Is a point-wise dissipation rate enough to show ISS for time-delay systems?

Antoine Chaillet, Pierdomenico Pepe, Paolo Mason, Yacine Chitour

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Abstract

In this paper, we address the question whether input-to-state stability (ISS) of nonlinear time-delay systems is guaranteed when a Lyapunov-Krasovskii functional (LKF) satisfies a dissipation inequality in which the dissipation rate involves solely the present value of the state. We do not yet confirm or infirm this conjecture, but rather identify growth restrictions, on the LKF or on the vector field ruling the dynamics, under which it holds true. An example taken from the neuroscience literature illustrates our findings. We also list a series of robustness properties that naturally hold under this point-wise dissipation inequality, and indicate possible research directions to confirm our conjecture.

1 Introduction

Input-to-state stability (ISS), introduced in Sontag (1989), is a powerful property for the analysis and control of nonlinear dynamical systems. Beyond global asymptotic stability in the absence of input (0-GAS), ISS induces several interesting robustness properties with respect to exogenous disturbances. In particular, solutions of ISS systems are bounded provided that the disturbance magnitude is bounded. Moreover, they converge to the origin in response to any vanishing input. More generally, they asymptotically converge to a neighborhood of the origin whose size is "proportional" to the magnitude of the applied input. See Sontag (2008) for a survey on the ISS property.

A crucial feature of ISS for finite-dimensional systems is its characterization in terms of Lyapunov-like dissipation inequalities (Sontag and Wang, 1995). This necessary and sufficient condition is that the derivative of a proper Lyapunov function candidate along the solutions of the considered system is upper bounded by a $\mathcal{K}_\infty$ dissipation rate of the state norm plus a $\mathcal{K}_\infty$ function of the input norm. Since any proper Lyapunov function candidate can be upper and

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lower bounded by $\mathcal{K}_\infty$ functions of the point-wise value of state norm, the existence of a $\mathcal{K}_\infty$ dissipation rate expressed in terms of the state norm is equivalent to that of a $\mathcal{K}_\infty$ dissipation rate expressed in terms of the Lyapunov function itself. This observation is instrumental in several proof techniques of ISS results for finite-dimensional systems, as it allows to rely on useful comparison lemmas.

Since time delays are widespread in control applications, several works have contributed to extending ISS and its subsequent analysis tools to nonlinear time-delay systems. This started with Teel (1998), in which a sufficient condition for ISS of time-delay systems was provided using Razumikhin techniques, which were in turn shown to boil down to small-gain results. In Pepe and Jiang (2006), ISS was later established using Lyapunov-Krasovskii functionals. These Razumikhin and Krasovskii sufficient conditions were shown in Karafyllis et al. (2008) to be also necessary for ISS. More precisely, it was shown there that ISS for time-delay systems is equivalent to the existence of a proper Lyapunov-Krasovskii functional V whose derivative along the solutions satisfies

$$\dot{V} \leq -\alpha(V) + \gamma(|u(t)|), \tag{1}$$

where u is the input, and α and γ denote class $\mathcal{K}_\infty$ functions. This dissipation inequality is identical to that obtained for finite-dimensional systems. However, unlike in the finite-dimensional case, it is not equivalent to requiring a dissipation rate involving the instantaneous value of the state (rather than V itself). This is due to the fact that, in general, Lyapunov-Krasovskii functionals cannot be upper bounded by the point-wise value of the state norm $|x(t)|$) in a time-delay context[2].

Although ISS is equivalent to requiring a dissipation inequality like (1), ensuring a dissipation rate involving the whole functional V is sometimes a difficult task in practice. For quadratic Lyapunov-Krasovskii functionals, this difficulty is typically tackled by adding an increasing linear or exponential term in the integral part of the functional; see e.g. Pepe and Jiang (2006); Ito et al. (2010); Mazenc et al. (2013). Some relaxed conditions were also proposed in Ito et al. (2010), where it is required that the dissipation rate should be expressed in terms of a functional M_a of the state prehistory, provided that V is upper- and lower-bounded by $\mathcal{K}_\infty$ functions of this same functional M_a.

Nonetheless, it has never been shown yet (nor infirmed) that ISS would also hold under the less restrictive dissipation inequality:

$$\dot{V} \leq -\alpha(|x(t)|) + \gamma(|u(t)|), \tag{2}$$

meaning with a *point-wise* dissipation rate, involving only the current value of the state norm. The aim of this paper is to provide preliminary results and open questions in this direction. We see two main motivations to study whether a dissipation inequality like (2) guarantees ISS for time-delay systems. The first one is of a technical nature: (2) is usually much easier to establish than (1) and does not require the addition of rather unnatural linear or exponential terms in

[2]They are rather bounded by a norm of the state prehistory $\|x_t\|$.

the Lyapunov-Krasovskii functional. The second one lies in the coherence with the original Lyapunov-Krasovskii theorem for the global asymptotic stability (GAS) of time-delay systems without disturbances (Hale and Lunel, 1993): this result does not require a dissipation inequality like $\dot{V} \leq -\alpha(V)$, but merely a point-wise dissipation rate $\dot{V} \leq -\alpha(|x(t)|)$, thus more coherent with (2).

We were not able yet to show that a dissipation inequality like (2) is enough to establish ISS. In Section 2, we start by recalling the necessary definitions in order to formally state this conjecture. In Section 3, we list a series of robustness properties that easily follow from the existence of a point-wise dissipation rate and relate them to existing concepts for finite-dimensional systems. In Section 4, we propose two types of growth restrictions under which a point-wise dissipation rate is indeed equivalent to ISS. The first one limits the growth of the integral part of the Lyapunov-Krasovskii functional in terms of the point-wise dissipation rate: under this condition, not only ISS is ensured, but we also provide an explicit construction of a Lyapunov-Krasovskii functional satisfying (1). This result proves especially useful for quadratic Lyapunov-Krasovskii functionals. The second growth restriction is on the vector field itself as compared to the point-wise dissipation rate. An example, inspired by neuronal population models, is provided in Section 5 to illustrate the results and their limitations. A detailed list of future research directions to establish our conjecture is provided in Section 7. The main proofs are provided in Section 6.

Notations. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a set $I \subset \mathbb{R}$ and a measurable signal $u : I \to \mathbb{R}^m$, $m \in \mathbb{N}_{\geq 1}$, $\|u\| := \operatorname{ess\,sup}_{t \in I} |u(t)|$. $\mathcal{U}$ denotes the set of all signals $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ that are measurable and locally essentially bounded. Given $u \in \mathcal{U}$ and $b > a > 0$, $u_{[a;b]} : [a;b] \to \mathbb{R}^m$ denotes the function defined as $u_{[a;b]}(t) = u(t)$ for all $t \in [a;b]$. Given $\theta > 0$, the set $C([-\theta;0], \mathbb{R}^n)$ of all continuous functions $\phi : [-\theta;0] \to \mathbb{R}^n$ is denoted by $\mathcal{X}$.

2 Problem statement and definitions

We consider time-delay systems of the form:

$$\dot{x}(t) = f(x_t, u(t)), \quad \forall a.e.\ t \geq 0, \tag{3}$$

where $u \in \mathcal{U}$ and, for each $t \geq 0$, $x_t \in \mathcal{X}$ denotes the prehistory function over $[t-\theta;t]$, $\theta > 0$:

$$x_t : \left\{ \begin{array}{ccc} [-\theta;0] & \to & \mathbb{R}^n \\ s & \mapsto & x(t+s). \end{array} \right.$$

Throughout the paper, the map $f : \mathcal{X} \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be Lipschitz on any bounded subset of $\mathcal{X} \times \mathbb{R}^m$, which ensures that, given any $x_0 \in \mathcal{X}$ and any $u \in \mathcal{U}$, (3) admits a unique and locally absolutely continuous solution on a maximal time interval $[0,b)$, $b \in (0;+\infty]$. Moreover, if $b < +\infty$, then the solution is unbounded on $[0,b)$. See Hale and Lunel (1993).

We recall the definition of ISS for time-delay systems.

Definition 1 (ISS) *The system* (3) *is input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\delta \in \mathcal{K}_\infty$ such that, given any $x_0 \in \mathcal{X}$ and any $u \in \mathcal{U}$, the corresponding solution exists for all $t \geq 0$ and satisfies*

$$|x(t)| \leq \beta(\|x_0\|, t) + \delta(\|u_{[0;t]}\|), \quad \forall t \geq 0.$$

This definition is the natural extension to time-delay systems of the ISS property originally introduced for finite-dimensional systems in Sontag (1989). It was the subject of an already wide literature on time-delay systems: see for instance Teel (1998); Pepe and Jiang (2006); Karafyllis and Jiang (2007); Mazenc et al. (2008); Karafyllis et al. (2008); Karafyllis and Jiang (2011); Dashkovskiy and Mironchenko (2013).

One reason for the success of ISS for the analysis of finite-dimensional systems probably lies on its Lyapunov characterization (Sontag and Wang, 1995). A generalization of this characterization to time-delay systems was provided in Karafyllis et al. (2008), based on Lyapunov-Krasovskii functionals. Given a locally Lipschitz functional $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$, we indicate its upper-right Dini derivative along the solutions of (3) as

$$\dot{V}|_{(3)} := \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h}, \quad \forall t \geq 0.$$

As detailed in Pepe (2007), local Lipschitzness of the functional V is instrumental in stability and ISS analysis.

Definition 2 (Strict/Relaxed ISS LKF) *Let $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be a locally Lipschitz functional for which there exists $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|), \quad \forall \phi \in \mathcal{X}. \tag{4}$$

Then V is said to be a strict ISS Lyapunov-Krasovskii functional (LKF) for (3) *if there exist $\alpha, \gamma \in \mathcal{K}_\infty$ such that, given any $x_0 \in \mathcal{X}$ and $u \in \mathcal{U}$, the corresponding solution $x(\cdot) := x(\cdot; x_0, u)$ of* (3) *satisfies, for almost all $t \geq 0$ in the maximum interval of solution's existence,*

$$\dot{V}|_{(3)} \leq -\alpha(V(x_t)) + \gamma(|u(t)|). \tag{5}$$

It is called a relaxed ISS LKF for (3) *if it satisfies* (4) *and, for almost all $t \geq 0$ in the maximum interval of solution's existence,*

$$\dot{V}|_{(3)} \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{6}$$

In both cases, α and γ are respectively referred to as a dissipation rate *and a* supply rate.

The difference between a strict LKF and a relaxed LKF stands in the way dissipation is achieved: for a strict LKF, the dissipation rate involves the value of the LKF itself, whereas for a relaxed LKF it merely involves the point-wise value of the state norm. In view of the bounds (4) on V, it clearly holds that any strict LKF is also a relaxed LKF.

It is known that ISS is equivalent to the existence of a *strict* ISS LKF (Karafyllis et al., 2008). The question we address here is whether the same holds for a *relaxed* ISS LKF. Ideally, we would like to solve the following conjecture.

Conjecture 1 *The system* (3) *is ISS if and only if it admits a relaxed ISS LKF.*

Since any strict ISS LKF is also a relaxed ISS LKF, the necessity part of this statement results straightforwardly from Karafyllis et al. (2008). The sufficiency part is not an easy problem, and we actually have no proof or disproof of it yet.

3 Properties induced by a relaxed ISS LKF

We start by listing some stability and robustness properties induced by the existence of a relaxed ISS LKF. These properties provide some possible research directions to prove or disprove Conjecture 1.

3.1 Forward completeness

First, we observe that the existence of a relaxed ISS LKF readily ensures that all solutions of (3) exist at all times.

Proposition 1 (Forward completeness) *If* (3) *admits a relaxed ISS LKF then, given any $x_0 \in \mathcal{X}$ and any $u \in \mathcal{U}$, its solution is unique, locally absolutely continuous, and defined over* $[0; +\infty)$.

Proof. Pick any $x_0 \in \mathcal{X}$ and any $u \in \mathcal{U}$. Then Hale and Lunel (1993) ensures that there exists $b \in (0; +\infty]$ such that (3) admits a unique and locally absolutely continuous solution over $[0; b)$, with $\lim_{t\to b^-} |x(t)| = +\infty$. By assumption, there exist $\underline{\alpha}, \overline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\begin{aligned} \underline{\alpha}(|x(t)|) &\leq V(x_t) \leq \overline{\alpha}(\|x_t\|), \quad \forall t \in [0; b) \\ \dot{V}|_{(3)} &\leq -\alpha(|x(t)|) + \gamma(|u(t)|), \quad \forall a.e.\ t \in [0; b). \end{aligned}$$

We directly get from these that $\underline{\alpha}(|x(t)|) \leq V(x_t) \leq V(x_0) + \gamma(\|u_{[0;t]}\|)t$ for all $t \in [0, b)$. Since this ensures that the solution is bounded over $[0; b)$, we conclude that $b = +\infty$. ■

3.2 0-GAS

A second straightforward consequence of the existence of a relaxed ISS LKF is the global asymptotic stability of the origin of the input-free system.

Proposition 2 (0-GAS) *If* (3) *admits a relaxed ISS LKF then the origin of the input-free system $\dot{x}(t) = f(x_t, 0)$ is globally asymptotically stable.*

Proof. The existence of relaxed ISS LKF V ensures, in view of Proposition 1, the existence of $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ such that, along all solutions of $\dot{x}(t) = f(x_t, 0)$

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|)$$
$$\dot{V} \leq -\alpha(|x(t)|),$$

for almost all $t \geq 0$. Global asymptotic stability of the origin of this input-free system then follows from the classical Lyapunov-Krasovskii theorem for global asymptotic stability (Hale, 1977, Theorem 2.1, p.105). ■

3.3 "L^2 to L^2" property

Another property that can be trivially derived from the existence of a relaxed ISS LKF is the following "L^2 to L^2" property.

Proposition 3 (L^2 to L^2 property) *If* (3) *admits a relaxed ISS LKF with dissipation rate $\alpha \in \mathcal{K}_\infty$ and supply rate $\gamma \in \mathcal{K}_\infty$, then there exists $\sigma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}$ and all $u \in \mathcal{U}$,*

$$\int_0^t \alpha(|x(s)|)ds \leq \sigma(\|x_0\|) + \int_0^t \gamma(|u(s)|)ds, \quad \forall t \geq 0. \tag{7}$$

Proof. By assumption and in view of Proposition 1, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}$, all $u \in \mathcal{U}$, and almost all $t \geq 0$,

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|) \tag{8}$$
$$\dot{V}|_{(3)} \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{9}$$

We directly get from these that $\int_0^t \alpha(|x(s)|)ds \leq V(x_0) - V(x_t) + \int_0^t \gamma(|u(s)|)ds \leq \overline{\alpha}(\|x_0\|) + \int_0^t \gamma(|u(s)|)ds$ for all $t \geq 0$. The conclusion follows with $\sigma = \overline{\alpha}$. ■

The property (7) can be seen as the natural extension of the "L^2 to L^2" property[3] introduced for non-delayed dynamics in Sontag (1998). It was shown in that reference that, for finite-dimensional systems, (7) is equivalent to ISS. The extension of that result to time-delay systems would thus provide an affirmative answer to Conjecture 1. Nonetheless, we are not aware of any work in the literature establishing that (7) implies ISS for time-delay systems.

3.4 Asymptotic gain of the average state power

Another straightforward property than can be derived from the existence of a relaxed ISS LKF is a bound on the average power of the state in terms of the magnitude of the input.

[3] This denomination is motivated by the fact that, when α and γ are square functions, (7) provides a link between the L^2 norm of the input and the L^2 norm of the state.

Proposition 4 (Asymptotic gain in average) *If* (3) *admits a relaxed ISS LKF with dissipation rate $\alpha \in \mathcal{K}_\infty$ and supply rate $\gamma \in \mathcal{K}_\infty$ then it holds that, for any $x_0 \in \mathcal{X}$ and any $u \in \mathcal{U}$,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \alpha(|x(s)|)ds \leq \gamma(\|u\|). \tag{10}$$

Proof. Integrating the dissipation inequality (9) yields, in view of Proposition 1, $\int_0^t \alpha(|x(s)|)ds \leq V(x_0) - V(x_t) + \gamma(\|u\|)t$ for all $t \geq 0$. Dividing by t and using the upper and lower bounds on V provided in (8), we get that

$$\frac{1}{t}\int_0^t \alpha(|x(s)|)ds \leq \frac{\overline{\alpha}(\|x_0\|)}{t} + \gamma(\|u\|), \quad \forall t > 0,$$

and the conclusion follows. ∎

3.5 Limit property

Another consequence of the existence of a relaxed ISS LKF is the following.

Proposition 5 (LIM property) *If* (3) *admits a relaxed ISS LKF then there exists $\delta \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}$ and all $u \in \mathcal{U}$,*

$$\inf_{t\geq 0} |x(t)| \leq \delta(\|u\|). \tag{11}$$

Proof. If $\|u\| = 0$, then it holds that $\dot{V}|_{(3)} \leq -\alpha(|x(t)|)$ for some $\alpha \in \mathcal{K}_\infty$. It follows from the classical Lyapunov-Krasovskii theorem (Hale, 1977, Theorem 2.1, p.105) that $\lim_{t\to\infty} |x(t)| = 0$, thus making (11) fulfilled. If $\|u\| \neq 0$, then the conclusion follows readily from Lemma 1 given in Section 6.2. ∎

The above property (11) is reminiscent of the so-called "limit property (LIM)" as introduced in Sontag and Wang (1996) for non-delayed systems. In that reference, it is shown that the combination of this property with 0-GAS is equivalent to ISS for finite-dimensional systems. Since the existence of a relaxed ISS LKF ensures 0-GAS, as seen in Proposition 2, the extension of that result to time-delay systems would give a positive answer to Conjecture 1. Nonetheless, we are not aware of any proof showing that the combination of 0-GAS and the LIM property (11) implies ISS for time-delay systems.

We also stress that we were not able yet to show that the stronger property

$$\inf_{t\geq 0} \|x_t\| \leq \delta(\|u\|)$$

implies ISS provided that the system is 0-GAS. Note that this latter property implies that the whole prehistory of the state eventually enters a neighborhood of the origin, whose size depends only of the magnitude of the applied input. We believe that establishing this fact would be a decisive step towards the establishment of Conjecture 1.

3.6 Bounded input with finite energy implies bounded converging state

The following result states that any system admitting a relaxed ISS LKF has a vanishing state in response to any input whose energy, measured through a specific $\mathcal{K}_\infty$ function, is bounded.

Proposition 6 *If* (3) *admits a relaxed ISS LKF with supply rate $\gamma \in \mathcal{K}_\infty$ then, for any $x_0 \in \mathcal{X}$ and any $u \in \mathcal{U}$ satisfying $\|u\| < +\infty$, the following implication holds:*

$$\int_0^\infty \gamma(|u(t)|)dt < \infty \quad \Rightarrow \quad \|x(\cdot)\| < \infty, \lim_{t \to +\infty} |x(t)| = 0. \tag{12}$$

Proof. Given any $u \in \mathcal{U}$ satisfying the left-hand side of (12), let $c > 0$ be such that $\max\left\{\|u\|\,;\, \int_0^{+\infty} \gamma(|u(s)|)ds\right\} \leq c$. By assumption, and in view of Proposition 1, there exist $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}$ and almost all $t \geq 0$,

$$\begin{aligned} \underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|) \\ \dot{V}|_{(3)} \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \end{aligned} \tag{13}$$

Integrating this dissipation inequality yields $V(x_t) - V(x_0) \leq -\int_0^t \alpha(|x(s)|)ds + c$ for all $t \geq 0$. In view of (13), this ensures in particular that

$$\underline{\alpha}(|x(t)|) \leq \overline{\alpha}(\|x_0\|) + c, \quad \forall t \geq 0 \tag{14}$$

$$\int_0^{+\infty} \alpha(|x(t)|)dt \leq \overline{\alpha}(\|x_0\|) + c. \tag{15}$$

(14) ensures that $\|x(\cdot)\| < +\infty$. Moreover, (14) and the fact that u is essentially bounded ensures that $t \mapsto f(x_t, u(t))$ is an essentially bounded function. Hence $\dot{x}(\cdot)$ is an essentially bounded function, implying that $x(\cdot)$ is uniformly continuous. Let $\tilde{\alpha}$ be any continuously differentiable $\mathcal{K}_\infty$ function such that $\tilde{\alpha}(s) \leq \alpha(s)$ for all $s \geq 0$, the existence of which is ensured by (Jiang et al., 1996, Lemma A.1). It then follows that the function $\tilde{\alpha}(|x(\cdot)|)$ is uniformly continuous and we get from (15) that $\int_0^{+\infty} \tilde{\alpha}(|x(t)|)dt < +\infty$. Invoking Barbalat's lemma, we conclude that $\lim_{t \to \infty} \tilde{\alpha}(|x(t)|) = 0$ and the proof follows by recalling that $\tilde{\alpha} \in \mathcal{K}_\infty$. ■

It was shown in Angeli et al. (2004) that, for finite dimensional systems, integral input-to-state stability (iISS: see Sontag (1998) for a precise definition) is equivalent to requiring 0-GAS and the following "Bounded Energy Weakly Converging State (BEWCS)" property:

$$\int_0^{+\infty} \gamma(|u(s)|)ds < +\infty \quad \Rightarrow \quad \liminf_{t \to +\infty} |x(t)| = 0,$$

for some $\gamma \in \mathcal{K}_\infty$. Since the property (12) implies BEWCS in our context, this suggests that at least iISS (see Pepe and Jiang (2006) for a definition of iISS for time-delay systems) could be derived from the existence of a relaxed ISS LKF. Nonetheless, we are not aware of any results in the literature establishing that 0-GAS and the BEWCS property (or the stronger requirement (12)) ensure iISS in presence of delays.

4 Relaxed ISS LKF implies ISS under growth restrictions

Since it was not possible at this point to prove or disprove Conjecture 1, we now identify particular classes of systems for which this conjecture holds true.

4.1 Growth restriction on the LKF upper bound

The following result focuses on a specific class of relaxed ISS LKF, which encompasses widely used functionals in the context of stability analysis of time-delay systems. It provides a sufficient condition for ISS linking the upper bound on the ISS KLF with its dissipation rate.

Theorem 1 (Restriction on the ISS LKF upper bound) *Assume that the system* (3) *admits a relaxed ISS Lyapunov-Krasovskii functional satisfying, for all* $\phi \in \mathcal{X}$*,*

$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}_1(|\phi(0)|) + \overline{\alpha}_2\left(\int_{-\theta}^{0} \overline{\alpha}_3(|\phi(s)|)ds\right) \tag{16}$$

and, along any solution of (3)*,*

$$\dot{V}|_{(3)} \leq -\alpha(|x(t)|) + \gamma(|u(t)|), \quad \forall a.e.\ t \geq 0, \tag{17}$$

for some $\underline{\alpha}, \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3, \alpha, \gamma \in \mathcal{K}_\infty$*. Assume further that* $\alpha \circ \overline{\alpha}_3^{-1}$ *is convex. Then the system* (3) *is ISS. Moreover, the functional* $\tilde{V}$ *defined, for all* $\phi \in \mathcal{X}$*, as*

$$\tilde{V}(\phi) := V(\phi) + \frac{1}{2\theta}\int_{-\theta}^{0}\int_{\tau}^{0} \alpha(|\phi(s)|)\, ds\, d\tau \tag{18}$$

is a strict LKF for (3)*.*

Typical techniques to "strictify" the ISS LKF (meaning to get a dissipation rate involving the whole ISS LKF, like in (5)) consist in adding an increasing linear or exponential term in the integral part of the relaxed ISS LKF: see for instance Pepe and Jiang (2006); Ito et al. (2010); Mazenc et al. (2013). The construction (18) rather adds the double integral of the point-wise dissipation rate α. See the proof, provided in Section 6.1, for more insights.

We stress that the above result imposes no growth restriction on the other two functions involved in the upper bound of the LKF (namely, $\overline{\alpha}_1$ and $\overline{\alpha}_2$).

The following result emphasizes the case of relaxed ISS LKF defined by quadratic terms. This class of LKF is widely used for the stability analysis of time-delay systems: see for instance the textbooks Hale (1977); Niculescu (2001); Gu et al. (2012); Fridman (2014). It implies that the LKF is bounded by a quadratic function of the M_2-norm of the state (Pepe and Jiang, 2006).

Corollary 1 (Relaxed quadratic ISS LKF) *Assume that there exist symmetric positive definite matrices $P_1, P_2, Q \in \mathbb{R}^{n\times n}$ and $\gamma \in \mathcal{K}_\infty$ such that the functional defined as*

$$V(\phi) = \phi(0)^T P_1\, \phi(0) + \int_{-\theta}^{0} \phi(s)^T P_2\, \phi(s) ds, \quad \forall \phi \in \mathcal{X},$$

satisfies, for all $x_0 \in \mathcal{X}$ and all $u \in \mathcal{U}$,

$$\dot{V}|_{(3)} \le -x(t)^T Q\, x(t) + \gamma(|u(t)|), \quad \forall a.e.\ t \ge 0.$$

Then the system (3) *is ISS.*

Proof. The proof follows straightforwardly from Theorem 1 by observing that $\overline{\alpha}_3$ and α in (16)-(17) can both be picked as squares in this context. Thus $\alpha \circ \overline{\alpha}_3^{-1}$ defines a linear (hence convex) function. ∎

4.2 Growth restriction on the vector field

Another way to ensure that the existence of an ISS LKF implies ISS is to restrict the growth of the vector field f. More precisely, we assume here that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, along all solutions of (3) and for all $t \ge 0$,

$$\left| \int_t^{t+\theta} f(x_s, u(s)) ds \right| \le \int_{t-\theta}^{t+\theta} \alpha_1(|x(s)|) ds + \int_t^{t+\theta} \alpha_2(|u(s)|) ds. \tag{19}$$

This assumption is little conservative *per se*, as illustrated in an example of Section 5. It is simply a way to estimate the maximum growth induced by the vector field. Then we have the following.

Theorem 2 (Restriction on the vector field) *Let V be a relaxed LKF for* (3) *and let α denote its dissipation rate. Assume that that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that the vector field f defining* (3) *satisfies* (19). *If $\alpha \circ \alpha_1^{-1}$ is convex then the system* (3) *is ISS.*

The proof of this result is provided in Section 6.2.

5 Example

In order to illustrate the main results of this paper, we consider the following example:

$$\tau \dot{x}(t) = -x(t) + S\big(cx(t-\theta) + u(t)\big), \quad \forall a.e.\ t \ge 0. \tag{20}$$

This time-delay system is used in the literature to model the activity of a neuronal population: see for instance Nevado-Holgado et al. (2010); Haidar et al. (2016). $x(t)$ then represents the average firing rate of the population at time t, meaning the instantaneous number of neuronal spikes per second. τ is the decay rate, and θ represents propagation delays along the axons. $S : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a nondecreasing, zero at zero, globally Lipschitz function. $c \in \mathbb{R}$ is a constant that models the average synaptic weight between neurons. u represents influence from other neuronal propagations or the effect of a stimulation signal. Then we have the following.

Proposition 7 *Let ℓ denote the Lipschitz constant of S and assume that $|c|\ell < 1$. Then the system* (20) *is ISS.*

Proof. This fact can be established using either Theorem 1 or Theorem 2. In both cases, the proof consists in showing that the functional defined as

$$V(\phi) = \tau\phi(0)^2 + \frac{1+\ell|c|}{2}\int_{-\theta}^{0} \phi(s)^2 ds, \quad \forall \phi \in \mathcal{X}, \tag{21}$$

is a relaxed ISS LKF. Here, we proceed by relying on Theorem 2. To that aim, observe first that, along the solutions of (20), V reads $V(x_t) = \tau x(t)^2 + \frac{1+\ell|c|}{2}\int_{t-\theta}^{t} x(s)^2 ds$ for all $t \geq 0$. It follows that

$$\tau x(t)^2 \leq V(x_t) \leq \left(\tau + \frac{1+\ell|c|}{2}\theta\right)\|x_t\|^2. \tag{22}$$

Moreover, using the fact that $ab \leq (\lambda a^2 + b^2/\lambda)/2$ for all $a, b \in \mathbb{R}$ and all $\lambda > 0$, it follows that, for almost all $t \geq 0$,

$$\begin{aligned}
\dot{V}|_{(20)} =& -2x(t)^2 + 2xS\big(cx(t-\theta) + u(t)\big) \\
&+ \frac{1+\ell|c|}{2}\big(x(t)^2 - x(t-\theta)^2\big) \\
\leq& -\left(2 - \frac{1+\ell|c|}{2}\right)x(t)^2 + 2\ell|x|\big(|c||x(t-\theta)| + |u(t)|\big) \\
&- \frac{1+\ell|c|}{2}x(t-\theta)^2 \\
\leq& -\left(2 - \frac{1+\ell|c|}{2} - \ell|c| - \frac{\ell}{\lambda}\right)x(t)^2 \\
&- \left(\frac{1+\ell|c|}{2} - \ell|c|\right)x(t-\theta)^2 + \lambda\ell u(t)^2 \\
\leq& -\Bigg(\frac{3(1-\ell|c|)}{2} - \frac{\ell}{\lambda}\Bigg)x(t)^2 - \frac{1-\ell|c|}{2}x(t-\theta)^2 + \lambda\ell u(t)^2,
\end{aligned}$$

where λ is any positive constant. Since $\ell|c| < 1$ by assumption, it follows that

$$\dot{V}|_{(20)} \leq -\left(\frac{3}{2}(1-\ell|c|) - \frac{\ell}{\lambda}\right)x(t)^2 + \lambda\ell u(t)^2, \quad \forall a.e.\ t \geq 0.$$

By picking $\lambda = \ell/(1-\ell|c|)$, we obtain that

$$\dot{V}|_{(20)} \leq -\frac{1}{2}\left(1-\ell|c|\right)x(t)^2 + \frac{\ell^2}{1-\ell|c|}u(t)^2. \tag{23}$$

It follows from (22) and (23) that V is a relaxed ISS LKF for (20), with dissipation rate α defined as $\alpha(s) = (1-\ell|c|)s^2/2$ for all $s \geq 0$. Furthermore, letting $f(x_t, u(t)) := \frac{1}{\tau}\left(-x(t) + S(cx(t-\theta) + u(t))\right)$, it holds that

$$\begin{aligned}
\tau\left|\int_t^{t+\theta} f(x_s, u(s))ds\right| &\leq \int_t^{t+\theta} \left|-x(s) + S\big(cx(s-\theta) + u(s)\big)\right| ds \\
&\leq \int_t^{t+\theta} |x(s)|ds + \ell|c|\int_t^{t+\theta} |x(s-\theta)|ds + \ell\int_t^{t+\theta} |u(s)|ds \\
&\leq (1+\ell|c|)\int_{t-\theta}^{t+\theta} |x(s)|ds + \ell\int_t^{t+\theta} |u(s)|ds. \qquad (24)
\end{aligned}$$

It follows that (19) holds with $\alpha_1(s) = (1+\ell|c|)s/\tau$ and $\alpha_2(s) = \ell s/\tau$ for all $s \geq 0$. Observing that

$$\alpha \circ \alpha_1^{-1}(s) = \frac{\tau^2(1-\ell|c|)}{2(1+\ell|c|)}s^2, \quad \forall s \geq 0,$$

the function $\alpha \circ \alpha_1^{-1}$ is convex and ISS of (20) follows from Theorem 2. ■

Let us now stress a limitation of Theorem 2. To that aim, consider the system (20) to which a cubic dissipative term is added, namely:

$$\tau\dot{x}(t) = -x(t) - x(t)^3 + S\big(cx(t-\theta) + u(t)\big).$$

Then it can easily be seen that the LKF V defined in (21) still satisfies the dissipation inequality (23), and hence has the same dissipation rate α. Nonetheless, the function α_1 involved in the bound (24) on the vector field can no longer be taken as a linear function, but rather involves cubic terms. It follows that $\alpha \circ \alpha_1^{-1}$ is no longer a convex function, thus making Theorem 2 inapplicable. This illustrates a serious limitation of Theorem 2 as, in the vector field bound (19), all terms are accounted as their worst-case contribution, should they contribute or not to stability. Nevertheless, in view of (21) and (23), ISS follows readily from Corollary 1, by picking $P_1 = 1/\tau$, $P_2 = (1+\ell|c|)/2$, $Q = (1-\ell|c|)/2$ and $\gamma(s) = \ell^2 s^2/(1-\ell|c|)$.

6 Main proofs

6.1 Proof of Theorem 1

The proof consists in showing that the function $\tilde{V}$ defined in (18) is a strict ISS Lyapunov-Krasovskii functional. ISS then follows readily from Pepe and Jiang

(2006). First notice that, in view of (18), it holds along the solutions of (3) that

$$\tilde{V}(x_t) = V(x_t) + \frac{1}{2\theta}\int_{-\theta}^{0}\int_{t+\tau}^{t}\alpha(|x(s)|)\,ds\,d\tau, \quad \forall t \geq 0. \tag{25}$$

It follows straightforwardly that $\tilde{V}$ satisfies bounds like (4). Moreover, it holds from (17) and Proposition 1 that, for almost all $t \geq 0$,

$$\begin{aligned}
\dot{\tilde{V}}|_{(3)} \leq& -\alpha(|x(t)|) + \gamma(|u(t)|) + \frac{1}{2\theta}\int_{-\theta}^{0}\left[\alpha(|x(t)|) - \alpha(|x(t+\tau)|)\right]d\tau \\
\leq& -\frac{1}{2}\alpha(|x(t)|) - \frac{1}{2\theta}\int_{t-\theta}^{t}\alpha(|x(\tau)|)d\tau + \gamma(|u(t)|).
\end{aligned} \tag{26}$$

In order to show that $\tilde{V}$ is a strict ISS LKF, it is thus sufficient to show that there exists $\tilde{\alpha} \in \mathcal{K}_\infty$ such that

$$\tilde{\alpha}(\tilde{V}(x_t)) \leq \frac{1}{2}\alpha(|x(t)|) + \frac{1}{2\theta}\int_{t-\theta}^{t}\alpha(|x(\tau)|)d\tau. \tag{27}$$

To show this, first observe that $\int_{t+\tau}^{t}\alpha(|x(s)|)\,ds \leq \int_{t-\theta}^{t}\alpha(|x(s)|)\,ds$ for all $\tau \in [-\theta;0]$. Consequently, it holds from (25) that

$$\begin{aligned}
\tilde{V}(x_t) &\leq V(x_t) + \frac{1}{2\theta}\int_{-\theta}^{0}\int_{t-\theta}^{t}\alpha(|x(s)|)\,ds\,d\tau \\
&\leq V(x_t) + \frac{1}{2}\int_{t-\theta}^{t}\alpha(|x(s)|)\,ds.
\end{aligned}$$

In view of (16), it follows that, given any $\tilde{\alpha} \in \mathcal{K}_\infty$,

$$\begin{aligned}
\tilde{\alpha}(\tilde{V}(x_t)) \leq& \,\tilde{\alpha}\left(V(x_t) + \frac{1}{2}\int_{t-\theta}^{t}\alpha(|x(s)|)\,ds\right) \\
\leq& \,\tilde{\alpha}\Big(\overline{\alpha}_1(|x(t)|) + \overline{\alpha}_2\Big(\int_{t-\theta}^{t}\overline{\alpha}_3(|x(s)|)ds\Big) + \frac{1}{2}\int_{t-\theta}^{t}\alpha(|x(s)|)\,ds\Big) \\
\leq& \,\tilde{\alpha}\circ 4\overline{\alpha}_1(|x(t)|) + \tilde{\alpha}\circ 4\overline{\alpha}_2\Big(\int_{t-\theta}^{t}\overline{\alpha}_3(|x(s)|)ds\Big) + \tilde{\alpha}\Big(\int_{t-\theta}^{t}\alpha(|x(s)|)\,ds\Big),
\end{aligned} \tag{28}$$

where we used twice the fact that $\tilde{\alpha}(a+b) \leq \tilde{\alpha}(2a)+\tilde{\alpha}(2b)$ for all $a, b \geq 0$. Recall that Jensen's inequality ensures that $\varphi\left(\int_a^b g(s)ds\right) \leq \frac{1}{b-a}\int_a^b \varphi((b-a)g(s))ds$ for all $a, b \in \mathbb{R}$ and all Lebesgue-integrable function $g : [a;b] \to \mathbb{R}_{\geq 0}$, provided that $\varphi : \mathbb{R} \to \mathbb{R}$ is convex. By assumption, the function defined as $\varphi : s \mapsto \alpha \circ \overline{\alpha}_3^{-1}(s/\theta)$ is convex. Applying this inequality to $g(\cdot) = \overline{\alpha}_3(|x(\cdot)|)$ yields

$$\alpha \circ \overline{\alpha}_3^{-1}\left(\frac{1}{\theta}\int_{t-\theta}^{t}\overline{\alpha}_3(|x(s)|)ds\right) \leq \frac{1}{\theta}\int_{t-\theta}^{t}\alpha(|x(s)|)ds.$$

It follows that

$$\int_{t-\theta}^{t} \overline{\alpha}_3(|x(s)|)ds \leq \theta\overline{\alpha}_3 \circ \alpha^{-1}\left(\frac{1}{\theta}\int_{t-\theta}^{t} \alpha(|x(s)|)ds\right).$$

Plugging this bound into (28) gives

$$\begin{aligned}\tilde{\alpha}(\tilde{V}(x_t)) \leq \tilde{\alpha} \circ 4\overline{\alpha}_1(|x(t)|) + \tilde{\alpha}\left(\int_{t-\theta}^{t} \alpha(|x(s)|)\, ds\right) \\ + \tilde{\alpha} \circ 4\overline{\alpha}_2 \circ \theta\overline{\alpha}_3 \circ \alpha^{-1}\left(\frac{1}{\theta}\int_{t-\theta}^{t} \alpha(|x(s)|)ds\right).\end{aligned} \tag{29}$$

By picking $\tilde{\alpha}$ as the $\mathcal{K}_\infty$ function defined, for all $s \geq 0$, as

$$\tilde{\alpha}(s) = \frac{1}{2}\min\left\{\alpha \circ \overline{\alpha}_1^{-1}(s/4)\,;\, \frac{1}{2}\alpha \circ \overline{\alpha}_3^{-1} \circ \frac{1}{\theta}\overline{\alpha}_2^{-1}(s/4)\,;\, \frac{s}{4\theta}\right\},$$

it can easily be seen that $\tilde{\alpha} \circ 4\overline{\alpha}_1(s) \leq \frac{1}{2}\alpha(s)$, $\tilde{\alpha} \circ 4\overline{\alpha}_2 \circ \theta\overline{\alpha}_3 \circ \alpha^{-1}(s/\theta) \leq \frac{s}{4\theta}$, and $\tilde{\alpha}(s) \leq \frac{s}{4\theta}$. It then follows from (29) that

$$\tilde{\alpha}(\tilde{V}(x_t)) \leq \frac{1}{2}\alpha(|x(t)|) + \frac{1}{2\theta}\int_{t-\theta}^{t} \alpha(|x(s)|)ds,$$

thus making (27) fulfilled. It then follows from (26) that, for almost all $t \geq 0$, $\dot{\tilde{V}}|_{(3)} \leq -\tilde{\alpha}(\tilde{V}(x_t)) + \gamma(|u(t)|)$, thus showing that $\tilde{V}$ is a strict ISS LKF for (3). ISS follows from the main result in Pepe and Jiang (2006).

6.2 Proof of Theorem 2

The proof relies on the following lemma, whose proof is provided in Section 6.3.

Lemma 1 *Suppose there exist a locally Lipschitz functional V and $\underline{\alpha}, \overline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ such that, for almost all $t \geq 0$,*

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|) \tag{30}$$

$$\dot{V}|_{(3)} \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{31}$$

Then, for any $c_1 > 3$, there exists $c_2 > 0$ such that for any $x_0 \in \mathcal{X}$ and any $u \in \mathcal{U}$ satisfying $\|u\| \neq 0$, there exists a time $T_0 \in [\theta; 2\theta + c_2(1 + \overline{\alpha}(\|x_0\|)/\gamma(\|u\|))]$ such that the solution of (3) satisfies

$$|x(T_0)| \leq \alpha^{-1} \circ c_1\gamma(\|u\|) \tag{32}$$

$$\int_{T_0-\theta}^{T_0+\theta} \alpha(|x(s)|)ds \leq c_1\theta\gamma(\|u\|). \tag{33}$$

$$\int_{t-2\theta}^{t+\theta} \alpha(|x(s)|)ds > c_1\theta\gamma(\|u\|), \quad \forall t \in [2\theta; T_0). \tag{34}$$

We stress that (34) provides no information in the case when $T_0 < 2\theta$, as the interval $[2\theta; T_0)$ is then empty.

First observe that, by assumption and in view of Proposition 1, there exist $\underline{\alpha}, \overline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}$, all $u \in \mathcal{U}$, and almost all $t \geq 0$,

$$\underline{\alpha}(|x(t)|) \leq V(x_t) \leq \overline{\alpha}(\|x_t\|) \tag{35}$$

$$\dot{V}|_{(3)} \leq -\alpha(|x(t)|) + \gamma(|u(t)|). \tag{36}$$

Applying Lemma 1 with $c_1 = 8$, it can be shown that there exists $c_2 > 0$ and a time

$$T_0 \in \left[\theta\,;\, 2\theta + c_2\left(1 + \frac{\overline{\alpha}(\|x_0\|)}{\gamma(\|u\|)}\right)\right] \tag{37}$$

such that

$$|x(T_0)| \leq \alpha^{-1} \circ 8\gamma(\|u\|)$$

$$\int_{T_0-\theta}^{T_0+\theta} \alpha(|x(s)|)ds \leq 8\theta\gamma(\|u\|).$$

Furthermore, considering x_{T_0} as the initial state and invoking again Lemma 1, we get that there exists a time $T_1 \in [T_0 + \theta; T_0 + 2\theta + c_2(1 + \overline{\alpha}(\|x_{T_0}\|)/\gamma(\|u\|))]$ such that

$$|x(T_1)| \leq \alpha^{-1} \circ 8\gamma(\|u\|)$$

$$\int_{T_1-\theta}^{T_1+\theta} \alpha(|x(s)|)ds \leq 8\theta\gamma(\|u\|).$$

Repeating this reasoning, we get that there exists a time sequence $\{T_k\}_{k\in\mathbb{N}}$ satisfying, for all $k \in \mathbb{N}$,

$$\theta \leq T_{k+1} - T_k \leq 2\theta + c_2\left(1 + \frac{\overline{\alpha}(\|x_{T_k}\|)}{\gamma(\|u\|)}\right), \tag{38}$$

such that

$$|x(T_k)| \leq \alpha^{-1} \circ 8\gamma(\|u\|) \tag{39}$$

$$\int_{T_k-\theta}^{T_k+\theta} \alpha(|x(s)|)ds \leq 8\theta\gamma(\|u\|). \tag{40}$$

This shows that solutions persistently visit the compact set $\{x \in \mathbb{R}^n : \alpha^{-1} \circ 8\gamma(\|u\|)\}$, at a rate that depends only on $\|u\|$ and on the value of the current state prehistory. Moreover, at these time instants T_k, the average value of $\alpha(|x(s)|)$ over an interval of length θ is "proportional" to the input's magnitude.

The next step consists in showing that, under the convexity assumption made on $\alpha \circ \alpha_1^{-1}$, (39)-(40) ensure that $\|x_{T_k+\theta}\|$ is bounded by a $\mathcal{K}_\infty$ function of $\|u\|$.

This can be done by relying on the growth restrictions made on f and the fact that $x(t) = x(t_0) + \int_{t_0}^t f(x_s, u(s))ds$, which implies that, for all $t \in [t_0; t_0 + \theta]$,

$$|x(t)| \leq |x(t_0)| + \left| \int_{t_0}^t f(x_s, u(s))ds \right| \tag{41}$$

$$\leq |x(t_0)| + \int_{t_0}^t |f(x_s, u(s))| \, ds \tag{42}$$

$$\leq |x(t_0)| + \int_{t_0}^{t_0+\theta} |f(x_s, u(s))| \, ds. \tag{43}$$

Relying on the assumption (19), we get that

$$|x(t)| \leq |x(t_0)| + \int_{t_0-\theta}^{t_0+\theta} \Big(\alpha_1(|x(s)|) + \alpha_2(|u(s)|) \Big) ds \tag{44}$$

$$\leq |x(t_0)| + \int_{t_0-\theta}^{t_0+\theta} \alpha_1(|x(s)|)ds + 2\theta\alpha_2(\|u\|). \tag{45}$$

Since $\alpha \circ \alpha_1^{-1}$ is convex, so is the function $s \mapsto \alpha \circ \alpha_1^{-1}(s/2\theta)$. Consequently, invoking Jensen's inequality, it holds that

$$\begin{aligned} \alpha \circ \alpha_1^{-1} \left(\frac{1}{2\theta} \int_{t_0-\theta}^{t_0+\theta} \alpha_1(|x(s)|)ds \right) &\leq \frac{1}{2\theta} \int_{t_0-\theta}^{t_0+\theta} \alpha \circ \alpha_1^{-1} \left(\frac{2\theta\alpha_1(|x(s)|)}{2\theta} \right) ds \\ &\leq \frac{1}{2\theta} \int_{t_0-\theta}^{t_0+\theta} \alpha(|x(s)|)ds. \end{aligned}$$

Plugging this into (45) then gives

$$|x(t)| \leq |x(t_0)| + 2\theta\alpha_1 \circ \alpha^{-1} \left(\frac{1}{2\theta} \int_{t_0-\theta}^{t_0+\theta} \alpha(|x(s)|)ds \right) + 2\theta\alpha_2(\|u\|), \quad \forall t \in [t_0; t_0 + \theta]. \tag{46}$$

In particular, for all $k \in \mathbb{N}$ and all $t \in [T_k; T_k + \theta]$, it holds that

$$|x(t)| \leq |x(T_k)| + 2\theta\alpha_1 \circ \alpha^{-1} \left(\frac{1}{2\theta} \int_{T_k-\theta}^{T_k+\theta} \alpha(|x(s)|)ds \right) + 2\theta\alpha_2(\|u\|).$$

In view of (39)-(40), it follows that

$$|x(t)| \leq \alpha^{-1} \circ 8\gamma(\|u\|) + 2\theta\alpha_1 \circ \alpha^{-1} \circ 4\gamma(\|u\|) + 2\theta\alpha_2(\|u\|), \quad \forall t \in [T_k; T_k + \theta], \, k \in \mathbb{N}.$$

Defining $\gamma_1(\cdot) := \alpha^{-1} \circ 8\gamma(\cdot) + 2\theta\alpha_1 \circ \alpha^{-1} \circ 4\gamma(\cdot) + 2\theta\alpha_2(\cdot)$, $\tilde{\gamma}$ is a $\mathcal{K}_\infty$ function and the above expression reads

$$\|x_{T_k+\theta}\| \leq \gamma_1(\|u\|), \quad \forall k \in \mathbb{N}. \tag{47}$$

Using this bound in (38), we obtain that the duration separating two successive instants T_k is bounded by a function of $\|u\|$ only. More precisely, for each $k \in \mathbb{N}$,

$$\theta \leq T_{k+1} - T_k \leq 2\theta + c_2 \left(1 + \frac{\overline{\alpha} \circ \gamma_1(\|u\|)}{\gamma(\|u\|)}\right). \tag{48}$$

Then, we study the solutions' behavior between $T_k + \theta$ and T_{k+1} by observing that (31) ensures in particular that $\dot{V} \leq \gamma(\|u\|)$, which, in view of (35), implies that, for all $t \in [T_k + \theta; T_{k+1}]$,

$$\begin{aligned}
\underline{\alpha}(|x(t)|) &\leq \overline{\alpha}(\|x_{T_k+\theta}\|) + \gamma(\|u\|)(t - T_k - \theta) \\
&\leq \overline{\alpha}(\|x_{T_k+\theta}\|) + \gamma(\|u\|)(T_{k+1} - T_k - \theta).
\end{aligned}$$

It then follows from (47)-(48) that, for each $k \in \mathbb{N}$,

$$\sup_{t \in [T_k+\theta; T_{k+1}]} |x(t)| \leq \gamma_2(\|u\|), \tag{49}$$

for some $\gamma_2 \in \mathcal{K}_\infty$. Defining $\overline{\gamma}(\cdot) := \underline{\alpha}^{-1}\Big((1 + c_2)\overline{\alpha} \circ \tilde{\gamma}(\cdot) + (\theta + c_2)\gamma(\cdot)\Big) + \tilde{\gamma}(\cdot)$, it can easily be seen that $\overline{\gamma} \in \mathcal{K}_\infty$ and, combining (47) and (49), we get that

$$\sup_{t \in [T_k; T_{k+1}]} |x(t)| \leq \overline{\gamma}(\|u\|), \quad \forall k \in \mathbb{N}.$$

Observing that (48) guarantees that $\bigcup_{k \in \mathbb{N}} [T_k; T_{k+1}] = [T_0; +\infty)$, we conclude that

$$\sup_{t \geq T_0} |x(t)| \leq \overline{\gamma}(\|u\|), \tag{50}$$

thus establishing the asymptotic gain property.

It is a known fact that, for finite-dimensional systems, (50) combined with 0-GAS imply ISS (Sontag and Wang, 1996). To the best of our knowledge, it has not yet been proved that the same holds true for time-delay systems. Thus, we need to proceed to the study of the transient dynamics. To that aim, considering three different cases.

Case 1: $T_0 < 2\theta$. In this case, it holds from (35)-(36) that, for all $t \in [0; T_0]$,

$$\begin{aligned}
\underline{\alpha}(|x(t)|) &\leq \overline{\alpha}(\|x_0\|) + \gamma(\|u\|)t \\
&\leq \overline{\alpha}(\|x_0\|)e^{-(t-T_0)} + \gamma(\|u\|)T_0 \\
&\leq \overline{\alpha}(\|x_0\|)e^{2\theta}e^{-t} + 2\theta\gamma(\|u\|).
\end{aligned}$$

This implies in particular that

$$|x(t)| \leq \beta_1(\|x_0\|, t) + \gamma_1(\|u\|), \quad \forall t \in [0; T_0], \tag{51}$$

where β_1 denotes the $\mathcal{KL}$ function defined as $\beta_1(s, t) := \underline{\alpha}^{-1}\left(2\overline{\alpha}(s)e^{2\theta}e^{-t}\right)$ for all $s, t \geq 0$ and γ_1 is the $\mathcal{K}_\infty$ function defined as $\gamma_1(\cdot) := \underline{\alpha}^{-1} \circ 4\theta\gamma(\cdot)$.

Case 2: $T_0 \geq 2\theta$ and $\|u\| > 1$. Then it holds from (37) that $T_0 \in [\theta; 2\theta + c_2(1 + \overline{\alpha}(\|x_0\|)/\gamma(1))]$. Moreover, proceeding as in Case 1, it follows from (35)-(36) that, for all $t \in [0; T_0]$,

$$\begin{aligned}
\underline{\alpha}(|x(t)|) &\leq \overline{\alpha}(\|x_0\|) + \gamma(\|u\|)T_0 \\
&\leq \overline{\alpha}(\|x_0\|) + \gamma(\|u\|)\left(2\theta + c_2\left(1 + \frac{\overline{\alpha}(\|x_0\|)}{\gamma(1)}\right)\right) \\
&\leq \overline{\alpha}(\|x_0\|) + (2\theta + c_2)\gamma(\|u\|) + \frac{c_2^2}{\gamma(1)^2}\gamma(\|u\|)^2 + \overline{\alpha}(\|x_0\|)^2 \\
&\leq \overline{\alpha}(\|x_0\|)\left(1 + \overline{\alpha}(\|x_0\|)\right) e^{T_0} e^{-t} + (2\theta + c_2)\gamma(\|u\|) + \frac{c_2^2}{\gamma(1)^2}\gamma(\|u\|)^2 \\
&\leq \overline{\alpha}(\|x_0\|)\left(1 + \overline{\alpha}(\|x_0\|)\right) \exp\left(2\theta + c_2\left(1 + \frac{\overline{\alpha}(\|x_0\|)}{\gamma(1)}\right)\right) e^{-t} \\
&\quad + (2\theta + c_2)\gamma(\|u\|) + \frac{c_2^2}{\gamma(1)^2}\gamma(\|u\|)^2.
\end{aligned}$$

Considering the functions defined for all $s, t \geq 0$ as

$$\begin{aligned}
\beta_2(s,t) &:= \underline{\alpha}^{-1}\left(2\overline{\alpha}(s)\left(1 + \overline{\alpha}(s)\right)\exp\left(2\theta + c_2\left(1 + \frac{\overline{\alpha}(s)}{\gamma(1)}\right)\right) e^{-t}\right) \\
\gamma_2(s) &:= \underline{\alpha}^{-1}\left(2(2\theta + c_2)\gamma(s) + \frac{2c_2^2}{\gamma(1)^2}\gamma(s)^2\right),
\end{aligned}$$

it can easily be seen that $\beta_2 \in \mathcal{KL}$ and $\gamma_2 \in \mathcal{K}_\infty$ and it holds that

$$|x(t)| \leq \beta_2(\|x_0\|, t) + \gamma_2(\|u\|), \quad \forall t \in [0; T_0]. \tag{52}$$

Case 3: $T_0 \geq 2\theta$ and $\|u\| \leq 1$. Using the function $N : \mathbb{R}_{\geq 0} \to \mathbb{N}$ defined in (62), namely

$$N(t) := \max\left\{k \in \mathbb{N} \,:\, 3(k+1)\theta \leq t\right\}, \quad \forall t \geq 0,$$

it holds from (36) that, for all $t \in [0; T_0]$,

$$\begin{aligned}
V(x_t) &\leq V(x_0) - \int_0^t \alpha(|x(s)|)ds + \gamma(\|u\|)t \\
&\leq V(x_0) - \frac{1}{4}\int_0^t \alpha(|x(s)|)ds - \frac{3}{4}\sum_{k=0}^{N(t)} \int_{3k\theta}^{3(k+1)\theta} \alpha(|x(s)|)ds + \sum_{k=0}^{N(t)+1} 3\theta\gamma(\|u\|) \\
&\leq V(x_0) - \frac{1}{4}\int_0^t \alpha(|x(s)|)ds - \frac{3}{4}\sum_{k=0}^{N(t)} \int_{3k\theta}^{3(k+1)\theta} \alpha(|x(s)|)ds + 3\theta(N(t)+2)\gamma(\|u\|).
\end{aligned}$$

Since $T_0 \geq 2\theta$ and we picked $c_1 = 8$, (34) ensures that

$$\int_{t-2\theta}^{t+\theta} \alpha(|x(s)|)ds > 8\theta\gamma(\|u\|), \quad \forall t \in [2\theta; T_0).$$

It follows that

$$
\begin{aligned}
V(x_t) &\leq V(x_0) - \frac{1}{4}\int_0^t \alpha(|x(s)|)ds - \frac{3}{4}\sum_{k=0}^{N(t)} 8\theta\gamma(\|u\|) + 3\theta(N(t)+2)\gamma(\|u\|) \\
&\leq V(x_0) - \frac{1}{4}\int_0^t \alpha(|x(s)|)ds - 6\theta(N(t)+1)\gamma(\|u\|) + 3\theta(N(t)+2)\gamma(\|u\|) \\
&\leq V(x_0) - \frac{1}{4}\int_0^t \alpha(|x(s)|)ds. \qquad (53)
\end{aligned}
$$

We next rely on the following result.

Lemma 2 (Integral version of Lyapunov-Krasovskii theorem) *Let $g : \mathbb{R}_{\geq 0} \times \mathcal{X} \to \mathbb{R}^n$ be locally Lipschitz and uniformly bounded in its first argument. Assume that there exist a functional $V : \mathbb{R}_{\geq 0} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ and $\underline{\alpha}, \overline{\alpha}, \alpha \in \mathcal{K}_\infty$ satisfying along any solution of $\dot{x}(t) = g(t, x_t)$,*

$$
\underline{\alpha}(|x(t)|) \leq V(t, x_t) \leq \overline{\alpha}(\|x_t\|) \qquad (54)
$$

$$
V(t, x_t) - V(t, x_0) \leq -\int_0^t \alpha(|x(s)|)ds. \qquad (55)
$$

Then there exists $\beta \in \mathcal{KL}$ such that, for all $t_0 \geq 0$ and all $x_{t_0} \in \mathcal{X}$,

$$
|x(t)| \leq \beta(\|x_{t_0}\|, t - t_0), \quad t \geq t_0. \qquad (56)
$$

In other words, the origin of $\dot{x}(t) = g(t, x_t)$ is uniformly globally asymptotically stable.

The proof of this lemma follows along that of the Lyapunov-Krasovskii theorem for uniform global asymptotic stability (Hale, 1977, Theorem 2.1, p. 105), and is therefore omitted.

Let $g(t, x_t) := f(x_t, \tilde{u}(t))$, where $\tilde{u}$ is defined as

$$
\tilde{u}(t) := \begin{cases} u(t) & \text{if } t \leq T_0 \\ 0 & \text{otherwise.} \end{cases} \qquad (57)
$$

Since $\|u\| \leq 1$, $\|\tilde{u}\| \leq 1$. Therefore g is locally Lipschitz and uniformly bounded in its first argument. Moreover, the solutions of $\dot{x}(t) = g(t, x_t)$ coincide with those of (3) over $[0; T_0]$. Invoking Lemma 2, we get from (35) and (53) that there exists a $\mathcal{KL}$ function β_3 such that

$$
|x(t)| \leq \beta_3(\|x_0\|, t), \quad \forall t \in [0; T_0]. \qquad (58)
$$

Thus, combining (51), (52), and (58), we get that, in the three cases,

$$
|x(t)| \leq \beta(\|x_0\|, t) + \max\{\gamma_1(\|u\|)\,;\, \gamma_2(\|u\|)\} \quad \forall t \in [0, T_0],
$$

where $\bar{\delta} := \gamma_1 + \gamma_2 \in \mathcal{K}_\infty$ and β is the $\mathcal{KL}$ function defined as $\beta(s,t) := \max_{i\in\{1,2,3\}} \beta_i(s,t)$ for all $s, t \geq 0$. With the asymptotic gain property (50), we finally get that

$$|x(t)| \leq \beta(\|x_0\|, t) + \delta(\|u\|), \quad \forall t \geq 0, \tag{59}$$

where δ is the $\mathcal{K}_\infty$ function defined as $\delta := \overline{\gamma} + \bar{\delta}$. This establishes ISS and concludes the proof.

6.3 Proof of Lemma 1

Consider any $c_1 > 0$. We start by showing that there exists $c_2 > 0$ such that, for all $x_0 \in \mathcal{X}$ and all $u \in \mathcal{U}$ satisfying $\|u\| \neq 0$, there exists $T' \in [2\theta; 2\theta + c_2(1 + \overline{\alpha}(\|x_0\|)/\gamma(\|u\|))]$ such that

$$\int_{T'-2\theta}^{T'+\theta} \alpha(|x(s)|)ds \leq c_1\theta\gamma(\|u\|). \tag{60}$$

To that aim, assume on the contrary that this does not hold, meaning that

$$\int_{t-2\theta}^{t+\theta} \alpha(|x(s)|)ds > c_1\theta\gamma(\|u\|), \quad \forall t \in [2\theta; 2\theta + c_2(1 + \overline{\alpha}(\|x_0\|)/\gamma(\|u\|))]. \tag{61}$$

Consider the non-decreasing unbounded piecewise constant function $N : \mathbb{R}_{\geq 0} \to \mathbb{N}$ defined as

$$N(t) := \max\left\{k \in \mathbb{N} \,:\, 3(k+1)\theta \leq t\right\}. \tag{62}$$

Integrating (31) then yields

$$\begin{aligned}
V(x_t) \leq & V(x_0) - \int_0^t \Big(\alpha(|x(s)|) - \gamma(|u(s)|)\Big)ds \\
\leq & V(x_0) - \sum_{k=0}^{N(t)} \int_{3k\theta}^{3(k+1)\theta} \Big(\alpha(|x(s)|) - \gamma(|u(s)|)\Big)ds \\
& - \int_{3(N(t)+1)\theta}^{t} \Big(\alpha(|x(s)|) - \gamma(|u(s)|)\Big)ds \\
\leq & V(x_0) - \sum_{k=0}^{N(t)} \int_{3k\theta}^{3(k+1)\theta} \Big(\alpha(|x(s)|) - \gamma(|u(s)|)\Big)ds \\
& + \int_{3(N(t)+1)\theta}^{t} \gamma(|u(s)|)ds.
\end{aligned}$$

Observe that, by the definition of N, it holds that

$$\frac{t}{3\theta} - 2 \leq N(t) \leq \frac{t}{3\theta} - 1, \quad \forall t \geq 0. \tag{63}$$

Consequently

$$\begin{aligned}
V(x_t) &\leq V(x_0) - \sum_{k=0}^{N(t)} \int_{3k\theta}^{3(k+1)\theta} \Big(\alpha(|x(s)|) - \gamma(|u(s)|)\Big)ds + \int_{3(N(t)+1)\theta}^{3(N(t)+2)\theta} \gamma(|u(s)|)ds \\
&\leq V(x_0) - \sum_{k=0}^{N(t)} \int_{3k\theta}^{3(k+1)\theta} \alpha(|x(s)|)ds + \sum_{k=0}^{N(t)+1} \int_{3k\theta}^{3(k+1)\theta} \gamma(|u(s)|)ds \\
&\leq V(x_0) - \sum_{k=0}^{N(t)} \int_{3k\theta}^{3(k+1)\theta} \alpha(|x(s)|)ds + 3(N(t)+2)\theta\gamma(\|u\|).
\end{aligned}$$

If (61) holds true, then $\int_{3k\theta}^{3(k+1)\theta} \alpha(|x(s)|)ds > c_1\theta\gamma(\|u\|)$ for all $k \in [1, N(t)]$ with $t \leq 2\theta + c_2(1+\overline{\alpha}(\|x_0\|)/\gamma(\|u\|))$. It follows that, for all $t \in [2\theta; 2\theta + c_2(1+\overline{\alpha}(\|x_0\|)/\gamma(\|u\|))]$,

$$\begin{aligned}
V(x_t) &< V(x_0) - \sum_{k=1}^{N(t)} c_1\theta\gamma(\|u\|) + 3(N(t)+2)\theta\gamma(\|u\|) \\
&\leq V(x_0) - c_1 N(t)\theta\gamma(\|u\|) + 3(N(t)+2)\theta\gamma(\|u\|) \\
&\leq \overline{\alpha}(\|x_0\|) - \big(N(t)(c_1-3) - 6\big)\theta\gamma(\|u\|),
\end{aligned} \tag{64}$$

where we used (30) for the latest bound. Furthermore, considering (64) for $t = 2\theta + c_2(1+\overline{\alpha}(\|x_0\|)/\gamma(\|u\|))$, it follows from (63) that

$$\begin{aligned}
V(x_t) &< \overline{\alpha}(\|x_0\|) - \left((c_1-3)\left(\frac{t}{3\theta} - 2\right) - 6\right)\theta\gamma(\|u\|) \\
&\leq \overline{\alpha}(\|x_0\|) - \left((c_1-3)\left(\frac{2\theta + c_2(1+\overline{\alpha}(\|x_0\|)/\gamma(\|u\|))}{3\theta} - 2\right) - 6\right)\theta\gamma(\|u\|) \\
&\leq \left(1 - (c_1-3)\frac{c_2}{3}\right)\overline{\alpha}(\|x_0\|) - \left((c_1-3)\left(\frac{2}{3} + \frac{c_2}{3\theta} - 2\right) - 6\right)\theta\gamma(\|u\|).
\end{aligned}$$

For any $c_1 > 3$, on might pick c_2 large enough that $\big(1 - (c_1-3)\frac{c_2}{3}\big) \leq 0$ and $(c_1-3)\left(\frac{c_2}{3\theta} - \frac{2}{3}\right) \geq 6$. With this choice of c_2, we thus get that $\dot{V} < 0$, which contradicts the fact that $V(x_t) \geq 0$ as ensured by (30). Thus, (60) is established by contradiction. Note that, without loss of generality, we may consider the smallest of all these time instants T', meaning that

$$\int_{t-2\theta}^{t+\theta} \alpha(|x(s)|)ds > c_1\theta\gamma(\|u\|), \quad \forall t \in [2\theta; T'). \tag{65}$$

Furthermore, (60) in turn implies that

$$\int_{T'-\theta}^{T'} \alpha(|x(s)|)ds \leq c_1\theta\gamma(\|u\|).$$

Invoking the mean value theorem, we get that there exists $T_0 \in [T' - \theta; T']$ such that

$$\alpha(|x(T_0)|)\theta \leq c_1\theta\gamma(\|u\|),$$

thus establishing (32), by observing that $T_0 \in [T' - \theta; T']$ ensures that $T_0 \in [\theta; 2\theta + c_2(1 + \overline{\alpha}(\|x_0\|)/\gamma(\|u\|))]$. With this particular T_0, it holds from (60) that

$$\begin{aligned} \int_{T_0-\theta}^{T_0+\theta} \alpha(|x(s)|)ds &\leq \int_{T'-2\theta}^{T'+\theta} \alpha(|x(s)|)ds \\ &\leq c_1\gamma(\|u\|), \end{aligned}$$

which establishes (33). Finally, since $T_0 \leq T'$, it holds form (65) that

$$\int_{t-2\theta}^{t+\theta} \alpha(|x(s)|)ds > c_1\theta\gamma(\|u\|), \quad \forall t \in [2\theta; T_0).$$

which establishes (34) and concludes the proof.

7 Conclusions and perspectives

The paper poses the conjecture that the existence of an ISS Lyapunov-Krasovskii functional with a point-wise dissipation rate is enough to guarantee ISS for nonlinear time-delay systems. It provides preliminary answers to this open question, by deriving stability and robustness properties that can be readily derived from such a relaxed ISS LKF and by identifying two classes of systems for which the conjecture holds true.

The paper raises more questions than answers. We believe in particular that proving that the existence of an ISS LKF combined with the asymptotic gain (AG) property would be a significant step towards proving the conjecture. Alternatively, demonstrating that the "L^2 to L^2" property is equivalent to ISS for time-delay systems would be enough to establish the conjecture. Future work will also address a similar question for ISS-related properties such as iISS (Sontag, 1998) and Strong iISS (Chaillet et al., 2014).

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