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Convex liftings based robust control design

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Abstract

This paper presents a new approach for control design of constrained linear systems affected by bounded additive disturbances and polytopic uncertainties. This method hinges on so-called *convex liftings* which emulate control Lyapunov function by providing a constructive framework for optimization based control implementation. It will be shown that this method can guarantee the recursive feasibility and robust stability in the sense of Lyapunov. Finally, a numerical example will be presented to illustrate this method.

Key words: Convex liftings, robust control, bounded additive disturbances, polytopic uncertainties, PWA Lyapunov function.

1 Introduction

Originated in Lyapunov (1907), Lyapunov stability becomes a fundamental concept in control theory. In stability analysis, a Lyapunov function is usually of use to prove closed loop stability or robust stability, see Molchanov and Pyatnitskiy (1989); Polanski (1995). On the other hand, in control design, control Lyapunov functions are usually employed to design stabilising/robust controllers, see among others Khalil (2002); Zubov and Boron (1964). Accordingly, whenever such control Lyapunov functions are used in optimization based strategies, these should be chosen such that the recursive feasibility and closed loop stability are all fulfilled. Different classes of control Lyapunov functions have been proposed in control theory. In the context of linear quadratic control, infinite/finite quadratic cost functions usually serve as control Lyapunov functions, as shown in Anderson and Moore (2007); Chmielewski and Manousiouthakis (1996). Particularly, in linear model predictive control, such a control Lyapunov function has been used to design robust controllers to cope with polytopic uncertainties, leading to a linear matrix inequality problem, see Kothare et al. (1996). Polyhedral control Lyapunov functions have also been exploited in several studies e.g. Gutman and Cwikel (1987); Blanchini (1995, 1994); Lazar (2010); Nguyen et al. (2015a), since they lead to simple design procedures; i.e. composed of linear constraints.

In the same line with the studies in Gutman and Cwikel (1987); Blanchini (1994); Nguyen (2014); Grammatico et al. (2014), this paper presents an attempt to use *convex liftings* in the design of robust controllers for discrete-time linear systems affected by bounded additive disturbances and polytopic uncertainties which can serve as control Lyapunov functions. This method is shown to guarantee the recursive feasibility and closed loop stability. In terms of implementation, this only requires resolution of a linear programming problem at each sampling instant.

2 Notation and Definitions

Throughout this paper, $\mathbb{N}, \mathbb{N}_{>0}, \mathbb{R}, \mathbb{R}_+$ denote the set of nonnegative integers, the set of positive integers, the set of real numbers and the set of nonnegative numbers, respectively. For ease of presentation, with a given $N \in \mathbb{N}_{>0}$, by $\mathcal{I}_N$, we denote the index set: $\mathcal{I}_N = \{i \in \mathbb{N}_{>0} \mid i \leq N\}$.

A polyhedron is the intersection of finitely many closed halfspaces. A polytope is a bounded polyhedron. If P is an arbitrary polytope, then by $\mathcal{V}(P)$, we denote the set of its vertices. If $\mathcal{S}$ is an arbitrary set, then $\mathrm{conv}(\mathcal{S})$ denotes the convex hull of $\mathcal{S}$. Also, for a full dimensional set $\mathcal{S}$, by $\mathrm{int}(S)$, we denote the interior of S. Further, we use $\dim(\mathcal{S})$ to denote the dimension of its affine hull.

Given a set $\mathcal{S} \subset \mathbb{R}^d$ and a matrix $A \in \mathbb{R}^{d \times d}$, then $A\mathcal{S}$ is defined as follows: $A\mathcal{S} = \{As \mid s \in \mathcal{S}\}$. Also, for any vector $x \in \mathbb{R}^d$, $\rho_{\mathcal{S}}(x)$ is defined as follows: $\rho_{\mathcal{S}}(x) = \min_{y \in \mathcal{S}} \sqrt{(y-x)^T(y-x)}$.

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Given two sets $\mathcal{S}_1, \mathcal{S}_2 \subset \mathbb{R}^d$, their Minkowski sum is denoted by $\mathcal{S}_1 \oplus \mathcal{S}_2$ and is defined by:

$$\mathcal{S}_1 \oplus \mathcal{S}_2 = \{ y_1 + y_2 \mid y_1 \in \mathcal{S}_1, y_2 \in \mathcal{S}_2 \} .$$

Also, $\mathcal{S}_1 \backslash \mathcal{S}_2$ is defined as follows:

$$\mathcal{S}_1 \backslash \mathcal{S}_2 := \left\{ x \in \mathbb{R}^d \mid x \in \mathcal{S}_1, \ x \notin \mathcal{S}_2 \right\} .$$

3 Problem settings

In this paper, we consider a discrete-time linear system:

$$x_{k+1} = A(k)x_k + B(k)u_k + w_k, \tag{1}$$

where x_k, u_k, w_k denote the state, control variables and additive disturbance at time k. The state space matrices $[A(k) \, B(k)]$ are time-varying and assumed to belong to an *uncertainty matrix polytope* denoted by Ψ and defined below:

$$[A(k) \, B(k)] \in \Psi = \operatorname{conv} \{ [A_1 \, B_1] , \dots, [A_L \, B_L] \} . \tag{2}$$

The state, control variables and disturbances are subject to constraints:

$$x_k \in \mathbb{X} \subset \mathbb{R}^{d_x}, \, u_k \in \mathbb{U} \subset \mathbb{R}^{d_u}, \, w_k \in \mathbb{W} \subset \mathbb{R}^{d_x}, \tag{3}$$

where $d_x, d_u \in \mathbb{N}_{>0}$, and $\mathbb{X}, \mathbb{U}, \mathbb{W}$ are polytopes containing the origin in their interior.

The objective is to find robust control laws which can cope with bounded additive disturbances and polytopic model uncertainties such that the closed loop is robustly stable. It is clear that if w_k is unknown, one cannot expect to guarantee asymptotic stability of the origin. In this case, asymptotic stability is replaced with an ultimate boundedness concept Khalil (2002); Kofman et al. (2007) or input to state stability Jiang and Wang (2001).

4 Robust control design based on convex liftings

4.1 Robust positively invariant sets

Positively invariant sets have been studied over several decades. Due to their relevance in control theory, they turn out to be useful in many control related studies e.g. Bitsoris (1988b,a); Bitsoris and Vassilaki (1995); Blanchini and Miani (2007); Kerrigan (2001). The definition of a robust positively invariant set for system (1) is recalled below.

Definition 4.1 Given an admissible control law $u_k = Kx_k \in \mathbb{U}$, a set $\Omega \subseteq \mathbb{X}$ is called *robust positively invariant* with respect to (1) if

$$(A(k) + B(k)K)\Omega \oplus \mathbb{W} \subseteq \Omega, \ \forall [A(k) \, B(k)] \in \Psi,$$

where Ψ is defined in (2).

To compute such a robust positively invariant set Ω, it is important to choose an appropriate unconstrained control law to cope with given bounded additive disturbances and polytopic uncertainties. In case polytopic uncertainties are not taken into account, such a control law $u_k = Kx_k$ can be computed from the Ricatti equation for some positive definite weighting matrices Q, R in the classical linear quadratic control design.

Otherwise, this control law should satisfy that there exists a Lyapunov function $V(x) : \mathbb{R}^{d_x} \to \mathbb{R}_+$ such that

$$V((A(k) + B(k)K)x_k) - V(x_k) < 0, \ \forall [A(k) \, B(k)] \in \Psi.$$

The computation of such a gain K was studied in e.g. Kothare et al. (1996). A simpler formulation is presented below:

$$\min_{Z,Y} -\operatorname{logdet}(Z)$$

subject to

$$Z = Z^T > 0$$

$$\begin{bmatrix} Z & (A_i Z + B_i Y)^T \\ A_i Z + B_i Y & Z \end{bmatrix} > 0, \ \forall \, i \in \mathcal{I}_L.$$

Then, gain K is determined by

$$K = YZ^{-1}.$$

It is already known that the above formulation is a linear matrix inequality (LMI) problem and is solvable by using semidefinite programming. Interested readers can find details in Boyd et al. (1994).

With respect to the state feedback $u_k = Kx_k$, the computation of a robust positively invariant set Ω for system (1) has been put forward in Nguyen (2014), as a simple extension of the idea presented in Gilbert and Tan (1991). Note also that prominent studies on the computation of the maximal and minimal positively invariant sets for a linear, discrete-time invariant system affected by bounded additive disturbances can be found in Kolmanovsky and Gilbert (1998); Rakovic et al. (2005). Still, in case system (1) is not affected by additive disturbances, then the minimal robust positively invariant set coincides with the origin due to its asymptotic stability i.e. $\Omega = \{0\}$.

Without loss of generality, we are interested hereafter in the case $\Omega \subseteq \mathbb{X} \subset \mathbb{R}^{d_x}$ represents a full-dimensional set.

4.2 Domain of attraction

Given a robust positively invariant set Ω associated with an admissible state feedback $u = Kx \in \mathbb{U}$ for all $x \in \Omega$, the domain of attraction is defined as the set of all points in

$\mathbb{X}$ which can be driven to Ω, see Khalil (2002). More precisely, the domain of attraction contains all points $x_0 \in \mathbb{X}$ such that there always exists control law satisfying constraints (3) which is able to steer the state to Ω as $k \to \infty$ i.e. $\lim_{k \to \infty} \rho_\Omega(x_k) = 0$. Computing exactly the domain of attraction is difficult. Instead, approximation of the domain of attraction is usually of use. For simplicity, in this paper, we restrict our attention to a contractive set. The definition of a contractive set for system (1) is recalled in the sequel.

Definition 4.2 Consider system (1) subject to model uncertainty (2) and constraints (3). A set $\mathcal{X} \subseteq \mathbb{X}$ is called $\lambda-contractive$ for a given $0 \le \lambda < 1$ if there exists a control law $u_k = \kappa(x_k) \in \mathbb{U}$ such that

$$(A(k)x_k + B(k)\kappa(x_k)) \oplus \mathbb{W} \subseteq \lambda \mathcal{X},$$
$$\forall x_k \in \mathcal{X}, \ \forall [A(k) \ B(k)] \in \Psi.$$

The *maximal $\lambda-contractive$ set*, denoted as P_λ, is defined as the set containing all $\lambda-contractive$ sets. An algorithm for the computation of the maximal $\lambda-contractive$ set has been put forward in Blanchini (1994). For completeness, this algorithm is recalled below.

$$\begin{aligned}
S_1 &= \mathbb{X}, \\
S_{i+1} &= \{x \in S_i \mid \exists u(x) \in \mathbb{U} \ \text{s.t.} \\
&\quad (A_j x + B_j u(x)) \oplus \mathbb{W} \subseteq \lambda S_i, \ \forall j \in \mathcal{I}_L\}, \\
P_\lambda &= S_\infty.
\end{aligned} \quad (4)$$

Hereafter, we will use the maximal $\lambda-contractive$ set as an estimation of the domain of attraction for a given $0 \le \lambda < 1$ i.e. $\mathcal{X} = P_\lambda \subseteq \mathbf{X}$. Without loss of generality, we assume that $\Omega \subset P_\lambda$.

4.3 Convex liftings construction

In control theory, convex liftings have been of use to facilitate implementation of piecewise affine control laws Baotic et al. (2008). Recently, they have been of use to solve the inverse parametric linear/quadratic programming problems Nguyen et al. (2014b,a, 2015b,c,d). In this paper, we will show that such convex liftings can also serve as control Lyapunov functions. Before recalling the definition of a convex lifting, additional definitions need to be recalled.

Definition 4.3 A collection of N full-dimensional polyhedra $\mathcal{X}_i \subset \mathbb{R}^{d_x}$, denoted by $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$, is called a *polyhedral partition of a polyhedron* $\mathcal{X} \subseteq \mathbb{R}^{d_x}$ if the following conditions hold:

- $\bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i = \mathcal{X}$,
- $\text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{X}_j) = \emptyset, \ \forall(i,j) \in \mathcal{I}_N^2, \ i \ne j$.

Two regions $\mathcal{X}_i, \mathcal{X}_j$ are called *neighboring* or *adjacent* if $i \ne j$, $(i,j) \in \mathcal{I}_N^2$, $\dim(\mathcal{X}_i \cap \mathcal{X}_j) = d_x - 1$. Further, if $\mathcal{X}$ is a polytope, then $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ is called a *polytopic partition*.

Definition 4.4 Given a polyhedral partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^d$, a *piecewise affine lifting* is described by function $z : \mathcal{X} \to \mathbb{R}$ with:

$$z(x) = a_i^T x + b_i \ \text{for any} \ x \in \mathcal{X}_i, \quad (5)$$

and $a_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$, $\forall i \in \mathcal{I}_N$.

Definition 4.5 Given a polyhedral partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subseteq \mathbb{R}^d$, a piecewise affine lifting $z(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$, is called *convex piecewise affine lifting* if the following conditions hold true:

- $z(x)$ is continuous over $\mathcal{X}$,
- for each $i \in \mathcal{I}_N$, $z(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \backslash \mathcal{X}_j$ and all $j \ne i$, $j \in \mathcal{I}_N$.

Note that the second condition in this definition implies that any pair of neighboring regions are lifted onto two distinct hyperplanes. Also, it implies the convexity of this piecewise affine lifting. For ease of presentation, a slight abuse of notation is used hereafter: a *convex lifting* will be understood as a convex piecewise affine lifting.

We present now an algorithm to construct a class of convex liftings which will be of use later in the proposed robust control design. Let $\ell(x)$ denote this convex lifting defined over an estimation of the domain of attraction $\mathcal{X}$. As discussed in Subsection 4.2, we restrict our attention to the maximal $\lambda-contractive$ set P_λ for a given $0 \le \lambda < 1$ i.e. $\mathcal{X} = P_\lambda$.

Algorithm 1 Construct a control Lyapunov function

Input: A given robust positively invariant set $\Omega \subset \mathbb{R}^{d_x}$, an estimation of the domain of attraction $\mathcal{X} = P_\lambda \subset \mathbb{R}^{d_x}$ with a given $0 \le \lambda < 1$ and a scalar $c > 0$.
Output: A convex lifting $\ell(x)$ such that $\ell(x) = 0$ for every $x \in \Omega$.

1: $V_1 = \mathcal{V}(\Omega)$, $\widehat{V}_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in V_1 \right\} \subset \mathbb{R}^{d_x+1}$.

2: $V_2 = \mathcal{V}(\mathcal{X})$, $\widehat{V}_2 = \left\{ \begin{bmatrix} x \\ c \end{bmatrix} \mid x \in V_2 \right\} \subset \mathbb{R}^{d_x+1}$.

3: $\Pi = \text{conv}(\widehat{V}_1 \bigcup \widehat{V}_2)$.
4: Solve the parametric linear programming problem:

$$z^*(x) = \min_z z \ \text{s.t.} \ \begin{bmatrix} x^T & z \end{bmatrix}^T \in \Pi. \quad (6)$$

5: $\ell(x) = z^*(x)$.

Steps 1-2 in Algorithm 1 aim to lift the vertices of Ω and $\mathcal{X}$ to $\mathbb{R}^{d_x+1}$ with appropriate heights. Namely, the vertices of Ω are lifted with heights equal to 0, whereas the vertices of $\mathcal{X}$ are lifted with heights equal to the given $c > 0$. Note that

(6) is a parametric linear programming problem, its optimal solution is thus a piecewise affine function defined over a polytopic partition denoted as follows: $\ell(x) = z^*(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$. Note also that by construction, there exists a region in the partition associated with $\ell(x)$ which coincides with Ω, since the vertices of Ω are lifted onto a lower facet of Π. The following observation describes the properties of such an $\ell(x)$, generated from Algorithm 1.

Lemma 4.6 *The function $\ell(x)$ over $\mathcal{X}$, generated from Algorithm 1, is continuous, convex, piecewise affine function.*

PROOF. $\ell(x)$ is a piecewise affine function since it is induced from a parametric linear programming problem. The continuity and convexity of $\ell(x)$ can be easily derived from Theorems IV.3 and IV.4 in Gal (1995). □

Lemma 4.7 *The function $\ell(x)$ over $\mathcal{X}$, generated from Algorithm 1, is a convex lifting over the associated partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$.*

PROOF. To prove that $\ell(x)$ is a convex lifting for $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$, we need to prove that for any pair of $(\mathcal{X}_i, \mathcal{X}_j)$ the associated optimal solutions are different i.e. $(a_i, b_i) \neq (a_j, b_j)$. Suppose the converse situation happens, more precisely, there exist two regions $(\mathcal{X}_i, \mathcal{X}_j)$ such that $(a_i, b_i) = (a_j, b_j)$.

First, it can be easily seen that the optimal solution to the parametric linear programming problem (6) is unique. In fact, suppose there exist two different optimal solutions to (6) i.e. $z_1^*(x)$ and $z_2^*(x)$. Consider a region $\mathcal{X}_i$ in the associated partition over which $z_1^*(x), z_2^*(x)$ are defined i.e. $z_1^*(x) = (a_i^{(1)})^T x + b_i^{(1)}$, $z_2^*(x) = (a_i^{(2)})^T x + b_i^{(2)}$. Since z is the cost function of (6), therefore, we obtain:

$$(a_i^{(1)})^T x + b_i^{(1)} = (a_i^{(2)})^T x + b_i^{(2)} \quad \text{for all} \ \ x \in \mathcal{X}_i. \quad (7)$$

Note that the set of all x satisfying (7) describes a set of dimension lower than d_x, whereas (7) also holds true for all $x \in \mathcal{X}_i$ as a full dimensional polyhedron. This case only holds if $(a_i^{(1)}, b_i^{(1)}) = (a_i^{(2)}, b_i^{(2)})$. This leads to the uniqueness of the optimal solution to (6).

Consider now two regions $(\mathcal{X}_i, \mathcal{X}_j)$ such that $(a_i, b_i) = (a_j, b_j)$. Let the optimization problem (6) be written in the following form:

$$\min_z z \ \ \text{s.t.} \ \ Gz \leq W + Ex. \quad (8)$$

Without loss of generality, the constraint set of (8) is assumed to be in minimal representation. Also, suppose the constraints active at $\begin{bmatrix} x^T & a_i^T x + b_i \end{bmatrix}^T$ and $\begin{bmatrix} x^T & a_j^T x + b_j \end{bmatrix}^T$ are respectively as follows:

$$G^{(i)} z = W^{(i)} + E^{(i)} x$$
$$G^{(j)} z = W^{(j)} + E^{(j)} x.$$

According to the uniqueness of the optimal solution to (8), $G^{(i)}, G^{(j)} \in \mathbb{R} \setminus \{0\}$. Also, since $\mathcal{X}_i \neq \mathcal{X}_j$, thus $G^{(j)} z \leq W^{(j)} + E^{(j)} x$ is not active at $\begin{bmatrix} x^T & a_i^T x + b_i \end{bmatrix}^T$ for $x \in \mathcal{X}_j$; more precisely

$$G^{(j)}(a_i^T x + b_i) < W^{(j)} + E^{(j)} x. \quad (9)$$

However, as assumed $(a_i, b_i) = (a_j, b_j)$, then $G^{(j)} z \leq W^{(j)} + E^{(j)} x$ becomes active at $\begin{bmatrix} x^T & a_i^T x + b_i \end{bmatrix}^T$ for $x \in \mathcal{X}_j$; namely,

$$G^{(j)}(a_i^T x + b_i) = W^{(j)} + E^{(j)} x. \quad (10)$$

Inclusions (9) and (10) are clearly contradictory. In other words, for any pair of different regions $(\mathcal{X}_i, \mathcal{X}_j)$, the optimal solution to (6) i.e. $\ell(x)$ satisfies $(a_i, b_i) \neq (a_j, b_j)$.

Additionally, Lemma 4.6 shows that $\ell(x)$ is a continuous, convex, piecewise affine function. Therefore, $\ell(x)$ is a convex lifting for $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ according to Definition 4.5. □

Lemma 4.8 *The function $\ell(x)$ over $\mathcal{X}$, generated from Algorithm 1, satisfies $\ell(x) = 0$ for every $x \in \Omega$ and $\ell(x) > 0$ for all $x \in \mathcal{X} \setminus \Omega$.*

PROOF. Indeed, consider $x \in \Omega$, then x can be written as a convex combination of the vertices of Ω as: $x = \sum_{v \in \mathcal{V}(\Omega)} \alpha(v) v$ with $\alpha(v) \geq 0$ and $\sum_{v \in \mathcal{V}(\Omega)} \alpha(v) = 1$. It is known that $\ell(x)$ over Ω is an affine function, then $\ell(x) = a_i^T x + b_i$ leads to $\ell(x) = 0$ for every $x \in \Omega$.

To complete the proof, it is necessary to show that $\ell(x) > 0$ for $x \in \mathcal{X} \setminus \Omega$. Indeed, as shown above, $\ell(x) = a_i^T x + b_i = 0$ for every $x \in \Omega$, then since Ω is full-dimensional, it follows $a_i = 0$, $b_i = 0$. Consider a region $\mathcal{X}_j \neq \Omega = \mathcal{X}_i$ of the polytopic partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ associated with $\ell(x)$, $\ell(x) = a_j^T x + b_j$ for every $x \in \mathcal{X}_j$. According to Lemma 4.7, $\ell(x)$ satisfies the convexity and continuity conditions of a convex lifting:

$$a_j^T x + b_j > a_i^T x + b_i = 0, \ \ \text{for every} \ \ x \in \mathcal{X}_j \setminus \mathcal{X}_i,$$
$$a_j^T x + b_j = a_i^T x + b_i = 0, \ \ \text{for every} \ \ x \in \mathcal{X}_j \cap \mathcal{X}_i.$$

The same inclusion for the other affine functions of $\ell(x)$, leads to the non-negativity of $\ell(x)$. Moreover, $\ell(x) > 0$ for every $x \in \mathcal{X} \setminus \Omega$. The proof is complete. □

Another property of $\ell(x)$ is presented as follows.

Lemma 4.9 *For any $x \in \mathcal{X}$ and $0 \leq \beta \leq 1$, $\ell(\beta x) \leq \beta \ell(x)$.*

PROOF. Due to the convexity of $\ell(x)$ over $\mathcal{X}$ as proved in Lemma 4.6, it leads to

$$\ell(\beta x + (1 - \beta)0) \leq \beta \ell(x) + (1 - \beta)\ell(0).$$

Due to the assumption that $0 \in \text{int}(\mathbb{W})$, then $0 \in \text{int}(\Omega)$, meaning that $\ell(0) = 0$ according to Lemma 4.8. This inclusion and the above one imply that $\ell(\beta x) \leq \beta \ell(x)$. $\quad\square$

4.4 Robust control design procedure

This subsection introduces the procedure for designing robust control laws based on convex liftings. This procedure can guarantee robust stability of the closed loop in the sense of Lyapunov. Therefore, a definition of this robust stability is recalled below.

Definition 4.10 Given a robust positively invariant set Ω and the domain of attraction $\mathcal{X} \subseteq \mathbb{X}$, consider the linear system (1) subject to constraints (3) and a control law $u = \kappa(x) \in \mathbb{U}$. The closed loop is called *robustly stable* if there exists a Lyapunov function $V(x) : \mathcal{X} \to \mathbb{R}_+$ and an $\alpha \in [0, 1)$ such that:

- $V(x) = 0$ for all $x \in \Omega$, $V(x) > 0$ for all $x \in \mathcal{X}\backslash\Omega$,
- $V(A(k)x_k + B(k)\kappa(x_k) + w_k) - \alpha V(x_k) \leq 0, \forall\, w_k \in \mathbb{W}$, $\forall\, x_k \in \mathcal{X}\backslash\Omega$ and $\forall\, [A(k)\ B(k)] \in \Psi$.

For robust design based on control Lyapunov function, it is important to find such a control Lyapunov function and use it for design procedure. Our design procedure based on a convex lifting, computed from Algorithm 1, is summarized in Algorithm 2.

Remark 4.11 Note that the task of verifying whether or not x_k belongs to Ω in Step 2 of Algorithm 2, can be easily carried out by checking whether or not $\ell(x_k) = 0$. This property is due to the construction of a convex lifting from Algorithm 1. Therefore, it is not necessary to store the constraints describing Ω in the implementation.

Natural questions arise here whether or not the linear programming problem (11) is feasible and whether closed loop stability is guaranteed by the proposed procedure. These questions are answered via the following theorem. Accordingly, it will be shown that convex lifting constructed in Algorithm 1 can serve as a Lyapunov function. Thus, the proposed control design can guarantee the robust stability as per Definition 4.10.

Algorithm 2 Robust control design procedure based on convex liftings

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Input: A robust positively invariant set Ω associated with a stabilizing control law $u = Kx$ over Ω. A convex lifting $\ell(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$, $i \in \mathcal{I}_N$ as in Algorithm 1.
Output: Control law $u^*(x_k)$ at each sampling time.
1: Compute $\ell(x_k)$.
2: **If** $x_k \in \Omega$ then $u^*(x_k) = Kx_k$, jump to Step 6.
3: **Else** Solve the following linear programming problem:

$$\left[\alpha^*\ (u_k^*)^T\right]^T = \arg\min_{\alpha,\, u_k} \alpha$$

$$\text{s.t. } a_i^T(A_j x_k + B_j u_k + w) + b_i \leq \alpha \ell(x_k) \quad (11)$$
$$\alpha \geq 0,\ u_k \in \mathbb{U},\ \forall i \in \mathcal{I}_N, \forall w \in \mathcal{V}(\mathbb{W}),$$
$$\forall\, [A_j\ B_j] \in \mathcal{V}(\Psi).$$

4:      Apply $u^*(x_k) = u_k^*$
5: **End**
6: $k \leftarrow k + 1$, return to Step 1.

Theorem 4.12 *Given a robust positively invariant set Ω associated with a robust control law gain K and an estimation of the domain of attraction $\mathcal{X} = P_\lambda$ for a given $0 \leq \lambda < 1$, if the initial condition $x_k \in \mathcal{X}$, then the linear programming problem (11) is recursively feasible. Furthermore, the closed loop is robustly stable in the sense of Lyapunov.*

PROOF. As for the feasibility of (11), one can easily see that $0 \leq \ell(x) \leq c$ by the construction in Algorithm 1. Therefore, due to the contractivity of $\mathcal{X}$, for any $x_k \in \mathcal{X}$ there always exists $u(x_k) \in \mathbb{U}$ such that:

$$A(k)x_k + B(k)u(x_k) + w_k \in \lambda\mathcal{X} \subset \mathcal{X}$$

for all $w_k \in \mathbb{W}$ and for all $[A(k)\ B(k)] \in \Psi$. Therefore, if $u^*(x_k)$ denotes an optimal solution to (11), then one has:

$$\begin{aligned} 0 \leq\ & \ell(A(k)x_k + B(k)u^*(x_k) + w_k) \\ \leq\ & \ell(A(k)x_k + B(k)u(x_k) + w_k) \\ \leq\ & c, \quad \forall\, w_k \in \mathbb{W},\ \forall\, [A(k)\ B(k)] \in \Psi. \end{aligned}$$

Due to this boundedness, the recursive feasibility of the linear programming problem (11) is ensured for a finite, large enough scalar α at each sampling time.

As for robust stability, it will be proved that for all $x_k \in \mathcal{X}\backslash\Omega$:

$$\ell(A(k)x_k + B(k)u^*(x_k) + w_k) < \ell(x_k),$$
$$\forall w_k \in \mathbb{W},\ \forall\, [A(k)\ B(k)] \in \Psi.$$

Indeed, due to the contractivity of $\mathcal{X}$, for any $v \in \mathcal{V}(\mathcal{X})$, there exists a control law, denoted by $u(v) \in \mathbb{U}$ such that $A(k)v + B(k)u(v) + w_k \in \lambda\mathcal{X}$ despite any disturbances $w_k \in \mathbb{W}$ and for all $[A(k)\ B(k)] \in \Psi$. For each $w_k \in \mathbb{W}$

and each $[A(k)\ B(k)] \in \Psi$, there exists $y(k, w_k) \in \mathcal{X}$ such that

$$A(k)v + B(k)u(v) + w_k = \lambda y(k, w_k).$$

Due to Lemma 4.9, this inclusion leads to

$$\ell(A(k)v + B(k)u(v) + w_k) = \ell(\lambda y(k, w_k)) \qquad (12)$$
$$\leq \lambda \ell(y(k, w_k)).$$

By the construction of $\ell(x)$ in Algorithm 1, the following is obtained:

$$\ell(y(k, w_k)) \leq c. \qquad (13)$$

Also, according to Algorithm 1,

$$\ell(v) = c. \qquad (14)$$

From (12), (13), (14), one can deduce that

$$\ell(A(k)v + B(k)u(v) + w_k) \leq \lambda \ell(v). \qquad (15)$$

Note that (15) holds for all $w_k \in \mathbb{W}$ and for all $[A(k)\ B(k)] \in \Psi$. Moreover, it can be observed that:

$$\ell(A(k)v + B(k)u^*(v) + w_k) \leq \ell(A(k)v + B(k)u(v) + w_k),$$
$$\forall w_k \in \mathbb{W}, \forall [A(k)\ B(k)] \in \Psi, \qquad (16)$$

where $u^*(x)$ denotes optimal control to (11) at x as used in Algorithm 2. (15) and (16) lead to the following fact:

$$\ell(A(k)v + B(k)u^*(v) + w_k) \leq \lambda \ell(v),$$
$$\forall w_k \in \mathbb{W}, \forall [A(k)\ B(k)] \in \Psi. \qquad (17)$$

Note that (17) holds true for all vertices of $\mathcal{X}$. Now, consider a point $x_k \in \mathcal{X}_i$ in the polytopic partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}_N}$ of $\mathcal{X}$ over which $\ell(x)$ is defined. Without loss of generality, suppose $\mathcal{X}_i \neq \Omega$, then x_k can be described via a convex combination of the vertices of $\mathcal{X}_i$, meaning:

$$x_k = \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)v, \text{ where } \alpha(v) \in \mathbb{R}_+, \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v) = 1.$$

Recall that due to the definition of convex lifting, $\ell(x)$ over $\mathcal{X}_i$ is an affine function, then $\ell(x_k)$ can be written in the following form:

$$\ell(x_k) = \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)\ell(v). \qquad (18)$$

If $v \in \mathcal{V}(\mathcal{X}_i)$ is a vertex of Ω, then due to the robust positive invariance of Ω with respect to a linear feedback $u^*(x) = Kx$, it satisfies

$$\ell(v) = 0 = \ell((A(k) + B(k)K)v + w_k),$$
$$\forall w_k \in \mathbb{W}, \forall [A(k)\ B(k)] \in \Psi. \qquad (19)$$

Otherwise, if $v \in \mathcal{V}(\mathcal{X}_i)$ is a vertex of $\mathcal{X}$, then it satisfies (17). Therefore, due to the convexity of $\ell(x)$ proved in Lemma 4.6 and (17), (18), (19), the following is obtained:

$$\lambda \ell(x_k) = \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)(\lambda \ell(v))$$
$$\geq \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)\ell(A(k)v + B(k)u^*(v) + w_k)$$
$$\geq \ell\left(A(k) \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)v + B(k) \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)u^*(v) + w_k\right)$$
$$= \ell\left(A(k)x_k + B(k) \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)u^*(v) + w_k\right). \qquad (20)$$

Recall that $u^*(v) \in \mathbb{U}, \forall v \in \mathcal{V}(\mathcal{X}_i) \cap \mathcal{V}(\mathcal{X})$ and $u^*(v) = Kv \in \mathbb{U}, \forall v \in \mathcal{V}(\mathcal{X}_i) \cap \mathcal{V}(\Omega)$, then it follows that

$$\sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)u^*(v) \in \mathbb{U}. \qquad (21)$$

Therefore, (21) leads to:

$$\ell\left(A(k)x_k + B(k) \sum_{v \in \mathcal{V}(\mathcal{X}_i)} \alpha(v)u^*(v) + w_k\right) \qquad (22)$$
$$\geq \ell(A(k)x_k + B(k)u^*(x_k) + w_k).$$

From (20) and (22), the following inclusion can be obtained:

$$\lambda \ell(x_k) \geq \ell(A(k)x_k + B(k)u^*(x_k) + w_k),$$
$$\forall w_k \in \mathbb{W}, \forall [A(k)\ B(k)] \in \Psi. \qquad (23)$$

Recall that $0 \leq \lambda < 1$, therefore

$$\ell(x_k) > \ell(A(k)x_k + B(k)u^*(x_k) + w_k),$$
$$\forall w_k \in \mathbb{W}, \forall [A(k)\ B(k)] \in \Psi, \qquad (24)$$

meaning $\{\ell(x_k)\}_{k=0}^{\infty}$ is a strictly decreasing sequence outside Ω and bounded in the interval $[0, c]$. Thus, this sequence is convergent to 0. In other words, $\ell(x)$ serves as a Lyapunov function according to Definition 4.10. $\square$

Remark 4.13 Note that by construction, the partition associated with a convex lifting in Algorithm 1, may not be a Delaunay decomposition as in Scibilia et al. (2009). This method does not rely on such a decomposition, but relies on a convex lifting defined over this partition. This approach is simple and only requires solving a linear programming problem at each sampling instant. However, the associated control law is not continuous at the moment the state switches into Ω (see step 2 of Algorithm 2). Note also that the checking whether the current state belongs to Ω can be relaxed. Accordingly, one can continue solving the problem (11) while trajectories still stay inside Ω. Indeed, if $x_k \in \Omega$, then due

to the construction $\ell(x_k) = 0$. Consider the next state, one can see that $Kx_k \in \mathbb{U}$, then it leads to:

$$0 \le \ell(A(k)x_k + B(k)u^*(x_k) + w_k)$$
$$\le \ell(A(k)x_k + B(k)Kx_k + w_k) = 0 = \ell(x_k).$$

This inclusion implies that optimal control law $u^*(x_k) \in \mathbb{U}$ to problem (11) also keeps the trajectories inside Ω, if x_k is inside Ω.

Remark 4.14 An open problem is to guarantee robust stability of the proposed method for another estimation of the domain of attraction as the $N-$*steps robust controllable set* denoted by $\mathcal{K}_N(\Omega)$ c.f. Kerrigan (2001). Note that in this case, proving the strict decrease of $\ell(x)$ becomes more difficult. Also, this strict decrease may not be successive.

Remark 4.15 Note that the explicit robust controller of (11) can be obtained by replacing $\alpha\ell(x_k)$ with a variable, denoted by e.g. z. Accordingly, the optimization problem (11) becomes a parametric linear programming problem with the decision argument to be $\begin{bmatrix} z & u_k^T \end{bmatrix}^T$ and the parameter as the current state x_k.

5 Numerical examples

To illustrate the proposed procedure, consider the following uncertain system:

$$x_{k+1} = A(k)x_k + B(k)u_k,$$

where

$$[A(k)\ B(k)] \in \text{conv} \left\{ \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \end{bmatrix}, \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix} \right\},$$

$$[A(k)\ B(k)] = \beta_k \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \end{bmatrix} + \gamma_k \begin{bmatrix} 1 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

and $\beta_k = \sin^2(p_k), \gamma_k = \cos^2(p_k)$, p_k represents a random scalar variable at time k. The present state and control variables are subject to the following constraints:

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x_k \le \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad -5 \le u_k \le 5.$$

An unconstrained controller is chosen as follows:

$$u = \begin{bmatrix} -3.2827 & -4.6780 \end{bmatrix} x.$$

Accordingly, the maximal robust positively invariant set associated with the above controller, i.e. Ω is shown in Fig. 1.

Also, the maximal $0.99-$contractive set $P_{0.99}$ is presented therein. This set is computed from procedure (4). A convex lifting $\ell(x)$, serving later as a control Lyapunov function, is visualized in Fig. 2 according to Algorithm 1 with $c = 10$. The closed loop trajectories are shown in Fig. 3 to be convergent to the origin, since the unconstrained control law can cope with the given set of polytopic uncertainties over Ω. Finally, the strict decrease of $\ell(x_k)$ over $\mathcal{X}\backslash\Omega$, is illustrated in Fig. 4. The numerical example of this paper has
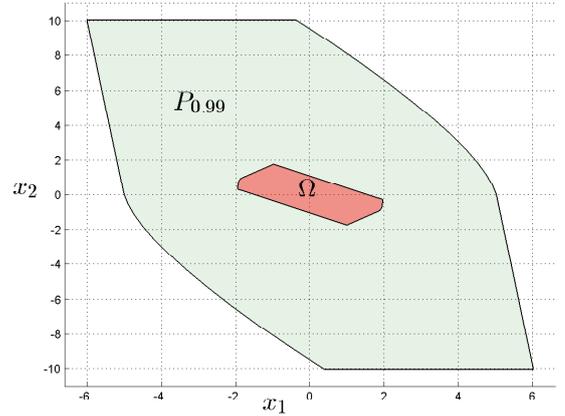

Fig. 1. The maximal robust positively invariant set Ω and an estimation of the domain of attraction $\mathcal{X} = P_{0.99}$.
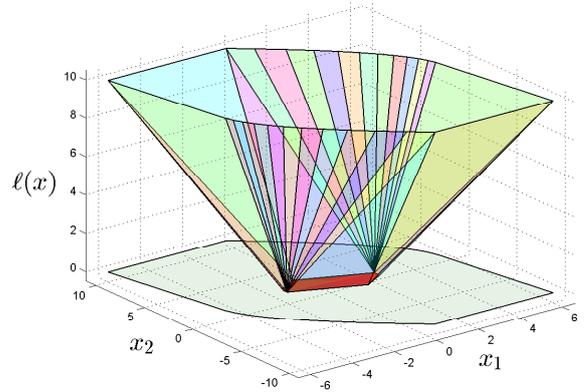

Fig. 2. A convex lifting $\ell(x)$ constructed by Algorithm 1 with $c = 10$.

been simulated in the environment of MPT 3.0 Herceg et al. (2013).

6 Conclusions

This paper presented a new method to design robust control law for constrained linear systems affected by bounded additive disturbances and polytopic uncertainties. This method was based on convex liftings. It was shown to guarantee the
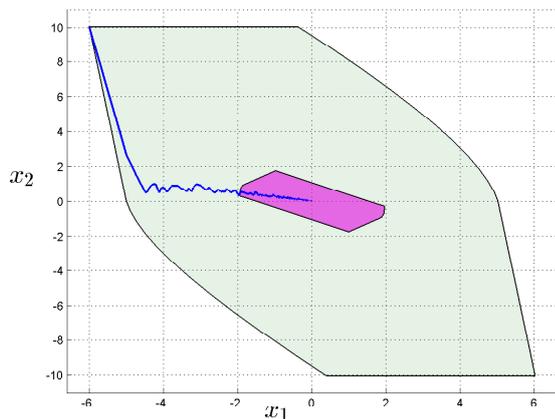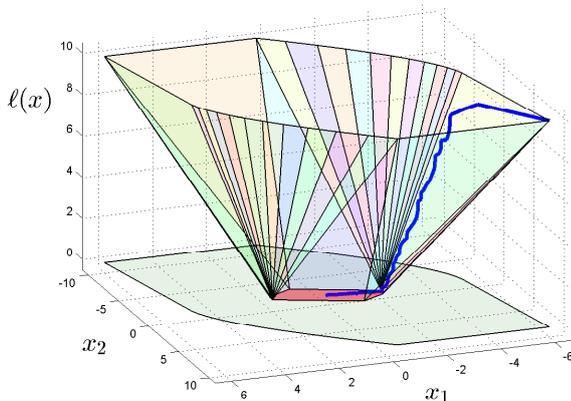
Fig. 3. Closed loop stability.

Fig. 4. The strict decrease of $\ell(x)$ over $\mathcal{X}\setminus\Omega$ along the state.

recursive feasibility and also robust stability in the sense of Lyapunov.

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