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Abstract: In this paper, a suboptimal approach to distributed closed-loop min-max MPC for uncertain systems consisting of polytopic subsystems with coupled dynamics subject to both state and input constraints is proposed. The approach applies the dynamic dual decomposition method and reformulates the original centralized min-max MPC problem into a distributed optimization problem. The suggested approach is illustrated on a simulation example of an uncertain system consisting of two interconnected polytopic subsystems.

Key words: Min-max Model Predictive Control, Distributed Control, Linear Interconnected Systems, Polytopic Uncertainty.

Introduction

Model Predictive Control (MPC) is an efficient methodology to solve complex constrained multivariable control problems in the absence, as well as in the presence of uncertainties [1-4]. MPC involves the solution at each sampling instant of a finite horizon optimal control problem subject to the system dynamics, and state and input constraints. The conceptual structure of MPC is given in Fig. 1 [1]. The MPC denomination stems from the idea of employing an explicit model of the plant to be controlled which is used to predict the future output behaviour. This prediction capability allows solving optimal control problems on line, where tracking error, namely the difference between the predicted output and the desired reference, is minimized over a future horizon [1].

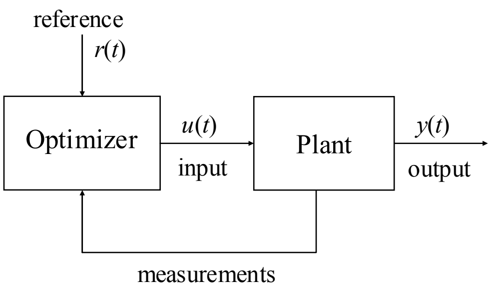

Fig. 1. Basic MPC structure [1].

The result of the optimization is applied according to a receding or moving horizon philosophy [1]: At time t only the first input of the optimal command sequence is actually applied to the plant. The remaining optimal inputs are discarded, and a new optimal control problem is solved at time $t+1$.

The model-based philosophy is the major advantage of the MPC philosophy but brings also the most important difficulties. Since the model is only an approximation of the real plant, it is important for the MPC to be *robust* with respect to model uncertainties and plant disturbances [1]. One possible strategy to obtain this is the design of min-max MPC, as was first proposed in [5]. There are two formulations of min-max MPC: the *open-loop* formulation (the optimization is performed over a sequence of control actions) and the *closed-loop* formulation (the optimization is performed over a sequence of feedback control laws) (see [6] for a review of the min-max MPC approaches). The *open-loop* min-max MPC [6] guarantees the robust stability and the robust feasibility of the system, but it may be very conservative since the control sequence has to ensure constraints fulfilment for all possible uncertainty scenarios without considering the fact that future measurements of the state contain information about past uncertainty values. The conservativeness of the *open-loop* approaches can be overcome by the *closed-loop* min-max MPC approaches [6].

However, solving in a centralized way MPC problems for medium- and large-scale systems may be

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impractical due to the large number of decision variables and the topology of the plant and data communication. Therefore, there is a strong motivation for development of methods for *distributed* solution of MPC problems. The *distributed* MPC has the advantage to reduce the original, large size, optimization problem into a number of smaller and more tractable ones. Recently, several approaches for *distributed /decentralized* MPC have been proposed [7, 8, 9].

There are only a few papers considering the problem of *robust distributed* MPC of interconnected polytopic systems. Thus, in [10] a *distributed* MPC algorithm for polytopic systems subject to actuator saturation is proposed. In [11], an online *distributed* MPC algorithm that deals explicitly with model errors is proposed. It is supposed that the subsystems are coupled through their inputs and only constraints on the inputs are considered. In [12, 13], an approach to *distributed open-loop* min-max MPC for interconnected polytopic systems is developed, which is based on the dual decomposition method [14, 15].

In this paper, an approach to *distributed closed-loop* min-max MPC for uncertain linear systems consisting of *polytopic* subsystems with coupled dynamics subject to both state and input constraints is proposed. The approach applies the dynamic dual decomposition method [14, 15] and reformulates the centralized closed-loop min-max MPC problem into a distributed closed-loop min-max MPC problem. It is based on distributed on-line optimization and can be applied to large-scale polytopic systems. In comparison with [12, 13], here the optimization of the feedback policies is described as an alternative to the optimization of the control actions.

Formulation of the Closed-Loop Min-Max MPC Problem

Consider a system composed by the interconnection of M linear uncertain subsystems (Fig. 2), described by the polytopic discrete-time models:

$$x_i(t+1) = A_i(t)x_i(t) + B_i(t)u_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij}x_j(t)$$
$$[A_i(t),\, B_i(t)] \in \Omega_i \qquad (1)$$
$$i = 1,2,\dots,M$$

where $x_i(t)\in\Re^{n_i}$ and $u_i(t)\in\Re^{m_i}$ are the state and control input vectors, related to the *i*-th subsystem, $A_i(t)\in\Re^{n_i\times n_i}$ and $B_i(t)\in\Re^{n_i\times m_i}$ are uncertain time-varying matrices, and $A_{ij}\in\Re^{n_i\times n_i}$ $j=1,2,\dots,M,\ j\neq i$ are known constant martices.

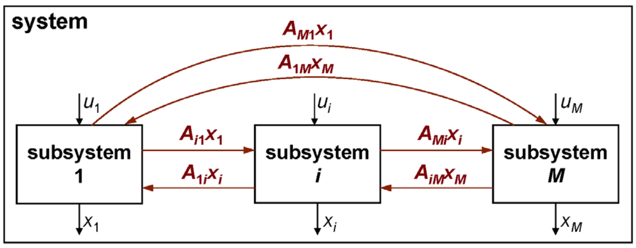

Fig. 2. System composed by the interconnection of M subsystems.

For a polytopic uncertainty description, Ω_i $i=1,2,\dots,M$ are polytopes:

$$\Omega_i = Co\{[A_i^1, B_i^1],[A_i^2, B_i^2],\dots,[A_i^{L_i}, B_i^{L_i}]\},\ i=1,\dots,M \quad (2)$$

where $Co\{\cdot\}$ denotes convex hull and $[A_i^r, B_i^r],\ r=1,2,\dots,L_i$ are its vertices. The following constraints are imposed on the subsystems:

$$u_{\min,i}\le u_i(t)\le u_{\max,i},\ x_{\min,i}\le x_i(t)\le x_{\max,i},\ i=1,\dots,M \quad (3)$$

The following assumption is made:

A1. $x_{\min,i}<0<x_{\max,i}$, $u_{\min,i}<0<u_{\max,i}$, $i=1,\dots,M$.

Let $x(t)$ and $u(t)$ denote the overall state and the overall control input, i.e.:

$$x(t) = [x_1(t), x_2(t),\dots,x_M(t)]\in\Re^n,\ n=\sum_{i=1}^{M} n_i \quad (4)$$

$$u(t) = [u_1(t), u_2(t),\dots,u_M(t)]\in\Re^m,\ m=\sum_{i=1}^{M} m_i \quad (5)$$

Another assumption will be made with respect to the rate of variation of parameters, mainly with respect to the prediction horizon as it will be shown next in the MPC design.

A2. The uncertain pairs $[A_i(t),\, B_i(t)]\in\Omega_i$, $i=1,\dots,M$ have infrequent changes in the sense that $[A_i(t),\, B_i(t)] = \text{const}$, $i=1,\dots,M$ for periods of time, which are not less than $\tilde N$ ($\tilde N\in\mathbb{N}$ is supposed to be sufficiently large).

Before formulating the robust MPC problem, a set $\tilde\Omega_i$ is introduced, which is a finite subset of Ω_i. The set $\tilde\Omega_i$ is defined by:

$$\tilde\Omega_i = \Omega_i^{\text{vert}} \cup \Omega_i^{\text{int}} \quad (6)$$

where $\Omega_i^{\text{vert}} = \{[A_i^r, B_i^r],\ r=1,2,\dots,L_i\}$ is the set of vertices of Ω_i and $\Omega_i^{\text{int}} = \{[A_i^{L_i+j}, B_i^{L_i+j}]\in \text{int}(\Omega_i)$, $j=1,2,\dots,K_i\}$ is a finite set which includes interior points of the set Ω_i.

It is supposed that a full measurement $x=[x_1,x_2,\dots,x_M]$ of the overall state is available at the current time t. The robust regulation problem is considered where the goal is to steer the overall state of the system (1) to the origin. Let N be a finite horizon such that $N<\tilde N$. The Assumption A2 characterizes a framework under which the model has a slow variation with respect to the time constants and prediction horizons. Thus, it can be accepted that $[A_i(t+k),\, B_i(t+k)] = \text{const} = [A_i,\, B_i]$, $k=0,1,\dots,N$, $i=1,\dots,M$. Let $F_i=[f_{i,0}, f_{i,1},\dots,f_{i,N-1}]$ be a sequence of feedback laws for the *i*-th subsystem, where $f_{i,k}:\Re^{n_i}\to\Re^{m_i}$, $k=0,1,\dots,N-1$ are known functions. Let $F=[F_1, F_2(t),\dots,F_M(t)]\in\Re^{Nm}$ contains the feedback laws for the whole system. Then, for the current x, the robust regulation MPC solves the optimization problem:

Problem P1 (Centralized min-max MPC):

$$V^*(x) = \min_F \max_{\substack{[A_i, B_i]\in\tilde{\Omega}_i \\ i=1,2,\dots,M}} J(U,x,[A_1,B_1],\dots,[A_M,B_M]) \quad (7)$$

subject to $x_{t|t} = x$ and:

$$x_{i,t+k|t} \in \mathrm{X}_i \;,\; \forall [A_i, B_i]\in\tilde{\Omega}_i \;,\; i=1,\dots,M \;,\; k=1,\dots,N \quad (8)$$

$$u_{i,t+k} \in \mathrm{U}_i \;,\; i=1,\dots,M \;,\; k=0,1,\dots,N-1 \quad (9)$$

$$u_{i,t+k} = f_{i,k}(x_{i,t+k|t}),\; k=0,1,\dots,N-1 \quad (10)$$

$$\begin{aligned} &x_{i,t+k+1|t} = A_i x_{i,t+k|t} + B_i u_{i,t+k} + \sum_{j=1,j\neq i}^{M} A_{ij} x_{j,t+k|t} \\ &[A_i, B_i]\in\tilde{\Omega}_i \;,\; i=1,\dots,M \;,\; k=0,1,\dots,N-1 \end{aligned} \quad (11)$$

$$x_{t+k|t} = [x_{1,t+k|t}, x_{2,t+k|t}, \dots, x_{M,t+k|t}],\; k=0,1,\dots,N \quad (12)$$

$$f_{t+k} = [f_{1,k}, f_{2,k}, \dots, f_{M,k}],\; k=0,1,\dots,N-1 \quad (13)$$

where $F = [f_t, f_{t+1}, \dots, f_{t+N}]$. The cost function is:

$$J(U,x,[A_1,B_1],\dots,[A_M,B_M]) = \sum_{k=0}^{N} l(x_{t+k|t}, u_{t+k}) \quad (14)$$

where:

$$l(x_{t+k|t}, u_{t+k}) = \sum_{i=1}^{M} l_i(x_{i,t+k|t}, u_{i,t+k}) \quad (15)$$

Here, $l_i(x_{i,t+k|t}, u_{i,t+k}) = x^T_{i,t+k|t} Q_i x_{i,t+k|t} + u^T_{i,t+k} R_i u_{i,t+k|t}$ is the stage cost for the *i*-th subsystem and $Q_i, R_i \succ 0$ are weighting matrices. The sets X_i and U_i are defined by:

$$\mathrm{X}_i = \{x_i \in \Re^{n_i} \mid x_{\min,i} \le x_i \le x_{\max,i}\} \quad (16)$$

$$\mathrm{U}_i = \{u_i \in \Re^{m_i} \mid u_{\min,i} \le u_i \le u_{\max,i}\} \quad (17)$$

It follows from (16)–(17) that X_i and U_i are convex (polyhedral) sets, which include the origin in their interior (due to Assumption A1). It should be noted that the state constraints (8) guarantee the robust feasibility of the solution in sense that the state constraints in (3) will be satisfied for the worst-case uncertainty realizations in $\tilde{\Omega}_i$, $i=1,\dots,M$.

Let $F^* = [f_0^*, f_1^*, \dots, f_{N-1}^*]$ denote the optimal solution of problem P1. According to the receding horizon strategy, the control action applied to the plant at time t is $u(x_{t|t}) = f_0^*(x_{t|t})$. The following assumption is also made:

A3. Each control law $f_{i,k}(x_i)$, $i=1,\dots,M$, $k=0,1,\dots,N-1$ is linear, i.e. it has the form:

$$f_{i,k}(x_{i,t+k|t}) = G_{i,t+k} x_{i,t+k|t} + g_{i,t+k} \quad (18)$$

where the matrix $G_{i,t+k} \in \Re^{m_i \times n_i}$ and the vector $g_{i,t+k} \in \Re^{m_i}$ are to be optimized. In general, the control law parameterization of the form (18) will result in a suboptimal solution of the min-max optimization problem (problem P1). Let $G = [G_1, G_2, \dots, G_M]$ and $g = [g_1, g_2, \dots, g_M]$, where:

$$\begin{aligned} G_i &= [G_{i,t}, G_{i,t+1}, \dots, G_{i,t+N}] \\ g_i &= [g_{i,t}, g_{i,t+1}, \dots, g_{i,t+N}] \\ i &= 1,2,\dots,M \end{aligned} \quad (19)$$

By taking into account (18)-(19), the equation (7) obtains the form:

$$V^*(x) = \min_{G,g} \max_{\substack{[A_i, B_i]\in\tilde{\Omega}_i \\ i=1,2,\dots,M}} J(U,x,[A_1,B_1],\dots,[A_M,B_M]) \quad (20)$$

It should be noted that the cost function (14) is in general non-convex with respect to the uncertain matrices A_i, B_i, $i=1,\dots,M$. Therefore, considering only the vertices of the sets Ω_i, $i=1,\dots,M$ when computing the worst-case cost would not have been sufficient. For this reason, the finite uncertainty sets $\tilde{\Omega}_i$, $i=1,\dots,M$ defined by (6) include some interior elements in addition to the vertices.

Distributed Closed-Loop Min-Max MPC of Linear Systems with Uncertainty

Problem P1 can be decomposed by using the dynamic dual decomposition approach [4, 5]. The following *decoupled* state equations can be formulated:

$$\begin{aligned} &x_i(t+1) = A_i(t) x_i(t) + B_i(t) u_i(t) + v_i(t) \\ &[A_i(t), B_i(t)] \in \Omega_i \\ &i=1,2,\dots,M \end{aligned} \quad (21)$$

with the additional constraints that [5]:

$$v_i(t) = \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij} x_j(t),\; i=1,\dots,M \text{ for all } t \quad (22)$$

Similar to [5], the constraints (22) are relaxed by introducing the corresponding Lagrange multipliers $p_i \in \Re^{n_i}$ (also referred to as prices) in the cost function (14). Further, define the following sequences:

$$\begin{aligned} &P = [p_t, p_{t+1}, \dots, p_{t+N}] \\ &p_{t+k} = [p_{1,t+k}, p_{2,t+k}, \dots, p_{M,t+k}],\; k=0,1,\dots,N \\ &V = [v_t, v_{t+1}, \dots, v_{t+N}], V_i = [v_{i,t}, v_{i,t+1}, \dots, v_{i,t+N}] \\ &v_{t+k} = [v_{1,t+k}, v_{2,t+k}, \dots, v_{M,t+k}],\; k=0,1,\dots,N \end{aligned} \quad (23)$$

Then, the original problem P1 is reformulated as a dual problem in view of *distributed* robust MPC:

Problem P2 (Distributed min-max MPC):

$$\begin{aligned} &\max_P \min_{G,g,V} \max_{\substack{[A_i, B_i]\in\tilde{\Omega}_i \\ i=1,2,\dots,M}} \\ &\sum_{k=0}^{N}\sum_{i=1}^{M} [l_i(x_{i,t+k|t}, G_{i,t+k}, g_{i,t+k}) + p^T_{i,t+k}(v_{i,t+k} - \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij} x_{j,t+k|t})] = \\ &\max_P \sum_{i=1}^{M} (\min_{G_i,g_i,V_i} \max_{[A_i, B_i]\in\tilde{\Omega}_i} \\ &\underbrace{\sum_{k=0}^{N} [l_i(x_{i,t+k|t}, G_{i,t+k}, g_{i,t+k}) + p^T_{i,t+k} v_{i,t+k} - x^T_{i,t+k|t} \sum_{\substack{j=1 \\ j\neq i}}^{M} A^T_{ji} p_{j,t+k}])}_{l_i^P(G_{i,t+k}, g_{i,t+k}, v_{i,t+k}, P, A_i, B_i)} \end{aligned} \quad (24)$$

subject to $x_{t|t} = x$, constraints (8)–(9) and:

$$\begin{aligned} &u_{i,t+k} = G_{i,t+k} x_{i,t+k|t} + g_{i,t+k} \\ &i=1,\dots,M \;,\; k=0,1,\dots,N-1 \end{aligned} \quad (25)$$

$$\begin{aligned} &x_{i,t+k+1|t} = A_i x_{i,t+k|t} + B_i u_{i,t+k} + v_{i,t+k} \;,\; [A_i, B_i]\in\tilde{\Omega}_i \\ &i=1,\dots,M \;,\; k=0,1,\dots,N-1 \end{aligned} \quad (26)$$

$$p_{t+N} = 0 \quad (27)$$

The inner *decoupled* optimization problems in problem P2 represent Quadratic Programming (QP) sub-problems. Each QP sub-problem is presented as follows:

Problem P3^i (*i*-th QP subproblem):

$$V_i^*(P, x_i) = \min_{G_i, g_i, V_i} \max_{[A_i, B_i] \in \tilde{\Omega}_i} \sum_{k=0}^{N} l_i^P(G_{i,t+k}, g_{i,t+k}, v_{i,t+k}, P, A_i, B_i) \quad (28)$$

subject to $x_{i,t|t} = x_i$ and:

$$x_{i,t+k|t} \in \mathrm{X}_i, \; \forall [A_i, B_i] \in \tilde{\Omega}_i, \; k = 1, ..., N \quad (29)$$

$$u_{i,t+k} \in \mathrm{U}_i, \; k = 0, 1, ..., N-1 \quad (30)$$

$$u_{i,t+k} = G_{i,t+k} x_{i,t+k|t} + g_{i,t+k}, \; k = 0, 1, \ldots, N-1 \quad (31)$$

$$x_{i,t+k+1|t} = A_i x_{i,t+k|t} + B_i u_{i,t+k} + v_{i,t+k}, \; [A_i, B_i] \in \tilde{\Omega}_i \quad k = 0, 1, ..., N-1 \quad (32)$$

Let $G_i^* = [G_{i,t}^*, G_{i,t+1}^*, ..., G_{i,t+N}^*]$, $g_i^* = [g_{i,t}^*, g_{i,t+1}^*, ..., g_{i,t+N}^*]$ and $V_i^* = [v_{i,t}^*, v_{i,t+1}^*, ..., v_{i,t+N}^*]$ be the optimal solution of the subproblem P3^i, and $X_i^* = [x_{i,t|t}^*, x_{i,t+1|t}^*, ..., x_{i,t+N|t}^*]$ denote the worst-case state trajectory corresponding to the optimal solution G_i^*, g_i^*, V_i^*, i.e.:

$$x_{i,t+k+1|t}^* = A_i^* x_{i,t+k|t}^* + B_i^*(G_{i,t+k} x_{i,t+k|t}^* + g_{i,t+k}) + v_{i,t+k}^* \quad k = 0, 1, ..., N-1 \quad (33)$$

where:

$$[A_i^*, B_i^*] = \arg \max_{[A_i, B_i] \in \tilde{\Omega}_i} \sum_{k=0}^{N} l_i^P(G_{i,t+k}^*, g_{i,t+k}^*, v_{i,t+k}^*, P, A_i, B_i) \quad (34)$$

It can be easily shown that under Assumption A3, the optimization problem P1 (where the equation (7) is represented equivalently as equation (20)) is convex with respect to the optimization variables G and g. Then it can be proved that if $x = [x_1, x_2, \ldots, x_M]$ is a feasible initial state of the system (1) (i.e. state for which there exists a sequence $F = [F_1, F_2(t), \ldots, F_M(t)]$ of feedback laws such that all constraints in the problem P1 are satisfied), then it holds:

$$V^*(x) = \max_P \sum_{i=1}^{M} V_i^*(P, x_i), \text{ with } p_{t+N} = 0 \quad (35)$$

It means that the solution of the centralized min-max MPC problem P1 is equivalent to the solution of the distributed dual min-max MPC problem P2. The proof of (35) follows similar arguments as in [12].

From (35) it follows that the computation of G_i^*, g_i^*, V_i^* for given prices P can be done in a decentralized way, but finding the optimal prices requires coordination. The prices P are found by applying the accelerated proximal gradient method to solve the dual problem to a convex primal optimization problem (see [16] and the references therein). Given a price prediction sequence $P^r = [p_t^r, \ldots, p_{t+N}^r]$ for the r-th iteration, the corresponding sequences $G_i^{*r} = [G_{i,t}^{*r}, ..., G_{i,t+N}^{*r}]$, $g_i^{*r} = [g_{i,t}^{*r}, ..., g_{i,t+N}^{*r}]$, $V_i^{*r} = [v_{i,t}^{*r}, ..., v_{i,t+N}^{*r}]$ are computed locally by solving problem P3^i and (33)–(34). Then, the prices can be updated distributedly with the following iteration for the *i*-th subsystem:

$$q_{i,t+k}^r = p_{i,t+k}^r + \frac{r-1}{r+2}(p_{i,t+k}^r - p_{i,t+k}^{r-1}), k = 0, 1, ..., N-1$$

$$p_{i,t+k}^{r+1} = q_{i,t+k}^r + \frac{1}{L} \nabla_{P_i} S(P, x)|_{P=Q^r}, k = 0, 1, ..., N-1 \quad (36)$$

with $p_{i,t+N}^{r+1} = p_{i,t+N}^r = 0$

where $Q^r = [Q_1^r, Q_2^r, \ldots, Q_M^r]$, $Q_i^r = [q_{i,t}^r, \ldots, q_{i,t+N}^r]$, $i = 1, 2, \ldots, M$ (with $q_{i,t+N}^r = p_{i,t+N}^r = 0$), and $S(P, x)$ is the dual function (cf. (24)):

$$S(P, x) = \min_{G, g, V} \max_{\substack{[A_i, B_i] \in \tilde{\Omega}_i \\ i=1,2,\ldots,M}} \sum_{k=0}^{N} \sum_{i=1}^{M} [l_i(x_{i,t+k|t}, G_{i,t+k}, g_{i,t+k}) + p_{i,t+k}^T(v_{i,t+k} - \sum_{\substack{j=1 \\ j \neq i}}^{M} A_{ij} x_{j,t+k|t})] \quad (37)$$

The gradient of the dual function $S(P, x)$ with respect to the prices P_i at $P = Q^r$ is:

$$\nabla_{P_i} S(P, x)|_{P=Q^r} = v_{i,t+k}^* - \sum_{\substack{j=1 \\ j \neq i}}^{M} A_{ij} x_{j,t+k|t}^* \quad k = 0, 1, ..., N-1 \quad (38)$$

From (38), it can be seen that in order to compute the gradient $\nabla_{P_i} S(P, x)|_{P=Q^r}$ in (36) it is necessary to have the worst-case state trajectories $X_j^* = [x_{j,t|t}^*, x_{j,t+1|t}^*, ..., x_{j,t+N|t}^*]$, $j = 1, \ldots, M$, $j \neq i$ of the interconnected subsystems, which on their hand depend on the values P_i of prices.

In (36), L is the Lipschitz constant to the gradient function $\nabla_P S(P, x)$. In [12], an off-line algorithm to obtain an estimate of L is provided.

The following suboptimal algorithm to distributed closed-loop min-max MPC of uncertain polytopic systems is proposed.

Algorithm 1 (Distributed closed-loop min-max MPC by on-line optimization):

1. Obtain an estimate $\hat{L}$ of the Lipschitz constant to the gradient $\nabla_P S(P, x)$, number R of iterations, and fix arbitrary guesses P_i^0, $i = 1, 2, \ldots, M$ for the price sequences. Let $t = 0$.

2. Let the state at time t be $x(t) = x = [x_1, \ldots, x_M]$.

3. for $r = 0, 1, \ldots, R-1$ **do**

4. For *i*-th subsystem, $i = 1, 2, \ldots, M$, communicate the price sequences $P_j^r = [p_{j,t}^r, \ldots, p_{j,t+N}^r]$, $j = 1, \ldots, M$, $j \neq i$ of the interconnected subsystems.

5. Compute the sequences $G_i^{*r} = [G_{i,t}^{*r}, \ldots, G_{i,t+N}^{*r}]$, $g_i^{*r} = [g_{i,t}^{*r}, \ldots, g_{i,t+N}^{*r}]$ and $V_i^{*r} = [v_{i,t}^{*r}, \ldots, v_{i,t+N}^{*r}]$, corresponding to the price sequence $P^r = [p_t^r, \ldots, p_{t+N}^r]$ by solving *distributedly* the QP sub-problems $P3^i$, $i = 1,2, \dots, M$. Compute the worst-case state trajectories $X_i^{*r} = [x_{i,t|t}^{*r}, \dots, x_{i,t+N|t}^{*r}]$, $i = 1,2, \dots, M$ from (33)–(34).

6. For i-th subsystem, $i = 1,2, \dots, M$, communicate the worst-case state trajectories $X_j^{*r} = [x_{j,t|t}^{*r}, \dots, x_{j,t+N|t}^{*r}]$, $j = 1, \dots, M$, $j \neq i$ of the interconnected subsystems.

7. Compute *distributedly* the updates $P_i^{r+1} = [p_{i,t}^{r+1}, \dots, p_{i,t+N}^{r+1}]$, $i = 1,2, \dots, M$ of the price sequences by applying (36) with $L = \hat{L}$ and using (38).

8. end

9. Let $P_i^0 = P_i^R$, $i = 1,2, \dots, M$.

10. Apply to the overall system the control inputs $u_i(t) = u_{i,t}^{*R-1}$, $i = 1,2, \dots, M$.

11. Let $t = t+1$ and go to step 2.

It is clear that the computational load required by the centralized solution of the robust MPC problem will increase rapidly both with the increase of the number of the subsystems constituting the overall system and with their dimensions. In contrast, with the distributed robust MPC approach the original, large size, optimization problem is decomposed into a number of smaller and more tractable ones, which can be solved in parallel. Therefore, the computational complexity depends only on the number of the variables and the number of the constraints associated to each subsystem. Another advantage of the distributed approach is that it allows the performance optimization of the subsystems to be done autonomously and it is not required to have a central optimization unit. In addition, the distributed MPC is characterized with reduced communicational requirements, because only the interconnected subsystems need to communicate in order to optimize the global performance.

With the proposed robust MPC approach the purpose is to achieve robust feasibility (satisfaction of the constraints for all possible uncertainties within the uncertainty set) and robust performance (optimizing the worst-case value of the performance index). It is known that the robust control system may have optimal performance only for some values of the uncertain parameters, i.e. the optimality is "sacrificed" in some extent in order to achieve robustness. It has to be kept in mind that if a nominal MPC controller is designed by ignoring the presence of uncertainty, this will lead to violation of constraints and possible poor performance for values of the uncertain parameters, which differ from the nominal ones. Therefore, it is necessary to guarantee robustness of the control system when there is uncertainty about plant dynamics. By using the closed-loop min-max MPC approach to obtain robustness, a less conservative solution is obtained in comparison to the open-loop min-max approach.

Simulation Example

Consider the following system composed of two interconnected polytopic subsystems S_1 and S_2:

$$S_1: \; x_1(t+1) = A_1(t)x_1(t) + B_1u_1(t) + A_{12}x_2(t), \quad A_1(t) \in \Omega_1 \tag{39}$$

$$S_2: \; x_2(t+1) = A_2(t)x_2(t) + B_2u_2(t) + A_{21}x_1(t), \quad A_2(t) \in \Omega_2 \tag{40}$$

where:

$$A_1(t) = \begin{bmatrix} \alpha_1(t) & -0.09 \\ 0.17 & 0.79 \end{bmatrix}, \; \alpha_1(t) \in [0.43, \; 0.83]$$
$$A_2(t) = \begin{bmatrix} \alpha_2(t) & -0.09 \\ 0.17 & 0.69 \end{bmatrix}, \; \alpha_2(t) \in [0.53, \; 0.93] \tag{41}$$
$$B_1 = \begin{bmatrix} 0.06 \\ 0.01 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.07 \\ 0.01 \end{bmatrix}; \; A_{12} = A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix}$$

Here, α_1 and α_2 are uncertain parameters. The sets Ω_1 and Ω_2 have two vertices corresponding to $\alpha_1 = 0.43$, $\alpha_1 = 0.83$ and $\alpha_2 = 0.53$, $\alpha_2 = 0.93$, respectively. The finite sets $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are defined as:

$$\tilde{\Omega}_1 = \{[A_1(\alpha_1), B_1], \; \alpha_1 \in \{0.43, 0.53, 0.63, 0.73, 0.83\}\}$$
$$\tilde{\Omega}_2 = \{[A_2(\alpha_2), B_2], \; \alpha_2 \in \{0.53, 0.63, 0.73, 0.83, 0.93\}\} \tag{42}$$

The following state and input constraints are imposed on the system (39)-(40):

$$\begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix} \leq x_i(t), \; -2 \leq u_i(t) \leq 2, \; i = 1,2 \tag{43}$$

The prediction horizon is $N = 5$ and the weighting matrices are $Q_i = I$, $R_i = 0.01$, $i = 1,2$. The centralized closed-loop min-max MPC problem (problem P1) is represented as a distributed problem (problem P2) by applying the dual decomposition approach. Then, Algorithm 1 with number of iterations $R = 10$ is used to generate the two control inputs for an initial state of the overall system $x(0) = [2 \;\; 2 \;\; 2 \;\; 2]$. The simulations are performed for the variations of the uncertain parameters, shown in Fig. 3.

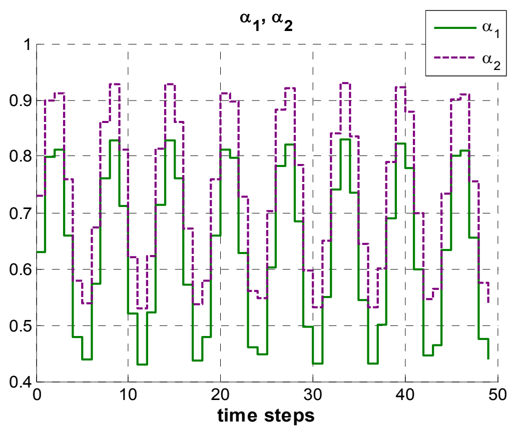

Fig. 3. The variation of parameters α_1, α_2.

The computed trajectories of the control inputs u_1, u_2 and the states x_1^1, x_2^1 and x_1^2, x_2^2, associated to the subsystems S_1 (i.e. $x_1 = [x_1^1, x_2^1]$) and S_2 (i.e. $x_2 = [x_1^2, x_2^2]$) are depicted in Fig. 4 to Fig. 6. The results show that the suboptimal trajectories obtained with the distributed min-max MPC keep both the state and input constraints.

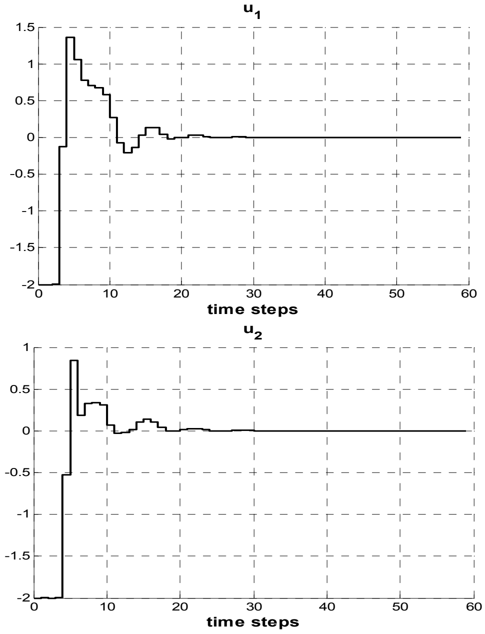

Fig. 4. The control inputs u_1, u_2.

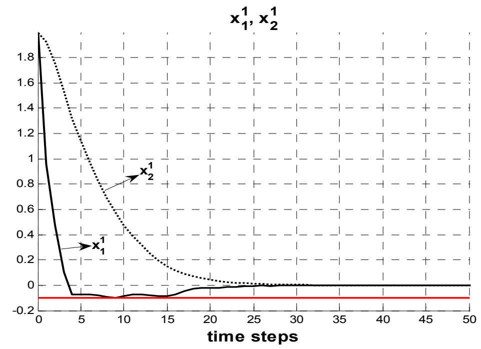

Fig. 5. The states x_2^1, x_2^2 of subsystem S_1.

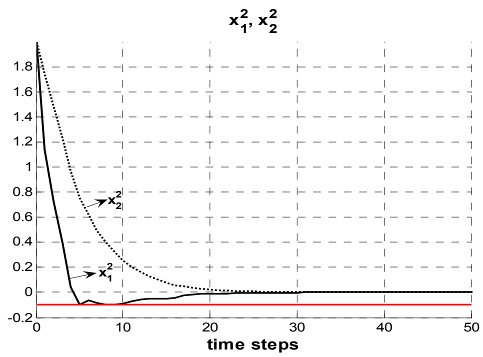

Fig. 6. The states x_2^1, x_2^2 of subsystem S_2.

Conclusions

In this paper, a suboptimal approach to distributed closed-loop min-max MPC for uncertain systems consisting of polytopic constrained subsystems is proposed and its performance is illustrated with a numerical example. The results show that the suboptimal trajectories obtained with the proposed approach keep both the state and input constraints.

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