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Bipartite and Cooperative Output Synchronizations of Linear Heterogeneous Agents: A Unified Framework

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Abstract

This paper investigates cooperative output synchronization and bipartite output synchronization of a group of linear heterogeneous agents in a unified framework. For a structurally balanced signed graph, we prove that the bipartite output synchronization is equivalent to the cooperative output synchronization over an unsigned graph whose adjacency matrix is obtained by taking the absolute value of each entry in the adjacency matrix of the signed graph. We obtain a new H_∞-criterion which is sufficient for both cooperative output synchronization and bipartite output synchronization.

Key words: Bipartite Output Synchronization, Cooperative Output Synchronization, Heterogeneous Multi-Agent Systems

1 Introduction

Cooperative consensus of multi-agent systems has been studied widely in the literature [10]. One particular interest is the Cooperative Output Synchronization (COS), where the outputs of the agents synchronize to each other or to a reference trajectory. There are several applications for COS like formation control, distributed control of UAVs, sensor networks, *etc* [10,6]. However, in a number of contexts such as social networks, marketing or games the interactions among agents are not necessarily cooperative [3], which are usually described by a signed graph, where positive and negative edge weights denote cooperation and competition among concerned nodes respectively.

One type of agreement over a signed graph is bipartite synchronization, where agents reach an agreement over the modulus of a variable. Bipartite Output Synchronization (BOS) studies output synchronization of the agents in modulus with possibly different signs. There are many engineering applications for BOS like analyzing trustworthiness of the nodes in a network [5] and anticipating unanimity of the opinions in a decision process in the presence of stubborn agents [4].

Comparing with [3,13,11,8], which consider a bipartite state synchronization problem, this paper studies a bipartite output synchronization problem. The BOS of linear heterogeneous agents is considered in [8] where the agents communicate the states of their dynamic compensetaors. The current paper is different from [8] in the sense that the agents communicate their outputs only, which is commonly considered in applications. Moreover, our suggested controller can be used for the COS and the BOS of uncertain agents in contrast to [8], as we incorporate a p-copy (Definition 1) of the desired synchronized trajectory in the dynamic controller and our controller design does not depend on a unique solution to the output regulation equation [8]. In contrast to other existing works which have restrictive assumptions such as homogeneity of the agents [13,11], undirected communication graphs [11] or first-order dynamics [3], our framework allows heterogeneity of the agents, and a general directed and time-invariant signed communication graph.

This paper is an extended version of [1] and has two main contributions. Firstly, we prove that the H_∞-criterion for the COS problems reported in [2,6] can be relaxed for some classes of communication graphs to ensure the existence of solutions for a larger set of problems. Secondly, we prove that the BOS problem is equivalent to the COS problem in the sense that a control solution to one problem induces a control solution to the other problem. In particular, that generalized H_∞ criterion introduced in the first contribution can also be applied to the BOS problem which is more relaxed than the H_∞ criterion reported in [1].

The rest of the paper is organized as follows. In Section 2 we introduce notations and preliminaries. In Section 3 we formulate the BOS and the COS problems, and show that they are equivalent via a novel transformation procedure. In Section 4 we propose a relaxed H_∞ criterion as a sufficient condition to ensure the existence of a solution to the COS problem, which is applicable to the BOS problem as well due to the aforementioned transformation procedure. Simulation results are shown in Section 5, and conclusions are drawn in Section 6.

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2 Preliminaries

Let $\mathbb{R}^{n\times m}$ be the set of $n \times m$ real matrices. I_n, $\mathbf{1}_n$ and $\mathbf{0}$ denote the identity matrix of dimension $n \times n$, an n-dimensional column vector of 1, and a matrix of zeros with a compatible dimension, respectively. The Kronecker product of two matrices A and B is denoted as $A \otimes B$. Let $A_i \in \mathbb{R}^{n_i \times m_i}$ for $i = 1, ..., N$. The operator $\underset{1:N}{Diag}\{A_i\}$ builds a block diagonal matrix with N diagonal blocks, whose ith diagonal block is A_i. The spectrum of matrix A is denoted by $spec(A)$ which is the multiset of its eigenvalues λ_i. The spectral radius of A is denoted as $\rho(A) = \max_{\lambda_i \in spec(A)} |\lambda_i|$. Given $A = [a_{ij}] \in \mathbb{R}^{n\times m}$, let $B := [A]_{n_1:n_2\times m_1:m_2} \in \mathbb{R}^{(n_2-n_1+1)\times(m_2-m_1+1)}$ be a matrix formed by rows $n_1, \cdots, n_2$ and columns $m_1, \cdots, m_2$ of A. The cardinality of a set V is denoted by $|V|$. A disjoint union of two sets V^1 and V^2 is denoted by $V^1 \dot\cup V^2$. The following definition is used throughout the paper.

Definition 1 ([7]) A pair of $M_1 = I_p \otimes \beta$, $M_2 = I_p \otimes \tau$ incorporate a *p-copy* internal model of a square matrix A if (β, τ) is controllable and the minimal polynomial of A divides the characteristic polynomial of β. □

By [12], a *signed graph* is represented by a touple $\mathcal{G}^s = (V, E, \theta)$, where $V = \{v_0, ..., v_N\}$ denotes a finite vertex set, $E \subseteq V \times V$ is a directed edge set, and $\theta : E \to \{+1, -1\}$ is a partial edge labeling function, which assigns either a positive or negative sign to each edge. We call $\mathcal{G}^u = (V, E)$ the corresponding *unsigned graph*. A (follower) subgraph of $\mathcal{G}^u$ obtained by removing the (leader) node v_0 can be represented by an $N \times N$ *adjacent* matrix $\mathcal{A}^u = [a^u_{ij}]$, where $a^u_{ij} = 1$ if $(v_j, v_i) \in E$, and $a^u_{ij} = 0$, otherwise. The adjacency of node v_0 and node v_i is denoted by a^u_{i0} and it is defined similarly. The *upper stream neighbor set* of a node $v \in V$ is defined as $N_v = \{v' \in V | (v', v) \in E\}$. The in-degree matrix F of that (follower) subgraph is defined as $F = \underset{1:N}{Diag}\{|N_{v_i}|\}$. The Laplacian of that (follower) subgraph is defined as $L^s = F - \mathcal{A}^s$, where $\mathcal{A}^s := [a^s_{ij} := \theta(v_j, v_i)a^u_{ij}]$ is the signed adjacent matrix. The signed pinning gain from the node v_0 to other nodes is denoted by the matrix $G^s = \underset{1:N}{Diag}\{g^s_i := \theta(v_0, v_i)a^u_{i0}\}$, and $G^u = \underset{1:N}{Diag}\{g^u_i := a^u_{i0}\}$ is the unsigned pinning gain. While the entries of the adjacency matrix $\mathcal{A}^u$ of the unsigned graph $\mathcal{G}^u$ are nonnegative, the entries of the adjacency matrix $\mathcal{A}^s$ of $\mathcal{G}^s$ can attain positive or negative values.

A directed graph is a *directed tree* if every node, except for one node called the root, has an in-degree equal to one, and the root node has its in-degree equal to zero, and in addition, each non-root node is reachable from the root node via a directed path. A directed graph has a *spanning tree* if it contains a directed tree over all nodes. A subgraph $\mathcal{G}^s_k = (V_k, E_k, \theta_k)$, where $V_k \subseteq V$, $E_k \subseteq E$ and θ_k being the restriction of θ over E_k, is called a *strongly connected subgraph* of $\mathcal{G}^s$ if each pair of different nodes $v_{ik}, v_{jk} \in V_k$ are reachable from each other via a directed path in the subgraph. In particular, a subgraph consisting of only one node, which is called a *single-node subgraph*, is always a strongly connected subgraph. $\mathcal{G}^s_k$ is *maximal* if there does not exist another strongly connected subgraph, which contains $\mathcal{G}^s_k$ as a subgraph.

Definition 2 (Structurally Balanced Graph [3]) A signed graph $\mathcal{G}^s = (V, E, \theta)$ is *structurally balanced* if it admits a bipartition of the nodes, $V = V^1 \dot\cup V^2$, such that (i) for all $(v_i, v_j) \in E \cap (V^q \times V^q)$ with $q = 1, 2$, $\theta(v_i, v_j) = 1$; and (ii) for all $v_i \in V^q$, $v_j \in V^r$ with $(v_i, v_j) \in E$, $q, r \in \{1, 2\}$, $q \neq r$, $\theta(v_i, v_j) = -1$. □

Let $\mathcal{D}$ be the set of gauge transformations $\mathcal{D} = \{\Sigma = \underset{1:N}{Diag}\{\sigma_i\} | \sigma_i \in \{\pm 1\}\}$. We define the following notations

$$\begin{aligned} H^s &= \underset{1:N}{Diag}\left\{\frac{1}{|N_{v_i}| + g^u_i}\right\}(F - \mathcal{A}^s + G^u), \\ H^u &= \underset{1:N}{Diag}\left\{\frac{1}{|N_{v_i}| + g^u_i}\right\}(F - \mathcal{A}^u + G^u). \end{aligned} \tag{1}$$

Lemma 1 ([13,3]) *Let $\mathcal{G}^s = (V, E, \theta)$ be a signed graph which is structurally balanced with the bipartition $V = V^1 \dot\cup V^2$, and $\mathcal{G}^u = (V, E)$ be its unsigned equivalent. Then $\Sigma_1 \mathcal{A}^s \Sigma_1 = \mathcal{A}^u$ and $\Sigma_1 H^s \Sigma_1 = H^u$ if and only if $\Sigma_1 = \underset{1:N}{Diag}\{\sigma_i\} \in \mathcal{D}$, where for all $v_i \in V^q, v_j \in V^r$ with $q, r \in \{1, 2\}$, we have $\sigma_i = \sigma_j$ if and only if $q = r$.* □

Lemma 2 ([9]) *Let a graph $\mathcal{G} = (V, E)$ contain K maximal strongly connected subgraphs $\mathcal{G}_k = (V_k, E_k)$, $k = 1, ..., K$. One can reorder the nodes such that the adjacent matrix $\mathcal{A}$ of $\mathcal{G}$ is lower block triangular and its m-th diagonal blocks is $\Xi_m \in \{\mathcal{A}_k | 1 \le k \le K\}$, where $\mathcal{A}_k$ is the adjacent matrix of $\mathcal{G}_k$.* □

3 Bipartite and Cooperative Output Synchronization Problems

Consider a group of $N+1$ linear heterogeneous agents consisting of N followers labeled as $i = 1, ..., N$ and a leader labeled as 0:

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{2}$$
$$y_i = C_i x_i, \tag{3}$$
$$z_i = D_i x_i,\ i = 1, ..., N \tag{4}$$
$$\dot{x}_0 = A_0 x_0, \tag{5}$$
$$y_0 = C_0 x_0, \tag{6}$$

where $x_i \in \mathbb{R}^{n_i}$, $y_i \in \mathbb{R}^p$, $u_i \in \mathbb{R}^{m_i}$ and $z_i \in \mathbb{R}^{q_i}$ are the state, the output, the control and the measured output of the agent i $(i = 0, ..., N)$, respectively. We make the following assumption.

Assumption 1 *The signed graph $\mathcal{G}^s = (V, E, \theta)$ associated with the multi-agent system is structurally balanced.*

Without loss of generality, let $\Sigma_1 = \underset{1:N}{Diag}\{\sigma_i\}$ be the gauge transformation introduced in Lemma 1, where $v_0 \in V^1$, $(\forall v_i \in V^1)\ \sigma_i = 1$, and $(\forall v_j \in V^2)\ \sigma_j = -1$.

Problem 1 ***Bipartite Output Synchronization (BOS) Problem:*** *Consider a group of $N + 1$ linear heterogeneous agents defined by (2-6). Assume that the agents communicate y_i's, over a structurally balanced signed graph $\mathcal{G}^s = (V, E, \theta)$. Design the matrices $K_{1i} \in \mathbb{R}^{m_i\times q_i}$, $K_{2i} \in \mathbb{R}^{m_i\times n_{\eta_i}}$, $R_i \in \mathbb{R}^{n_{\eta_i}\times n_{\eta_i}}$, $S_i \in \mathbb{R}^{n_{\eta_i}\times p}$ for each $i = 1, ..., N$, such that*

$$\begin{aligned} u_i &= K_{1i} z_i + K_{2i}\eta_i, \\ \dot\eta_i &= R_i \eta_i + S_i \delta_i, \text{ where } \eta_i \in \mathbb{R}^{n_{\eta_i}} \\ \delta_i &= \frac{1}{|N_{v_i}| + g^u_i}[\sum_{j=1}^{N}(a^u_{ij} y_i - a^s_{ij} y_j) + g^u_i y_i - g^s_i y_0], \end{aligned} \tag{7}$$

render $\lim_{t\to+\infty} e_{bi}(t) = y_i(t) - \sigma_i y_0(t) = 0$. □

In this paper, we transform the BOS problem into another problem called cooperative output synchronization problem, which is defined below.

Problem 2 ***Cooperative Output Synchronization (COS) Problem:*** *Consider a group of $N+1$ linear heterogeneous agents defined by (2-6). Assume that the agents communicate y_i's over an unsigned graph $\mathcal{G}^u = (V, E)$. Design the matrices K_{1i}, K_{2i}, R_i, S_i such that for each $i = 1, ..., N$,*

$$\begin{aligned}
&\bar{u}_i = K_{1i} z_i + K_{2i} \bar{\eta}_i, \\
&\dot{\bar{\eta}}_i = R_i \bar{\eta}_i + S_i \bar{\delta}_i, \qquad \bar{\eta}_i \in \mathbb{R}^{n_{\eta_i}} \\
&\bar{\delta}_i = \frac{1}{|N_{v_i}| + g_i^u} \left[\sum_{j=1}^{N} a_{ij}^u (y_i - y_j) + g_i^u (y_i - y_0) \right],
\end{aligned} \tag{8}$$

render $\lim_{t \to +\infty} e_{ci}(t) = y_i(t) - y_0(t) = 0$. □

The controls (7-8) reduce to a state-feedback for $z_i = x_i$ and an explicit output-feedback for $z_i = y_i$. We now show the equivalence of the COS and the BOS problems by means of a similarity transformation. Let

$$\begin{aligned}
&\tilde{H}^s = H^s \otimes I_p, \; \tilde{H}^u = H^u \otimes I_p, \; Z = \tilde{H}^u - I_{Np}, \\
&\tilde{X} = \underset{1:N}{Diag}\{X_i\}, \; X_i \in \{A_i, B_i, C_i, D_i, R_i, S_i, K_{1i}, K_{2i}\}, \\
&\tilde{A}_0 = I_N \otimes A_0, \; \tilde{C}_0 = I_N \otimes C_0, C_c = \begin{bmatrix} \tilde{C} & \mathbf{0} \end{bmatrix}, \\
&A_c = \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \tilde{S}\tilde{H}^s\tilde{C} & \tilde{R} \end{bmatrix}, \; \bar{A}_c = \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \tilde{S}\tilde{H}^u\tilde{C} & \tilde{R} \end{bmatrix}, \\
&B_c = \begin{bmatrix} \mathbf{0} \\ -\tilde{S}(G^s \otimes I_p)\tilde{C}_0 \end{bmatrix}, \; \bar{B}_c = \begin{bmatrix} \mathbf{0} \\ -\tilde{S}\tilde{H}^u\tilde{C}_0 \end{bmatrix}.
\end{aligned}$$

The overall closed-loop system of all agents (2) seeking the BOS over the structurally balanced signed graph $\mathcal{G}^s$ via controllers (7) is given by

$$\begin{aligned}
\dot{\xi}_b &= A_c \xi_b + B_c r_G, \\
\dot{r_G} &= \tilde{A}_0 r_G, \\
e_b &= C_c \xi_b - \Sigma_1 \otimes C_0 r_G,
\end{aligned} \tag{9}$$

where $\xi_b = [x_1^T, \ldots, x_N^T, \eta_1^T, \ldots, \eta_N^T]^T$, $r_G = \mathbf{1}_N \otimes x_0$. The closed-loop system of all agents (2) seeking the COS over $\mathcal{G}^u$ via the control signal (8) is given by

$$\begin{aligned}
\dot{\xi}_c &= \bar{A}_c \xi_c + \bar{B}_c r_G, \\
\dot{r_G} &= \tilde{A}_0 r_G, \\
e_c &= C_c \xi_c - \tilde{C}_0 r_G,
\end{aligned} \tag{10}$$

where $\xi_c = [x_1^T, ..., x_N^T, \bar{\eta}_1^T, ..., \bar{\eta}_N^T]^T$.

The equivalence between the cooperative state synchronization and the bipartite state synchronization of first-order homogeneous agents is shown in [3]. Next, we show the equivalence between the COS and the BOS for a general linear heterogeneous multi-agent system.

Theorem 1 *The control signal u_i in (7) solves the BOS problem over a structurally balanced signed graph $\mathcal{G}^s = (V, E, \theta)$ if and only if the control signal $\bar{u}_i$ in (8) solves the COS problem over the unsigned graph $\mathcal{G}^u = (V, E)$.* □

PROOF. Denote $\Sigma_2 = \underset{1:N}{Diag}\{\sigma_i \otimes I_{n_i}\}$, $\Sigma_3 = \underset{1:N}{Diag}\{\sigma_i \otimes I_{n_{\eta_i}}\}$, $\Sigma_4 = \Sigma_1 \otimes I_p$, $\bar{\Sigma} = \begin{bmatrix} \Sigma_2 & \mathbf{0} \\ \mathbf{0} & \Sigma_3 \end{bmatrix}$. Clearly $\bar{\Sigma}^{-1} = \bar{\Sigma}$. Let $\bar{\xi} = \bar{\Sigma}\xi_b$. Then

$$\begin{aligned}
\dot{\bar{\xi}} &= \bar{\Sigma} A_c \bar{\Sigma} \bar{\xi} + \bar{\Sigma} B_c r_G, \\
e_b &= C_c \bar{\Sigma} \bar{\xi} - \Sigma_1 \otimes C_0 r_G.
\end{aligned} \tag{11}$$

According to the definition of $\bar{\Sigma}$ we have

$$\bar{\Sigma} A_c \bar{\Sigma} = \begin{bmatrix} \tilde{A} + \tilde{B}\tilde{K}_1\tilde{D} & \tilde{B}\tilde{K}_2 \\ \Sigma_3(\tilde{S}\tilde{H}^s\tilde{C})\Sigma_2 & \tilde{R} \end{bmatrix}.$$

It is easy to show that $\Sigma_3(\tilde{S}\tilde{H}^s\tilde{C})\Sigma_2 = \tilde{S}\Sigma_4\tilde{H}^s\Sigma_4\tilde{C} = \tilde{S}(\Sigma_1 H^s \Sigma_1) \otimes I_p \tilde{C}$. According to Lemma 1, $\Sigma_1 H^s \Sigma_1 = H^u$. Hence $\bar{\Sigma} A_c \bar{\Sigma} = \bar{A}_c$. Similarly,

$$\bar{\Sigma} B_c = \begin{bmatrix} \mathbf{0} \\ -\tilde{S}(\Sigma_1 G^s \otimes I_p)\tilde{C}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\tilde{S}(G^u \otimes I_p)\tilde{C}_0 \end{bmatrix}.$$

Noting that $L^u \mathbf{1}_N = \mathbf{0}$, (11) reads

$$\dot{\bar{\xi}} = \bar{A}_c \bar{\xi} + \bar{B}_c r_G. \tag{12}$$

It remains to show the error (9) in the new coordinate

$$e_b = \begin{bmatrix} \Sigma_4 \tilde{C} & \mathbf{0} \end{bmatrix} \bar{\xi} - \Sigma_4 \tilde{C}_0 r_G = \Sigma_4 (C_c \bar{\xi} - \tilde{C}_0 r_G) = \Sigma_4 e_c.$$

Let $\bar{\xi} = [x_1^T, ..., x_N^T, \bar{\eta}_1^T, ..., \bar{\eta}_N^T]^T$ and $\Sigma_4^{-1} e_b = e_c = [e_{c1}^T, ..., e_{cN}^T]^T$. The system in (12) with property $e_{ci} \to \mathbf{0}$ represents a COS problem for (2-6) over $\mathcal{G}^u$ using controller (8). Hence, the matrices $(K_{1i}, K_{2i}, R_i, S_i)$ in (8) solves the COS problem if and only if, the same matrices in (7) solves the BOS problem. ■

4 A relaxed H_∞ Criterion for COS and BOS problems

In this section, we extend a result of [2,1] and obtain a sufficient condition to ensure the existence of a control solution to both the BOS and the COS problems. Let graph $\mathcal{G}^u$ contain K maximal strongly connected subgraphs $\{\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_K\}$. Order the nodes in $\mathcal{G}^u$ such that the adjacency matrix of $\mathcal{G}^u$ is lower block triangular. For each $\mathcal{G}_k$ we assume that its vertex set $V_k = \{v_{s_{C_k}}, \cdots, v_{s_{C_k}+n_{C_k}-1}\}$, where $|V_k| = n_{C_k}$. Define

$$\begin{aligned}
&Z_k := [Z]_{(s_{C_k}:(s_{C_k}+n_{C_k}-1)) \times (s_{C_k}:(s_{C_k}+n_{C_k}-1))}, \\
&\underline{A}_i := \begin{bmatrix} A_i + B_i K_{1i} D_i & B_i K_{2i} \\ S_i C_i & R_i \end{bmatrix}, \; \hat{B}_i := \begin{bmatrix} 0 \\ S_i \end{bmatrix}, \\
&\hat{C}_i := \begin{bmatrix} C_i & 0 \end{bmatrix}, \; h_i(s) := -\hat{C}_i (sI - \underline{A}_i)^{-1} \hat{B}_i.
\end{aligned} \tag{13}$$

Matrix Z_k is the normalized adjacency matrix of the kth maximal strongly connected subgraph. We consider the following assumptions [2].

Assumption 2 *The graph $\mathcal{G}^u$ contains a directed spanning tree with the leader as its root node.*

Assumption 3 *rank* $\begin{bmatrix} A_i - \lambda I_{n_i} & B_i \\ C_i & \boldsymbol{0} \end{bmatrix} = n_i + p$, $\forall \lambda \in$ *spec*(A_0) *for* $i = 1, ..., N$.

Assumption 4 *The pair* (R_i, S_i) *contains a p-copy of the leader dynamics* A_0.

Assumption 5 *The triple* $\left(\begin{bmatrix} A_i & \mathbf{0} \\ S_iC_i & R_i \end{bmatrix}, \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \begin{bmatrix} D_i & \mathbf{0} \\ \mathbf{0} & I_{n_{\eta_i}} \end{bmatrix}\right)$ *is output-feedback stabilizable.*

Theorem 2 *Consider Assumptions 2-5. If* K_{1i} *and* K_{2i} *are selected in such a way that for all* $k \in \{1, \cdots, K\}$,

$$\max \|h_i\|_\infty < \frac{1}{\rho(Z_k)}, \ \forall v_i \in V_k \tag{14}$$

then (8) solves the COS problem over $\mathcal{G}^u$. □

PROOF. The characteristics equation of $\bar{A}_c$ reads

$$\begin{aligned}\Delta =& det(sI - \bar{A}_c) = det(X - \begin{bmatrix} 0 & \tilde{S} \end{bmatrix}^T Z \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}) \\ =& det(X) det(I - \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} X^{-1} \begin{bmatrix} 0 & \tilde{S} \end{bmatrix}^T Z),\end{aligned}$$

where $X = \begin{bmatrix} sI - (\tilde{A} + \tilde{B}\tilde{K}_1\tilde{D}) & -\tilde{B}\tilde{K}_2 \\ -\tilde{S}\tilde{C} & sI - \tilde{R} \end{bmatrix}$ and we have used the Sylvester's determinant Theorem to obtain the last equation. Let $T = [c_1^T, c_{N+1}^T, c_2^T, c_{N+2}^T, \ldots, c_{2N}^T]$

$$c_i \in \mathbb{R}^{\varrho_i \times (\sum_{i=1}^N n_i + \sum_{i=1}^N n_{\eta_i})}, \ i = 1, \ldots 2N$$

where $\varrho_i = n_i$ if $i = 1, \ldots, N$ and $\varrho_i = n_{\eta_i}$ otherwise. The matrix c_i is a block row matrix with $2N$ blocks which all are zero except the ith row block which is I_{ϱ_i}. Then,

$$\begin{aligned}\Delta =& det(T^{-1}TXT^{-1}T) \times \\ & det(I - \begin{bmatrix} \tilde{C} & 0 \end{bmatrix} T^{-1}TX^{-1}T^{-1}T \begin{bmatrix} 0 \\ \tilde{S} \end{bmatrix} Z) \\ =& det(T^{-1}) det(TXT^{-1}) det(T) det(I + h(s)Z) \\ =& det(\underset{1:N}{Diag}\{sI - \underline{A}_i\}) \times det\left(I + h(s)Z\right),\end{aligned}$$

where $h(s) = \underset{1:N}{Diag}\{h_i\}$. Noting that Δ is the characteristics equation of $\bar{A}_c$, this matrix is stable if all roots of Δ are strictly negative. The stability of $\mathcal{F}(s) = I + h(s)Z$ guarantees that all poles of $h(s)$ are stable which are given by the eigenvalues of $\underline{A}_i$. Hence, the stability of $\mathcal{F}(s)$ implies the stability of $\underline{A}_i$ and totally, they guarantee the stability of $\bar{A}_c$. Since the transfer matrix $h(s)$ is diagonal and the nodes are numbered such that Z is lower block triangular (Lemma 2), $\mathcal{F}(s)$ is lower block triangular and the stability of $\mathcal{F}(s)$ is given by the stability of its diagonal blocks. For the nodes in $\mathcal{G}_k$, the stability of $\mathcal{F}_k(s) = I + h_{n_{C_k}} Z_k$ is of interest, where $h_{n_{C_k}}(s) = \underset{s_{C_k}:(s_{C_k}+n_{C_k}-1)}{Diag}\{h_l(s)\}$. By the small gain theorem, $\mathcal{F}_k(s)$ is stable if the condition in (14) is satisfied. Thus, $\bar{A}_c$ is stable and the COS is achieved. ■

Remark 1 *The sufficient condition in Theorem 2 is more relaxed than Theorem 2 of [2]. According to Theorem 2,* $\|h_i\|_\infty$ *should be bounded by* $\frac{1}{\rho(Z_k)}$. *However by Theorem 2 of [2], this should be bounded by* $\frac{1}{\max_k \rho(Z_k)}$. *Clearly, the result by Theorem 2 of [2] is more stringent than Theorem 2. Moreover, according to Theorem 2, if a disjoint strongly connected subgraph* $\mathcal{G}_k$ *contains a single node,* $Z_k = 0$ *and (14) simplifies to the stability of* $\underline{A}_i$.

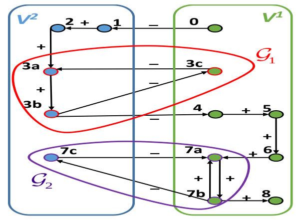


Fig. 1. The communication graph

Corollary 1 *Consider Assumptions 1-5. Assume that* K_{1i} *and* K_{2i} *are selected such that (14) holds. Then (7) solves the BOS problem over* $\mathcal{G}^s$. □

PROOF. It follows from Theorem 1 and Theorem 2.■

5 Simulation Results

Consider the signed graph $\mathcal{G}^s$ shown in Fig. 1. The signed graph $\mathcal{G}^s$ satisfies Assumptions 1-2. Also, it has two non-single-node maximal strongly connected subgraphs containing the nodes $V_1 = \{3a, 3b, 3c\}$ and $V_2 = \{7a, 7b, 7c\}$ respectively. We have $\rho(Z_1) = 0.7937$ and $\rho(Z_2) = 0.8514$ for $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. Consider the dynamics of the leader and the followers as

$$0 : \left\{ \dot{x}_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_0, \ y_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_0, \right.$$

$$1, 3a, 3c, 7a, 7c : \left\{ \begin{aligned} \dot{x}_1 &= \begin{bmatrix} -0.3 & -2 \\ 0.1 & -0.2 \end{bmatrix} x_1 + \begin{bmatrix} 1.8 & -0.8 \\ 0.9 & 1.6 \end{bmatrix} u_1, \\ y_1 &= \begin{bmatrix} -0.1 & 1.2 \\ 0.4 & 1.4 \end{bmatrix} x_1, \ z_1 = x_1, \end{aligned} \right.$$

$$2, 3b, 4, 6, 7b : \left\{ \begin{aligned} \dot{x}_2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 6 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u_2, \\ y_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_2, \ z_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} x_2 \end{aligned} \right.$$

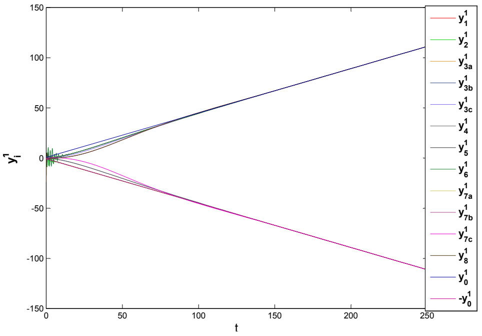
$$5, 8 : \left\{ \begin{aligned} \dot{x}_5 &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x_5 + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u_5, \\ y_5 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_5, \ z_5 = x_5. \end{aligned} \right.$$

Select the p-copy of the leader as $R_i = I_2 \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $S_i = I_2 \otimes \begin{bmatrix} 0 & 1 \end{bmatrix}^T$, $i = 1, \ldots, N$. Select the controller gains as

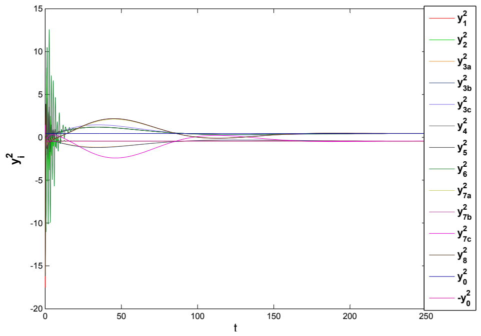
$$K_1 = \begin{bmatrix} -10 & -4 & 460 & 172 & -497 & -183 \\ 6 & -13 & -591 & -203 & 246 & 87 \end{bmatrix},$$

$$K_{3a,3c} = \begin{bmatrix} -0.35 & -0.86 & 0 & -0.09 & -0.01 & -0.3 \\ 0.14 & -1.5 & -0.01 & -0.34 & 0 & -0.01 \end{bmatrix},$$

$$K_{3b} = \begin{bmatrix} -1.93 & 1.59 & -0.47 & -2.91 & -0.09 & -2.19 \\ -10.9 & 12.45 & -2.97 & -16.12 & -0.84 & -18.36 \end{bmatrix},$$



(a) The first outputs of all agents



(b) The second outputs of all agents

Fig. 2. The outputs of all followers, the output of the leader and its mirror

$$K_{7a,7c} = \begin{bmatrix} -0.37 & -0.87 & -0.003 & -0.08 & -0.02 & -0.34 \\ 0.16 & -1.55 & -0.02 & -0.43 & 0.001 & 0.011 \end{bmatrix},$$

$$K_{7b} = \begin{bmatrix} -2.64 & 1.48 & -0.68 & -4.8 & -0.14 & -5.04 \\ -17.17 & 11.76 & -4.91 & -32.84 & -1.25 & -41.67 \end{bmatrix},$$

$$K_{2,4,6} = \begin{bmatrix} -0.39 & 0.29 & -0.15 & -2.13 & -0.04 & -1 \\ -2.17 & 1.14 & -0.41 & -5.28 & -0.83 & -16.32 \end{bmatrix},$$

$$K_{5,8} = \begin{bmatrix} -26.98 & -1.22 & -718 & -241 & -17.7 & -3.97 \\ 0.22 & 26 & 17.7 & 3.977 & 562 & 206 \end{bmatrix}.$$

To have a summary, we brought $\|h_i\|_\infty$, their authorized relaxed upper bounds by Corollary 1 and the authorized upper bound by Theorem 2 of [2] in Table 1.

Table 1
$\|h_i\|_\infty$s and their upper bounds

i	$\|h_i\|_\infty$	Upper bound by Corollary 1	Upper bound by Theorem 2 of [2]
1	1.3794	$< \infty$	1.1746
2, 4, 6	5.0357	$< \infty$	1.1746
3a,3c	1.1017	1.2599	1.1746
3b	1.2377	1.2599	1.1746
5, 8	1.2941	$< \infty$	1.1746
7a, 7c	1.1685	1.1746	1.1746
7b	1.1559	1.1746	1.1746

As a comparison, according to Theorem 2 of [2], we need $\|h_i\|_\infty < \frac{1}{\max_k \rho(Z_k)} = 1.1746$, $i = 1 : 12$. However, the selected gains do not satisfy this requirement, hence, there is no guarantee for the existence of a solution to the COS problem. But under our generalized criterion, *i.e.*, Theorem 2, a solution does exist. Then by Corollary 1, a control solution to the BOS problem is also guaranteed.

In Fig. 2 the outputs of all followers, y_i, $,i = 1, \dots, 12$, the output of the leader, y_0, and its mirror, $-y_0$, are sketched. From the figure we can see that the agents achieve the BOS with two subgroups $V^1 = \{0, 3c, 4, 5, 6, 7a, 7b, 8\}$ and $V^2 = \{1, 2, 3a, 3b, 7c\}$, where $y_i \to y_0, \forall i \in V^1$ and $y_j \to -y_0, \forall j \in V^2$. The initial conditions are selected randomly.

6 Conclusion

In this paper we have investigated the COS and the BOS problems of a group of $N+1$ linear heterogeneous agents consisting of one leader and N followers. We have obtained a relaxed H_∞-criterion as a sufficient condition to ensure the existence of a control solution to the COS problem by using the concept of the maximal strongly connected subgraphs. Moreover, we have shown that the BOS problem over a signed graph is equivalent to the COS problem over an unsigned graph in the sense that a control solution to one problem induces a control solution to the other, which allows all sorts of controller synthesis techniques developed for one problem to be applicable to the other, *e.g.*, that relaxed H_∞ criterion.

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