{j=1}^{L} \alpha_j (A_j + B_j G_i) w_{il} + \alpha_j B_j g_i = V\gamma_{il}. \quad (17)$$

Equation (17) holds $\forall i \in \mathcal{I}_N$ and $\forall l \in \mathcal{I}_{q_i}$ which means that it will hold for all the columns of the matrix W as defined in (9). Exploiting the PWA mapping of the columns of W as in (9), equation (17) leads to the matrix formulation of the inclusion: $\sum_{j=1}^{L} \alpha_j A_j W + \alpha_j B_j U = V\Gamma$, with an important restriction that each column of Γ is restricted to the simplex $\mathcal{S}_q$, which can be expressed as: $\mathbf{1}^T\Gamma = \mathbf{1}^T$, $\Gamma \in \mathbb{R}_+^{q\times p}$.

These elements prove that $\mathcal{R}$ in (12) represents a *parameterized* set of robustness margin over all the model uncertainties guaranteeing the positive invariance of the closed loop. In order to complete the proof, the set $\mathcal{R}$ is projected on the space of the parameters α in (12). $\square$

3.3 Construction based on the half-space representation

This subsection presents another result related to the robustness margin through the halfspace description of a polytope. The notation of interest are already defined by (6). The main result towards the explicit robustness margin description, is summarized by the next theorem.

Theorem 3.3 *Consider the system* (10) *affected by a parametric uncertainty polytope* (5). *For a given piecewise affine control law* (3)*, the robustness margin is obtained as the projection*

$$\Omega_{rob}^{\alpha} = \text{Proj}_{\mathbb{R}^L}\mathcal{P} \quad (18)$$

where $\mathcal{P}$ represents the polytope:

$$\mathcal{P} = \Big\{(\alpha, \Gamma_1 ... \Gamma_N) \in \mathcal{S}_L \times \mathbb{R}_+^{r\times r_1} \times \ldots \times \mathbb{R}_+^{r\times r_N} \mid \sum_{j=1}^{L} \alpha_j F(A_j + B_j G_i) = \Gamma_i F_i, \quad \Gamma_i h_i \leq h - F\sum_{j=1}^{L} \alpha_j B_j g_i,\ \forall i \in \mathcal{I}_N\Big\}, \quad (19)$$

where $\Gamma_i, i \in \mathcal{I}_N$ represent suitable matrices with the non-negative elements, satisfying the above constraints.

PROOF. It is clear that for every $[A\ B] \in \Omega_{rob}$ and $\forall i \in \mathcal{I}_N$: $(A + BG_i)\mathcal{X}_i \oplus Bg_i \subseteq \mathcal{X}$. Note also that $\forall i \in \mathcal{I}_N, \mathcal{X}_i \subseteq \mathcal{X}$, the above inclusion is equivalent to: $\mathcal{X}_i \subseteq \{x \in \mathcal{X} \mid F[(A + BG_i)x + Bg_i] \leq h\}$. In this form, the inclusion has the advantage of an explicit halfspace representation for both terms:

$$\{x : F_i x \leq h_i\} \subseteq \{x \in \mathcal{X} \mid F[(A + BG_i)x + Bg_i] \leq h\}.$$

Using the Extended Farkas Lemma [14,24], there exists a matrix Γ_i with non-negative elements such that:

$$F(A + BG_i) = \Gamma_i F_i,\ \Gamma_i h_i \leq h - FBg_i,\ \forall i \in \mathcal{I}_N. \quad (20)$$

The proof is complete by observing that all the realizations of $[A\ B] \in \Omega_{rob}$ are spanned by convex combinations of the extreme realizations in the polytopic uncertainty set (5):

$$\begin{cases} \sum_{j=1}^{L} \alpha_j F(A_j + B_j G_i) = \Gamma_i F_i \\ \Gamma_i h_i \leq h - F\sum_{i=1}^{L} \alpha_j B_j g_i \end{cases} \forall i \in \mathcal{I}_N. \quad (21)$$

One can observe that (21) defines a polyhedron in the extended space of the elements of α and of the matrices Γ_i, therefore, the set Ω_{rob}^{α} is obtained by the projection onto the space of α as specified by (18). $\square$

The robustness margin, constructed via the halfspace representation, can be technically computed by the intersection of the sets of $\alpha = [\alpha_1 \ldots \alpha_L]^T \in \mathcal{S}_L$ corresponding to the regions in the polytopic partition $\mathcal{P}_N(\mathcal{X})$, satisfying (21). This technical aspect will be detailed in the computational aspects section.

3.4 Further properties of the robustness margin

The convexity of the set Ω_{rob} is confirmed by the construction (12) which expresses an isomorphic relation with the set Ω_{rob}^{α} in terms of operations of projection and intersection. Both operations are closed over the space of convex sets. The following corollary characterizes in a formal manner the structural properties of the robustness margin.

Corollary 3.4 *The set Ω_{rob} representing the robustness margin of PWA dynamics* (10) *is a polytope.*

PROOF. The sets $\mathcal{S}_L$ and $\mathcal{R}$ used in the construction (12) are polytopes because of their boundedness, as a consequence Ω_{rob}^{α} inherits this structural property. By virtue of the isomorphism, the set Ω_{rob} is also a polytope. $\square$

Theorem 3.2 was stated under Assumptions 1-3 but its formulation can be relaxed if additional properties are considered.

Corollary 3.5 *Under the hypotheses of Theorem 3.2, if in addition Assumption 4 holds, then Ω_{rob}^{α} is obtained as*

$\Omega^{\alpha}_{rob} = \text{Proj}_{\mathbb{R}^L} \mathcal{R}^*$ *with*

$$\mathcal{R}^* = \big\{(\alpha, \Gamma) \in \mathcal{S}_L \times \mathbb{R}_+^{q\times p} \mid \mathbf{1}^T\Gamma \leq \mathbf{1}^T, \sum_{j=1}^{L} \alpha_j (A_j W + B_j U) = V\Gamma\big\}. \quad (22)$$

PROOF. The result is in direct relationship with the one in Theorem 3.2 except the relaxation of the equality $\mathbf{1}^T\Gamma = \mathbf{1}^T$ to inequality. This inequality can be readily transformed into $\mathbf{1}^T\Gamma \leq \beta\mathbf{1}^T$ for some $0 \leq \beta \leq 1$. Subsequently, it ensures that: $(A + BG_i)x + Bg_i \in \beta\mathcal{X}$. By the fact $0 \in \text{int}(\mathcal{X})$, it follows that $\beta\mathcal{X} \subseteq \mathcal{X}$ and the proof is complete. □

Note that this corollary may be of help for further development of robustness margin while guaranteeing the asymptotic stability of the origin. More precisely, the contractiveness condition of $\mathcal{X}$ may be required when appropriate constraints are imposed, whereby $\mathbf{1}^T\Gamma \leq \mathbf{1}^T$ is replaced with $\mathbf{1}^T\Gamma \leq \beta\mathbf{1}^T$, with a scalar $0 \leq \beta < 1$.

The role of Assumption 3 in Theorem 3.2 is marginal and only allows a compact treatment of the vertices in the state space partition. The continuity can be dropped, as shown in the next result. Despite the relative complication of the notation, none of the fundamental properties is lost and the result is equivalent to the one in Theorem 3.3 which makes abstraction of the continuity in the proof.

Corollary 3.6 *Under the hypotheses of Theorem 3.2, if Assumption 3 is dropped, then* Ω^{α}_{rob} *is obtained as* $\Omega^{\alpha}_{rob} = \text{Proj}_{\mathbb{R}^L} \mathcal{R}^c$ *with*

$$\mathcal{R}^c = \big\{(\alpha, \Gamma_1, \dots, \Gamma_N) \in \mathcal{S}_L \times \mathbb{R}_+^{q\times q_1} \times \dots \times \mathbb{R}_+^{q\times q_N} \mid \mathbf{1}^T\Gamma_i \leq \mathbf{1}^T, \textstyle\sum_{j=1}^{L} \alpha_j (A_j W_i + B_j U_i) = V\Gamma_i, \forall i \in \mathcal{I}_N\big\}.$$

PROOF. The argument follows the same line as the one of Theorem 3.2 with the particularity that the image of the vertices via the forward mapping becomes multi-valued due to the presence of common vertices in the set of generators for neighbor regions, but associated with different control values. This has to be considered consequently in the robustness margin description which contains explicitly the inclusion of the image of each region in the set $\mathcal{X}$. □

4 Explicit fragility margin for PWA control laws

The present section aims to provide a measure of the maximal set of admissible variation in the piecewise affine control law coefficients, also denoted as the *fragility margin* such that the closed-loop positive invariance is guaranteed.

4.1 Problem formulation

Given the nominal dynamics (2) and a continuous PWA control law (3) such that the set $\mathcal{X}$ is positively invariant [4,6,7,14,5,27,1], a *fragility margin* problem aims to characterize the set of admissible parametric variations on the local control gains such that the positive invariance property is preserved. Indeed, due to the piecewise affine characteristic of the controller, the fragility margins of control gains for each region are independent and the global study reduces to the superposition (intersection) of the margins for each such region in the partition.

Starting from the description of the closed-loop nominal PWA: $x_{k+1} = (A + BG_i)x_k + Bg_i$, for $x_k \in \mathcal{X}_i$ guaranteeing the positive invariance of $\mathcal{X}$, one can consider a set of parametric errors of the PWA control law gains for each region $\mathcal{X}_i \subseteq \mathcal{X}$, denoted as $\Delta_i \subset \mathbb{R}^{mn+m}$ such that:

$$x_{k+1} = (A + B(G_i + \delta_{G_i,k}))x_k + B(g_i + \delta_{g_i,k}) \subseteq \mathcal{X} \quad (23)$$

with i such that $x_k \in \mathcal{X}_i$ and $\begin{bmatrix}\text{vec}^T(\delta_{G_i,k}) & \delta^T_{g_i,k}\end{bmatrix}^T \in \Delta_i$.

The approach will be similar to the one adopted for the robustness margin. Thus in the preamble, the following theorem can be stated.

Theorem 4.1 *The sets* $\Delta_i, \forall i \in \mathcal{I}_N$ *characterizing the fragility margin for a PWA controller are convex.*

PROOF. See the proof of Theorem 3.1. □

4.2 Construction based on the vertex representation

Using the dual representation, the fragility problem can be treated in the same positive invariance framework. The matrix notation in (7)−(9) will be used next.

Theorem 4.2 *Consider a discrete LTI system* (2) *and a piecewise affine state feedback* (3) *over a polytopic partition* $\mathcal{P}_N(\mathcal{X})$ *of the set* $\mathcal{X}$ *such that Assumptions 1-3 are fulfilled. The fragility margin of the controller defined over* $\mathcal{X}_i$ *is obtained as*

$$\Delta^G_i = \text{Proj}_{(\delta_{G_i}, \delta_{g_i})} \mathcal{F}_i, \quad (24)$$

where $\mathcal{F}_i$ *represents the polyhedron:*

$$\mathcal{F}_i = \Big\{(\delta_{G_i}, \delta_{g_i}, \Gamma_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}_+^{q\times q_i} \mid \mathbf{1}^T\Gamma_i = \mathbf{1}^T, \begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} V_i \\ U_i\end{bmatrix} + B\delta_{G_i}V_i + B\delta_{g_i}\mathbf{1}^T = V\Gamma_i\Big\}. \quad (25)$$

PROOF. By the positive invariance of $\mathcal{X}$, $\forall x \in \mathcal{X}_i$

$$Ax + B((G_i + \delta_{G_i})x + (g_i + \delta_{g_i})) \subseteq \mathcal{X}.$$

By a simple transformation, one can obtain

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ f_{pwa}(x) \end{bmatrix} + B\delta_{G_i} x + B\delta_{g_i} \subseteq \mathcal{X}.$$

From the boundedness and convexity of $\mathcal{X}_i$, it follows that $\forall w_{il} \in \mathcal{W}_i$

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} w_{il} \\ f_{pwa}(w_{il}) \end{bmatrix} + B\delta_{G_i} w_{il} + B\delta_{g_i} = y_{il}. \tag{26}$$

$y_{il} \in \mathcal{X}$ has another description via the generators of $\mathcal{X}$

$$y_{il} = V\gamma_{il} \text{ for } \gamma_{il} \in \mathbb{R}^q_+, \text{ satisfying } \mathbf{1}^T\gamma_{il} = 1. \tag{27}$$

(26), (27) lead directly to the following

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} w_{il} \\ f_{pwa}(w_{il}) \end{bmatrix} + B\delta_{G_i} w_{il} + B\delta_{g_i} = V\gamma_{il}. \tag{28}$$

Equation (28) holds $\forall w_{il} \in \mathcal{W}_i$, thus by completing the matrix $V_i = [\mathcal{W}_i]$ which has its columns as the vertices of $\mathcal{X}_i$, and U_i being its images via the map f_{pwa}, one can easily see that $\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} V_i \\ U_i \end{bmatrix} + B\delta_{G_i} V_i + B\delta_{g_i}\mathbf{1}^T = V\Gamma_i$, where $\mathbf{1}^T\Gamma_i = \mathbf{1}^T$ and $\Gamma_i \in \mathbb{R}^{q\times q_i}_+$, $q_i = \text{Card}(\mathcal{W}_i)$. □

Remark 4.3 The fragility study cannot be extended concomitantly to uncertainties in the state space partition and the associated feedback gains without loosing the linear formulations in (25) and (30). Indeed, to study the impact of the uncertainties in the partition, the matrices F_i and h_i in (6) need to be perturbed and consequently, equations (25), (30) become bilinear in the unknowns. The fragility margin with respect to the state space partition will be studied independently in Section 5.

It can be observed that the sets $\Delta_i^G, \forall i \in \mathcal{I}_N$ in (24), are polyhedra. This property is related to the linearity of the constraints in the set description (see the argument used in Corollary 3.4) and can be officially stated as follows:

Corollary 4.4 *The set Δ_i^G in* (24) *is a polyhedron $\forall i \in \mathcal{I}_N$.*

PROOF. The proof is similar to the one of Corollary 3.4. □

Corollary 4.5 *Under the hypotheses of Theorem 4.2, if Assumption 4 holds, then the fragility margin of the controller associated with the region $\mathcal{X}_i, i \in \mathcal{I}_N$ can be obtained as $\Delta_i^{G*} = \text{Proj}_{(\delta_{G_i},\delta_{g_i})} \mathcal{F}_i^*$ whose definition is below*

$$\mathcal{F}_i^* = \big\{(\delta_{G_i}, \delta_{g_i}, \Gamma_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^{q\times q_i}_+ \mid \mathbf{1}^T\Gamma_i \le \mathbf{1}^T, \\ \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} V_i \\ U_i \end{bmatrix} + B\delta_{G_i} V_i + B\delta_{g_i}\mathbf{1}^T = V\Gamma_i\big\}.$$

PROOF. The proof is similar to the one of Corollary 3.5. □

Remark 4.6 The following observations can clarify the implications of the above results:

- Corollary 4.5 describes a relaxation in the formulation of the set Γ_i. Analyzing exclusively the constraints, it naturally leads to a larger set Δ_i^{G*} as the result of Corollary 4.5 relative to Δ_i^G in Theorem 4.2. Note however that under Assumptions 1–4 the sets are equivalent as long as the relaxation to the inequality extends the inclusion $f_{pwa}(\mathcal{X}) \subseteq \mathcal{X}$ to $f_{pwa}(\mathcal{X}) \subseteq \beta\mathcal{X}$ for some $0 \le \beta \le 1$.
- The fragility margin obtained by the above results can be used in the context of the explicit MPC design under finite precision arithmetic discussed in [26].

4.3 Construction based on the half-space representation

Using the halfspace representation of the polytopes in the partition, the following result can be stated:

Theorem 4.7 *Consider a discrete LTI system* (2) *and a piecewise affine control law* (3) *satisfying Assumptions 1-3. For each region $\mathcal{X}_i$ of the partition $\mathcal{P}_N(\mathcal{X})$ in the controller definition, the fragility margin is defined by the set:*

$$\Delta_i^G = \text{Proj}_{\,(\delta_{G_i},\delta_{g_i})} \mathcal{Q}_i \tag{29}$$

where $\mathcal{Q}_i$ represents the polyhedron:

$$\begin{aligned} \mathcal{Q}_i = \big\{(\delta_{G_i},\delta_{g_i}, H_i) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m \times \mathbb{R}^{r\times r_i}_+ \mid \\ F(A + B(G_i + \delta_{G_i})) = H_i F_i, \\ H_i h_i \le h - FB(g_i + \delta_{g_i})\big\}. \end{aligned} \tag{30}$$

PROOF. For $i \in \mathcal{I}_N$ and $\forall x \in \mathcal{X}_i$

$$(A + B(G_i + \delta_{G_i}))x + B(g_i + \delta_{g_i}) \subseteq \mathcal{X}$$

From the halfspace representation of the polytope $\mathcal{X}$, it follows that $\forall x \in \mathcal{X}_i$

$$F((A + B(G_i + \delta_{G_i}))x + B(g_i + \delta_{g_i})) \le h.$$

In other words, $\mathcal{X}_i \subseteq \mathcal{H}_i$, where they are defined below

$$\begin{aligned}\mathcal{X}_i &= \{x \in \mathbb{R}^n \mid F_i x \leq h_i\}\\ \mathcal{H}_i &= \big\{x \in \mathbb{R}^n \mid F(A + B(G_i + \delta_{G_i}))x \\ &\qquad \leq h - FB(g_i + \delta_{g_i})\big\}.\end{aligned}$$

The Extended Farkas Lemma [14,24] leads directly to the result

$$F(A + B(G_i + \delta_{G_i})) = H_i F_i,\ H_i h_i \leq h - FB(g_i + \delta_{g_i}).$$

This inclusion completes our proof. $\square$

5 Explicit fragility of state space partition

In this section, a so-called *explicit fragility of the state space partition* problem stemming from the implementation of a piecewise affine controller, is tackled. It aims to compute the set of tolerable errors for the description of the regions in the polytopic partition $\mathcal{P}_N(\mathcal{X})$ of the set $\mathcal{X}$ provided the positive invariance property of $\mathcal{X}$ is preserved.

Note that if the halfspace representation is considered, the linearity of imposed constraints will be lost. Instead, we compute this margin via the vertex representation, whereby the errors on the halfspace description are implicitly deduced.

Consider an LTI dynamic (2) and a continuous PWA control law (3), this state feedback controller is defined over a polytopic partition $\mathcal{P}_N(\mathcal{X})$ of the state space $\mathcal{X}$. Consider the vertex representation of $\mathcal{X}_i$ as in (7), the description of $\mathcal{X}_i$ in the presence of coefficient errors can be presented as follows $\widetilde{\mathcal{X}}_i := \operatorname{conv}\{w_{i1} + \delta_{i1}, \ldots, w_{iq_i} + \delta_{iq_i}\}$. A solution to the explicit fragility margin of the components in the polytopic partition $\mathcal{P}_N(\mathcal{X})$ will be provided next in terms of the admissible errors $\delta_{il}, l \in \mathcal{I}_{q_i}$ for each region $\mathcal{X}_i$. The polytope $\mathcal{X}$ is under the following assumption:

Assumption

(5) The boundary of the polytope $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i$ is not subject to uncertainty [2] which is equivalent to $\mathcal{X} = \bigcup_{i \in \mathcal{I}_N} \widetilde{\mathcal{X}}_i$.

This assumption ensures that the positive invariance can be stated and analyzed in terms of an explicit inclusion:

$$(A + BG_i)x + Bg_i \subseteq \mathcal{X}, \quad \forall x \in \widetilde{\mathcal{X}}_i \subseteq \mathcal{X}. \tag{31}$$

with a right hand side represented by a set $\mathcal{X}$ free of uncertainties. The set of admissible errors $\delta_i = \left[\delta_{i1}^T \ldots \delta_{iq_i}^T\right]^T \in \mathbb{R}^{nq_i}$ of the vertices of $\mathcal{X}_i$ can be computed through the following result by exploiting the notation introduced in (8), (9):

[2] In the light of the present result this assumption is related to the vertex representation but the duality makes the assumption hold in the dual representation

Theorem 5.1 *Consider a polytopic partition $\mathcal{P}_N(\mathcal{X})$ of $\mathcal{X}$ over which a PWA controller (3) is defined. The controller, designed with respect to a nominal LTI dynamic (2), satisfies Assumptions 1-3 and 5. The fragility margin of the vertex representation of the polytopic partition $\mathcal{P}_N(\mathcal{X})$ is described for each region $\mathcal{X}_i$ via:*

$$\begin{aligned}\Delta_i^v = \Bigg\{\delta_i \in \mathbb{R}^{nq_i} \mid &\begin{bmatrix}\mathbf{I} \otimes F \\ \mathbf{I} \otimes F(A + BG_i)\end{bmatrix} \delta_i \leq \\ &\begin{bmatrix}\mathbf{1} \otimes h - (\mathbf{I} \otimes F)\mathrm{vec}(V_i) \\ \mathbf{1} \otimes h - (\mathbf{I} \otimes F\,[A\ B])\mathrm{vec}\left(\begin{bmatrix}V_i \\ U_i\end{bmatrix}\right)\end{bmatrix}\Bigg\},\end{aligned} \tag{32}$$

where $\mathbf{1} \in \mathbb{R}^{q_i}$ and $\mathbf{I} \in \mathbb{R}^{q_i \times q_i}$.

PROOF. From Assumption 5 we have that $\forall x \in \mathcal{X}$, there exists $\gamma_i \in \mathcal{S}_{q_i}$ such that $\forall x \in \widetilde{\mathcal{X}}_i \subseteq \mathcal{X}$ and subsequently: $x = \sum_{l=1}^{q_i} \gamma_{il}(w_{il} + \delta_{il})$. Then we can easily see due to the halfspace representation of $\mathcal{X}$ that: $F(w_{il} + \delta_{il}) \leq h, \quad \forall l \in \mathcal{I}_{q_i}$. It follows that:

$$(\mathbf{I} \otimes F)\delta_i \leq \mathbf{1} \otimes h - (\mathbf{I} \otimes F)\mathrm{vec}(V_i). \tag{33}$$

In addition, (31) holds true only if it holds also $\forall w_{il} \in \mathcal{V}(\mathcal{X}_i)$. More clearly,

$$(A + BG_i)(w_{il} + \delta_{il}) + Bg_i \subseteq \mathcal{X}, \forall l \in \mathcal{I}_{q_i}. \tag{34}$$

From the half-space representation of $\mathcal{X}$, (34) is equivalent to:

$$F(A + BG_i)\delta_{il} \leq h - F\,[A\ B]\begin{bmatrix}w_{il} \\ f_{pwa}(w_{il})\end{bmatrix}, \forall l \in \mathcal{I}_{q_i}.$$

The above inclusion leads directly to the following:

$$(\mathbf{I} \otimes F(A + BG_i))\delta_i \leq \mathbf{1} \otimes h - (\mathbf{I} \otimes F\,[A\ B])\mathrm{vec}\left(\begin{bmatrix}V_i \\ U_i\end{bmatrix}\right).$$

Finally, (32) is found by the concomitant satisfaction of (33) and the above inclusion. $\square$

From the above result, the following set:

$$\widehat{\mathcal{X}}_i = \operatorname{conv}\left\{\bigcup_{l \in \mathcal{I}_{q_i}} w_{il} \oplus \mathrm{Proj}_{\delta_{il}} \Delta_i^v\right\}, \tag{35}$$

represents the maximal erroneous halfspace representation of $\mathcal{X}_i$. More clearly, if $\widetilde{\mathcal{X}}_i$ stands for the implemented halfspace representation of $\mathcal{X}_i$, then any implemented $\widetilde{\mathcal{X}}_i \subseteq \widehat{\mathcal{X}}_i$

can guarantee the positive invariance of $\mathcal{X}$ with respect to the given PWA control law.

6 Computational aspects

The theoretical aspects introduced in the previous sections establish the existence and point to the effective constructions of robustness and fragility margins in connection with the closed-loop PWA dynamics. Nevertheless, an analysis of systems of linear equalities/inequalities involved in the parameterized set description may provide an insight on the practical computational aspects.

6.1 Explicit robustness margin of PWA controller

6.1.1 The vertex representation

The construction in (12) and (13) represents a polyhedral set in high dimensional spaces and their treatment in these original formulations will lead to a computational complexity which is difficult to handle [3]. In order to scale this barrier, let us consider (13) element by element for $l \in \mathcal{I}_p$:

$$\Omega_l^\alpha = \Big\{\alpha \in \mathcal{S}_L \mid \mathbf{1}^T\Gamma(\cdot,l) = 1, \Gamma(\cdot,l) \in \mathbb{R}_+^q, \\ \sum_{j=1}^{L}\alpha_j(A_jW(\cdot,l) + B_jf_{pwa}(W(\cdot,l))) = V\Gamma(\cdot,l)\Big\}. \tag{36}$$

Then the robustness margin can also be defined: $\Omega_{rob}^\alpha = \bigcap_{l\in\mathcal{I}_p}\Omega_l^\alpha$. Recall that V is the matrix having the columns composed of the vertices of $\mathcal{X}$. If $\widehat{w}_l = \left[W^T(\cdot,l)\ f_{pwa}^T(W(\cdot,l))\right]^T$, then (36) can be rewritten in the form of a matrix equation where the variable is $\beta_l = [\alpha_1 \ldots \alpha_L\ \ \Gamma^T(\cdot,l)]^T \geq 0$

$$\begin{bmatrix} \begin{bmatrix}A_1 & B_1\end{bmatrix}\widehat{w}_l \ldots \begin{bmatrix}A_L & B_L\end{bmatrix}\widehat{w}_l & -V \\ 0_L^T & 1_q^T \\ 1_L^T & 0_q^T \end{bmatrix}\beta_l = \begin{bmatrix}0_n \\ 1 \\ 1\end{bmatrix} \tag{37}$$

This system of equations in the form $\mathcal{A}\beta_l = \mathcal{B}$, has a family of solutions: $\beta_l = \mathcal{A}_st + \mathcal{B}_s$, where $\mathcal{A}_s$ is an orthonormal basis for the null space of $\mathcal{A}$ (satisfying $\mathcal{A}\mathcal{A}_s = 0$), $\mathcal{B}_s$ denotes a feasible solution of equation (37) and t stands for a vector of appropriate dimension. Due to the non-negativity of all elements in β_l, we obtain the admissible set of t, denoted by Φ_t i.e. $\Phi_t = \{t \mid -\mathcal{A}_st \leq \mathcal{B}_s\}$. It is observed that $\Phi_{\beta_l} := \{\beta_l \mid (37) \text{ holds}\} = \mathcal{A}_s\Phi_t \oplus \mathcal{B}_s$ represents a polytope. Therefore, due to the above relation, Φ_t also represents a polytope. So one needs only to calculate all vertices of Φ_{β_l} by applying the transformation to the vertices of Φ_t. Finally, the set Ω_l^α of coefficients α for which (36) holds is obtained via the orthogonal projection of Φ_{β_l} on the space of α: $\Omega_l^\alpha = \mathrm{Proj}_{\mathbb{R}^L}\Phi_{\beta_l}$.

[3] We refer here to the computational complexity of the orthogonal projection of a polytope [11].

6.1.2 The halfspace representation

From equation (18), it follows that $\Omega_{rob}^\alpha = \bigcap_{i\in\mathcal{I}_N}\mathrm{Proj}_{\mathbb{R}^L}\mathcal{P}_i$, where $\mathcal{P}_i \subset \mathbb{R}_+^L \times \mathbb{R}_+^{r\times r_i}$ are derived from the definition of $\mathcal{P}$ in (19) for each $i \in \mathcal{I}_N$:

$$\mathcal{P}_i = \Big\{(\alpha,\Gamma_i) \big| \sum_{j=1}^{L}\alpha_jF(A_j + B_jG_i) = \Gamma_iF_i, \\ \Gamma_ih_i \leq h - F\sum_{j=1}^{L}\alpha_jB_jg_i\Big\}. \tag{38}$$

To facilitate the computation, one needs to transform the above conditions into a polyhedral form with the meaningful variables for each region. Indeed, the equation in (38) needs to be decoupled row by row $\forall k \in \mathcal{I}_r$:

$$\Gamma_i(k,\cdot)F_i = [\alpha_1 \ldots \alpha_{L-1}]Z_k + F(k,\cdot)(A_L + B_LG_i)$$
$$Z_k = \begin{bmatrix} F(k,\cdot)(A_1 - A_L + B_1G_i - B_LG_i) \\ \ldots \\ F(k,\cdot)(A_{L-1} - A_L + B_{L-1}G_i - B_LG_i)\end{bmatrix}.$$

Denote the following vector: $z = \left[\mathrm{vec}^T(\Gamma_i^T)\, \alpha_1 \ldots \alpha_{L-1}\right]^T$, then the following can be obtained:

$$D_1z = E_1,\ D_1 = \begin{bmatrix} F_i & \ldots & 0_{r_i\times n} \\ \vdots & \ddots & \vdots \\ 0_{r_i\times n} & \ldots & F_i \\ -Z_1 & \ldots & -Z_r\end{bmatrix}^T, \\ E_1 = (\mathbf{I}_r \otimes (A_L + B_LG_i)^T)\mathrm{vec}(F^T). \tag{39}$$

In the same way, an equivalent representation of the inequality in (38) can be presented below:

$$\begin{matrix} D_2z \leq E_2, \\ E_2 = h - FB_Lg_i, \end{matrix}\quad D_2 = \begin{bmatrix} h_i & \ldots & 0_{r_i\times 1} \\ \vdots & \ddots & \vdots \\ 0_{r_i\times 1} & \ldots & h_i \\ Y_1 & \ldots & Y_r\end{bmatrix}^T, \tag{40}$$

with $Y_k = [F(k,\cdot)(B_1 - B_L)g_i \ldots F(k,\cdot)(B_{L-1} - B_L)g_i]^T$ $\forall k \in \mathcal{I}_r$. The solution of (39) is a set of z which depends on t s.t. $z = \mathcal{D}_1t + \mathcal{E}_1$, where $\mathcal{D}_1$ is an orthonormal basis for the null space of D_1 and $\mathcal{E}_1$ is a feasible solution of (39). Due to the non-negativity of z, the values of t satisfy $-\mathcal{D}_1t \leq \mathcal{E}_1$. Also, due to (40), the set of t denoted by Φ_t, can be described by: $\Phi_t = \{t \mid -\mathcal{D}_1t \leq \mathcal{E}_1,\ D_2\mathcal{D}_1t \leq E_2 - D_2\mathcal{E}_1\}$. Recall that the set of z denoted by Φ_z, can be described via Φ_t as: $\Phi_z = \mathcal{D}_1\Phi_t \oplus \mathcal{E}_1$. Then $\mathrm{Proj}_{\mathbb{R}^L}\mathcal{P}_i$ can be computed from $\mathrm{Proj}_{\mathbb{R}^{L-1}}\Phi_z$.

6.2 Explicit fragility margin of PWA controller

For simplicity, without loss of generality, variations in G_i are exclusively considered.

6.2.1 The vertex representation

Consider the fragility margin for the controller of the region $\mathcal{X}_i$. Define also the following set for $l \in \mathcal{I}_{q_i}$

$$\begin{aligned}&\Delta^G_{il} = \big\{\delta_{G_i} \in \mathbb{R}^{m\times n} \mid \mathbf{1}^T\Gamma_i(\cdot,l) = 1, \Gamma_i(\cdot,l) \in \mathbb{R}^q_+\\ &\begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} V_i(\cdot,l)\\ U_i(\cdot,l)\end{bmatrix} + B\delta_{G_i}V_i(\cdot,l) = V\Gamma_i(\cdot,l)\big\}.\end{aligned} \tag{41}$$

The fragility margin can also be defined as follows: $\Delta^G_i = \bigcap_{l\in\mathcal{I}_{q_i}} \Delta^G_{il}$. If we denote $\widehat{w}_{il} = \begin{bmatrix} V_i^T(\cdot,l)\, U_i^T(\cdot,l)\end{bmatrix}^T$, then (41) can be rewritten as a system of linear equations where the variable is $\beta_{il} = \begin{bmatrix}\mathrm{vec}^T(\delta_{G_i})\ \Gamma_i^T(1:q-1,l)\end{bmatrix}^T \in \mathbb{R}^{nm+q-1}$ ($\Gamma_i(q,l) = 1 - \mathbf{1}^T_{q-1}\Gamma_i(1:q-1,l)$):

$$\begin{bmatrix} V_i^T(\cdot,l)(\mathbf{I}_n\otimes B(1,\cdot)) & \\ \vdots & -\widetilde{V}\\ V_i^T(\cdot,l)(\mathbf{I}_n\otimes B(n,\cdot)) & \end{bmatrix}\beta_{il} = v_q - \begin{bmatrix} A & B\end{bmatrix}\widehat{w}_{il}, \tag{42}$$

with $\widetilde{V} = [v_1 - v_q \dots v_{q-1} - v_q]$ (recall that $V = [\mathcal{V}(\mathcal{X})] = [v_1 \dots v_q]$.) Equation (42) in the form $\mathcal{A}\beta_{il} = \mathcal{B}$, has a family of solutions: $\beta_{il} = \mathcal{A}_s t + \mathcal{B}_s$, where $\mathcal{A}_s$ is an orthonormal basis for the null space of $\mathcal{A}$ and $\mathcal{B}_s$ denotes a feasible solution of (42). Due to the non-negativity $\beta_{il}(nm+1 : nm+q-1) = \Gamma_i(1:q-1,l) \geq 0$, the values of t satisfy: $-\mathcal{A}_s^{(2)}t \leq \mathcal{B}_s^{(2)}$ where the matrices $\mathcal{A}_s^{(1)}, \mathcal{B}_s^{(1)}, \mathcal{A}_s^{(2)}, \mathcal{B}_s^{(2)}$ are defined below:

$$\begin{aligned}\begin{bmatrix}\mathcal{A}_s^{(1)} & \mathcal{B}_s^{(1)}\end{bmatrix} &= \begin{bmatrix}\mathcal{A}_s & \mathcal{B}_s\end{bmatrix}(1:nm,\cdot),\\ \begin{bmatrix}\mathcal{A}_s^{(2)} & \mathcal{B}_s^{(2)}\end{bmatrix} &= \begin{bmatrix}\mathcal{A}_s & \mathcal{B}_s\end{bmatrix}(nm+1:nm+q-1,\cdot).\end{aligned}$$

Also, $\Gamma_i(1:q-1,l)$ satisfies the constraint: $\mathbf{1}^T\Gamma_i(1:q-1,l) \leq 1$. Thus, the set of t denoted by Φ_t can be presented as: $\Phi_t = \left\{t \mid -\mathcal{A}_s^{(2)}t \leq \mathcal{B}_s^{(2)}, \mathbf{1}^T\mathcal{A}_s^{(2)}t \leq 1 - \mathbf{1}^T\mathcal{B}_s^{(2)}\right\}$, with the remark that $\Phi_{\beta_{il}} = \{\beta_{il} \mid (42)\text{ holds}\} = \mathcal{A}_s\Phi_t \oplus \mathcal{B}_s$ represents a polyhedral set. Therefore, due to the boundedness of $\Gamma_i(1:q-1,l)$, Φ_t is a polytope, meaning so is $\Delta^G_{il} = \mathcal{A}_s^{(1)}\Phi_t \oplus \mathcal{B}_s^{(1)}$. Repeat the same computation for all $l \in \mathcal{I}_{q_i}$, then the fragility margin for G_i i.e. Δ^G_i can be obtained.

6.2.2 The halfspace representation

From equation (30), it follows that for each $i \in \mathcal{I}_N$ the fragility margin can be described in terms of a set:

$$\begin{aligned}&\mathcal{Q}_i = \big\{(\delta_{G_i}, H_i) \in \mathbb{R}^{m\times n}\times\mathbb{R}^{r\times r_i}_+ \mid\\ &H_i h_i \leq h - FBg_i, F(A + B(G_i + \delta_{G_i})) = H_iF_i\big\}.\end{aligned} \tag{43}$$

In order to facilitate the computation, one has to transform the above conditions into a polytope formulation with a reduced set of meaningful variables for each region. Define z as: $z_1 = \mathrm{vec}(H_i^T), z_2 = \mathrm{vec}(\delta_{G_i}), z = \begin{bmatrix} z_1^T & z_2^T\end{bmatrix}^T$. The equality in (43) allows the iterative elimination (step by step for each row) of dependent variables:

$$H_i(k,\cdot)F_i = F(k,\cdot)B\delta_{G_i} + F(k,\cdot)(A + BG_i).$$

and leads to the following set of relationships:

$$\begin{aligned}&D_1 z = E_1, \quad D_1 = \begin{bmatrix} F_i & \dots & 0_{r_i\times n}\\ \vdots & \ddots & \vdots\\ 0_{r_i\times n} & \dots & F_i\\ Z_1 & \dots & Z_r\end{bmatrix}^T,\\ &E_1 = (\mathbf{I}_r\otimes(A+BG_i)^T)\mathrm{vec}\left(F^T\right),\\ &Z_k = \mathbf{I}_n\otimes(-B^TF^T(k,\cdot)),\ \forall k\in\mathcal{I}_r.\end{aligned} \tag{44}$$

Similarly, the inequality in (43) is equivalent to:

$$\begin{aligned}&D_2 z_1 \leq E_2,\\ &E_2 = h - FBg_i,\end{aligned}\ D_2 = \begin{bmatrix} h_i & \dots & 0_{r_i\times 1}\\ \vdots & \ddots & \vdots\\ 0_{r_i\times 1} & \dots & h_i\end{bmatrix}^T. \tag{45}$$

The family of solutions in (44) has the following form: $z = \mathcal{A}_s\tilde{z} + \mathcal{B}_s$, where $\mathcal{A}_s$ is an orthonormal basis for the null space of D_1, $\mathcal{B}_s$ is a feasible solution of $D_1 z = E_1$. Define the following matrices:

$$\begin{aligned}&\mathcal{A}_s^{(1)} = \mathcal{A}_s(1:rr_i,\cdot), \quad \mathcal{A}_s^{(2)} = \mathcal{A}_s(rr_i+1:rr_i+nm,\cdot)\\ &\mathcal{B}_s^{(1)} = \mathcal{B}_s(1:rr_i), \quad \mathcal{B}_s^{(2)} = \mathcal{B}_s(rr_i+1:rr_i+nm).\end{aligned}$$

Due to the non-negativity of $z_1 = \mathrm{vec}(H_i^T)$ and (45), the set of $\tilde{z}$ denoted by $\Phi_{\tilde{z}}$, can be described as: $\Phi_{\tilde{z}} = \left\{\tilde{z} \mid -\mathcal{A}_s^{(1)}\tilde{z} \leq \mathcal{B}_s^{(1)},\ D_2\mathcal{A}_s^{(1)}\tilde{z} \leq E_2 - D_2\mathcal{B}_s^{(1)}\right\}$. Consequently, Δ^G_i can be obtained as: $\Delta^G_i = \mathcal{A}_s^{(2)}\Phi_{\tilde{z}}\oplus\mathcal{B}_s^{(2)}$.

7 Numerical example

In the present section, several examples allow the previous theoretical results to be illustrated. Note that every simulation in the present article has been carried out in MPT 3.0 (see [15]).

7.1 *Explicit robustness margin of PWA controllers*

An illustration is carried out on a linear system with uncertainty set described by:

$$[A_1\ B_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 1 & 1.5 \end{bmatrix}, [A_2\ B_2] = \begin{bmatrix} 1 & 0 & 1.5 \\ 0.5 & 1.5 & 1 \end{bmatrix},$$
$$[A_3\ B_3] = \begin{bmatrix} 1.5 & 0 & 1 \\ 3.8 & 1 & 1 \end{bmatrix},$$

in the presence of constraints on the control variable and the output variable: $-5 \leq u_k \leq 5, -5 \leq y_k \leq 5$, with the nominal model chosen to synthesize a PWA control law:

$$A = 0.3A_1 + 0.2A_2 + 0.5A_3,$$
$$B = 0.3B_1 + 0.2B_2 + 0.5B_3,\ C = [1\,0]\,.$$

A continuous PWA state feedback control law is designed with prediction horizon 2, weighting matrices $Q = \mathbf{I}_2, R = 1$ and the terminal constraint chosen as the maximal output admissible set [12]. The state space partition is presented in Figure 1.

Figure 2 shows the image of Ω_{rob}^{α} via the orthogonal projection on the plane $[\alpha_1\ \ \alpha_2]$. Note that the shaded violet region presents the whole region of α_1, α_2. The blue point denotes the considered nominal system, this point coincides with a vertex of this robustness margin set. It is observed that this robustness margin differs from the classical notion, because the given control law cannot guarantee the positive invariance of the feasible region $\mathcal{X}$ if the nominal system is perturbed away from the robustness margin.

7.2 *Explicit fragility margin of PWA controllers*

Region 6 has the halfspace representation and its corresponding controller as follows:

$$F_6 = \begin{bmatrix} -1 & 1 & -0.2073 & 0.2073 \\ 0 & 0 & -0.9783 & 0.9783 \end{bmatrix}^T,$$
$$h_6 = \begin{bmatrix} -0.8 & 5 & 23.6177 & -17.9116 \end{bmatrix}^T,$$
$$u(x) = \begin{bmatrix} -1.5625 & 0 \end{bmatrix} x + 6.25.$$

The fragility margin for the control law of region $\mathcal{X}_6$ is illustrated in Figure 3. Note that this margin via two different approaches is theoretically identical. It can be seen that the slope gain G_6 without parametric error of the control law associated with this region is pointed out at point $(0, 0)$ in blue which is a vertex of the fragility margin set. It is easy

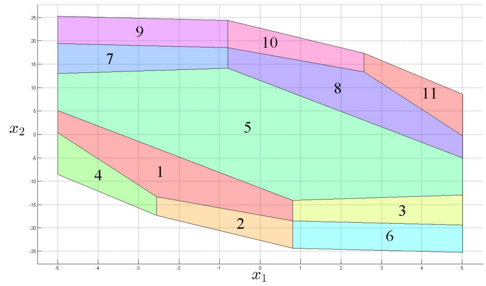

Fig. 1. State space partition.

to see that this control law is fragile due to the fact that if the control law gain G_6 is perturbed away from the fragility set, then closed loop stability may be lost.

7.3 *Explicit fragility of state space partition*

Again, the state space partition and the PWA control law designed above, are considered. The outer blank polytope in Figure 4, represents $\mathcal{X}$. For illustration, we focus on the unconstrained region $\mathcal{X}_5$, which is the orange polytope. The pink polytope represents $\widehat{\mathcal{X}}_5$, defined in (35). It implies that for any implemented representation $\widetilde{\mathcal{X}}_5$ of $\mathcal{X}_5$, satisfying $\widetilde{\mathcal{X}}_5 \subseteq \widehat{\mathcal{X}}_5$, the positive invariance of $\mathcal{X}$ is ensured with respect to the above PWA control law.

8 Conclusions

A measure of the robustness and fragility of the positive invariance for a piecewise affine system has been introduced in the present paper. Two points of view have been presented with respect to the closed-loop dynamics of a linear system with a piecewise affine control law: the robustness with respect to parametric model uncertainties and the fragility of the piecewise affine feedback control function. For both cases it has been shown that the margins are represented by convex sets of admissible parameter variations. Following this idea, the extension to the explicit fragility margin of the state space partition has been also tackled. This problem also leads to polyhedral set descriptions making the analysis very attractive from the computational point of view. The approach allows one to have a generic vision about the margins related to continuous PWA state feedback control laws and also provides new insight in the implementation limitations for this type of controllers.

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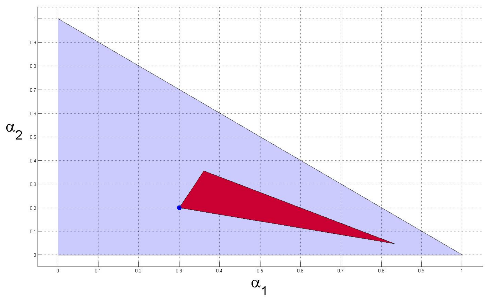

Fig. 2. Robustness margin in the plane of α_1, α_2.

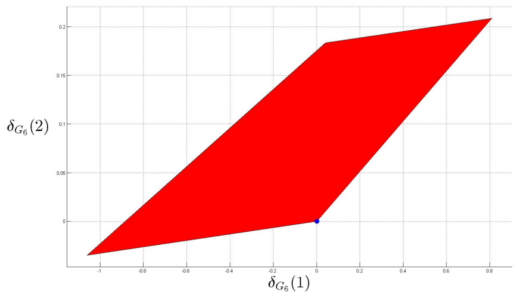

Fig. 3. Fragility margin of the controller in region $\mathcal{X}_6$.

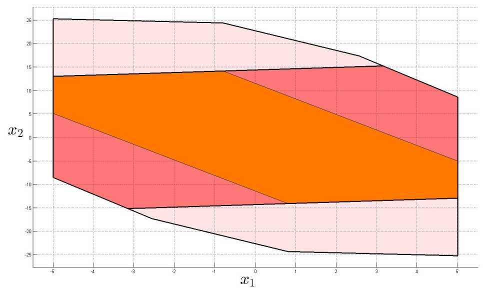

Fig. 4. The shaded pink region is $\widehat{\mathcal{X}}_5$, defined in (35).

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