HAL
archives-ouvertes.fr

Incremental stability of spatiotemporal delayed dynamics and application to neural fields

Georgios Detorakis, Antoine Chaillet

To cite this version:

Georgios Detorakis, Antoine Chaillet. Incremental stability of spatiotemporal delayed dynamics and application to neural fields. This document is an extended version of the eponym paper published at the IEEE Conference on Deci.. 2017. <hal-01587961>

HAL Id: hal-01587961

https://hal-centralesupelec.archives-ouvertes.fr/hal-01587961

Submitted on 14 Sep 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Incremental stability of spatiotemporal delayed dynamics and application to neural fields

Georgios Is. Detorakis and Antoine Chaillet

***Abstract*— We propose a Lyapounov-Krasovskii approach to study incremental stability in spatiotemporal dynamics with delays and their capability to be entrained by a periodic input. We then focus on delayed neural fields, which describe the spatiotemporal evolution of neuronal activity. We provide an explicit condition, involving the slope of the activation functions and the strength of coupling, under which delayed neural fields are incrementally stable regardless of communication delays. We finally show how this approach can be used to draw frequency profiles of neuronal populations.**

I. INTRODUCTION

Incremental stability characterizes systems whose solutions all converge to one another, with a transient overshoot "proportional" to the distance between initial states. It can be established through the study of an incremental Lyapunov function provided that its derivative along any two solutions of the system is negative as long as they do not coincide [2].

Incremental stability shares strong similarities with convergent [30], [10] and contractive [26], [17] systems. These similarities have been studied in details in [35]. A key feature that can be deduced from these properties is the ability of the system to be entrained by its input. More precisely, if any of these properties holds uniformly in the applied input then, under mild conditions, the response of the system to any T-periodic input is asymptotically T-periodic [26], [30], [36], [8], [27]. While this property is natural for asymptotically stable linear systems, it is far from granted when dynamics are nonlinear. More precisely, examples of systems that have a globally asymptotically stable equilibrium for each constant value of the input, but for which this entrainment property does not hold, can be found in [33], [36].

As observed in [31], the capacity of a system to be entrained by its input is the key ingredient from which one can derive frequency profiles. These take the form of magnitude Bode plots, reflecting the asymptotic amplification of the system in response to harmonic excitations at various frequencies. Unlike in the linear case, their shape may depend on the input amplitude.

These frequency profiles are especially attractive for neuroscience applications. Indeed, oscillations play a fundamental role in brain coordination [6] and altered oscillations in specific structures correlate with pathological symptoms, for instance in Parkinson's disease [21]. The possibility to predict which frequency band is preferably amplified by a given brain structure would therefore constitute a valuable tool for computational neuroscientists, and could lead to the development of stimulation strategies that aim at restoring healthy frequency profiles.

The spatiotemporal dynamics of brain oscillations can be captured by neural fields models [9]. These models take the form of integro-differential equations that abstract the brain structure to a continuous medium. Communication delays can be taken into account in this model, giving rise to delayed neural fields [16], [3], [38]. This feature is particularly relevant as the non-instantaneous communication between neurons, due to axonal propagation and synaptic delays, is believed to play a significant role in the onset and features of brain oscillations [28], [29], [24].

To the best of our knowledge, no systematic tool to guarantee incremental stability of neural fields (with or without delays) has yet been proposed. Incremental stability of delayed systems can be established using Lyapunov-Krasovskii [34] or Razumikhin [8], [12] approaches, but these results have not yet been extended to spatiotemporal dynamics.

Here, building up on the works [16], [2], we propose a Lyapunov-Krasovskii approach to establish incremental stability of spatiotemporal delayed dynamics and provide conditions under which it guarantees the entrainment property. We then focus on delayed neural fields to show that, under the stability condition derived in [16], the system happens to be also incrementally stable. An example motivated by the study of the brain circuitry involved in the motor symptoms of Parkinson's disease demonstrates the possibility to draw frequency profiles of the dynamics involved, which happen to exhibit a resonance in the frequency band associated to motor symptoms.

II. SPATIOTEMPORAL DELAYED DYNAMICS

Throughout this paper, we make use of the following notation. Ω denotes a compact subset of $\mathbb{R}^q$, $q \in \mathbb{N}_{\geq 1}$. Given $n \in \mathbb{N}_{\geq 1}$ and $\bar{d} \in \mathbb{R}_{\geq 0}$, $\mathcal{U}^n := C^0(\mathbb{R}_{\geq 0}, \mathcal{F}^n)$, $\mathcal{F}^n := L_2(\Omega, \mathbb{R}^n)$, $\mathcal{C}^n := C^0([-\bar{d}; 0], \mathcal{F}^n)$ if $\bar{d} > 0$, and $\mathcal{C}^n := \mathcal{F}^n$ if $\bar{d} = 0$. Accordingly, we defined the norms $\|\cdot\|_{\mathcal{F}}$ and $\|\cdot\|_{\mathcal{C}}$ as $\|x\|_{\mathcal{F}} := \sqrt{\int_\Omega x(r)^2 dr}$ for all $x \in \mathcal{F}^n$ and $\|x\|_{\mathcal{C}} := \sup_{t\in[-\bar{d};0]} \|x(t)\|_{\mathcal{F}}$ for all $x \in \mathcal{C}^n$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is increasing and satisfies $\alpha(0) = 0$. $\alpha \in \mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$. A continuous function

This work is supported by a public grant overseen by the French National Research Agency (ANR) as part of the "Investissement d'Avenir" program, through the "iCODE Institute project" funded by the IDEX Paris-Saclay, ANR-11-IDEX-0003-02, and by the ANR JCJC project SynchNeuro.

G. Detorakis is with Department of Cognitive Sciences, UC Irvine, USA gdetorak@uci.edu

A. Chaillet is with Laboratoire des Signaux et Systèmes, CentraleSupélec, Univ. Paris Saclay, Gif sur Yvette, France and with Institut Universitaire de France antoine.chaillet@centralesupelec.fr

$\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for any $t \in \mathbb{R}_{\geq 0}$ and, given any $s \in \mathbb{R}_{\geq 0}$, $\beta(s, \cdot)$ is non-increasing and tends to zero as its argument tends to infinity.

A. Definition

We consider dynamical systems defined as

$$\dot{x}(t) = f(x_t, u(t)), \tag{1}$$

where $f : \mathcal{C}^n \times \mathcal{F}^m \to \mathcal{F}^n$ and $u \in \mathcal{U}^m$. $x(t) \in \mathcal{F}^n$ represents the instantaneous value of the state: at each time instant t, it is a *function* of the space variable rather than a single point of $\mathbb{R}^n$. $x_t \in \mathcal{C}^n$ represents the history of this function over the latest time interval of length $\bar{d}$; in other words, for each fixed $\theta \in [-\bar{d}; 0]$, $x_t(\theta) := x(t + \theta)$. Given any $t \in \mathbb{R}_{\geq 0}$, $\dot{x}(t) \in \mathcal{F}^n$ is the function defined as $r \mapsto \frac{\partial}{\partial t}\big([x(t)](r)\big)$. System (1) is thus rules by a differential equation over the Banach space $\mathcal{C}^n$. From now on, unless explicitly stated, $\mathcal{C}^n$, $\mathcal{F}^n$, and $\mathcal{U}^m$ will be simply written as $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{U}$. Throughout the paper, we assume the following.

Assumption 1 (Uniform regularity) *f is locally Lipschitz and there exists $\rho \in \mathcal{K}_\infty$ such that, for all $\phi, \varphi \in \mathcal{C}$ and all $v \in \mathcal{F}$, $\|f(\phi, v) - f(\varphi, v)\|_{\mathcal{F}} \leq \rho(\|\phi - \varphi\|_{\mathcal{C}})$.*

We stress that Assumption 1 ensures that f maps bounded sets of $\mathcal{C} \times \mathcal{F}$ to bounded sets of $\mathcal{F}$. In other words, f is completely continuous.

Uniform incremental stability was originally studied in a finite-dimensional context [2]. A natural extension for the class of systems (1) is as follows.

Definition 1 (δGAS) *The system* (1) *is* uniformly incrementally stable (δGAS) *if it is forward complete and there exists $\beta \in \mathcal{KL}$ such that, for any pair of initial conditions $x_0, y_0 \in \mathcal{C}$ and any input $u \in \mathcal{U}$, the corresponding solutions of (1) satisfy, for all $t \in \mathbb{R}_{\geq 0}$,*

$$\|x(t; x_0, u) - x(t; y_0, u)\|_{\mathcal{F}} \leq \beta(\|x_0 - y_0\|_{\mathcal{C}}, t). \tag{2}$$

Similarly to incremental stability for finite-dimensional systems [2] and time-delay systems [8], [12], δGAS imposes that any two solutions with the same applied input eventually converge to one another at a rate uniform in the applied input, and that the maximum distance between them in the transients is small if the distance between initial states is small. Similar ingredients are present in the input-output formulation of incremental stability [19], [18], [37].

B. Krasovskii-Lyapunov approach for incremental stability

Given $V : \mathcal{C} \to \mathbb{R}$ locally Lipschitz, we indicate its upper-right Dini derivative along solutions of (1) by

$$\dot{V}^{(1)}(t, \phi, u) := \limsup_{h \to 0^+} \frac{V(x_{t+h}) - V(x_t)}{h},$$

on the interval of existence of the solution $x(\cdot)$ of (1) with initial state $\phi \in \mathcal{C}$ and input $u \in \mathcal{U}$. As discussed in [22], Assumption 1 and the fact that all considered inputs are continuous in time ensure absolute continuity of the map $t \mapsto x_t$. Consequently, the map $t \mapsto V(x_t)$ is also absolutely continuous, ensuring that the negativity of $\dot{V}$ for almost all t in an interval $[t_1; t_2]$ implies $V(x_{t_2}) < V(x_{t_1})$. Reasoning as in [32], it might be possible to extend the results presented here to merely measurable locally essentially bounded inputs: see [7] for a spatiotemporal version of this observation.

The following result provides a way to establish δGAS of spatiotemporal delayed dynamics.

Theorem 1 (Lyapunov-Krasovskii for δGAS) *Assume that there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}$, and a locally Lipschitz functional $V : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_{\geq 0}$ such that, for any $x_0, y_0 \in \mathcal{C}$ and any $u \in \mathcal{U}$, the corresponding solutions of* (1) *$x(\cdot) := x(\cdot; x_0, u)$ and $y(\cdot) := x(\cdot; y_0, u)$ satisfy, for almost all $t \in \mathbb{R}_{\geq 0}$,*

$$\underline{\alpha}(\|x(t) - y(t)\|_{\mathcal{F}}) \leq V(x_t, y_t) \leq \overline{\alpha}(\|x_t - y_t\|_{\mathcal{C}}) \tag{3}$$

$$\dot{V}^{(1)}(t, x_0, y_0, u) \leq -\alpha(\|x(t) - y(t)\|_{\mathcal{F}}). \tag{4}$$

Then, under Assumption 1, the system (1) *is δGAS.*

A specificity of the above result lies in the fact that the dissipation rate in (4) involves the $\mathcal{F}$-norm of the incremental state error, rather than its $\mathcal{C}$-norm. In other words, it depends on the current value of the incremental state error rather than its prehistory. If the right-hand side of (4) was $-\alpha(\|x_t - y_t\|_{\mathcal{C}})$, then it would hold that $\dot{V}^{(1)} \leq -\alpha \circ \overline{\alpha}^{-1}(V)$ and uniform incremental stability would readily follow from the comparison lemma. Here, the proof is slightly more involved, but follows along the lines of the original Lyaponov-Krasovskii approach [20], [25]: see Section VI-A.

C. Entrainment

One interesting feature of δGAS systems is their capability to be entrained by their input.

Definition 2 (Entrainment) *The system* (1) *is said to be* entrained by its input *if, given any $T \geq 0$ and any T-periodic*[1] *$u \in \mathcal{U}$, there exists a T-periodic solution $\bar{x}^u : \mathbb{R}_{\geq 0} \to \mathcal{F}$ of* (1) *such that, for any $x_0 \in \mathcal{C}$,*

$$\lim_{t \to \infty} \|x(t; x_0, u) - \bar{x}^u(t)\|_{\mathcal{F}} = 0.$$

The entrainment property thus ensures that, after transients, the system evolves in a periodic fashion at the rhythm of the applied input. We stress that entrainment ensures in particular that, in response to any constant input, there exists a unique equilibrium to which all solutions converge.

In line with [8] with an incremental stability approach, [33], [30] with a convergent systems approach, and [36], [26] with a contraction approach, the following result provides a condition under which δGAS ensures entrainment.

Theorem 2 (Entrainment by periodic inputs) *Assume that Assumption 1 holds and that* (1) *is δGAS. If there exists*

[1] When $T = 0$, u is called T-periodic if it is constant.

a continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that, given any $x_0 \in \mathcal{C}$ and any periodic $u \in \mathcal{U}$, the solution of (1) *satisfies*

$$\limsup_{t\to\infty} \|x(t)\|_{\mathcal{F}} \leq \sigma\Big(\sup_{t\geq 0} \|u(t)\|_{\mathcal{F}}\Big), \tag{5}$$

then (1) *is entrained by its input u.*

The proof of Theorem 2, provided in Section VI-C, relies on the following result.

Lemma 1 (Existence of a periodic solution) *Let $\mathcal{X} := C([-\bar{d}; 0], X)$ be a Banach space and let $g : \mathbb{R} \times \mathcal{X} \to X$ be a completely continuous function, locally Lipschitz in its second argument, for which there exists $T > 0$ such that $g(t + T, \phi) = g(t, \phi)$ for all $\phi \in \mathcal{X}$ and all $t \in \mathbb{R}$. If there exists a bounded set $\mathcal{A} \subset \mathcal{X}$ attracting all solutions of*

$$\dot{x}(t) = g(t, x_t), \tag{6}$$

then (6) *admits a T-periodic solution.*

Thus, δGAS ensures entrainment provided that a compact set, whose size may depend on the input magnitude, attracts all solutions. The proof of this result, provided in Section VI-B, is derived from [20].

III. DELAYED NEURAL FIELDS

A. Considered dynamics

We now focus on a particular class of spatiotemporal dynamics for which the entrainment property constitutes an interesting feature. Delayed neural fields are integro-differential equations representing the spatiotemporal activity of neuronal populations and accounting for the non-instantaneous communication between neurons. They read:

$$\tau_i \frac{\partial z_i}{\partial t}(r,t) = -z_i(r,t) + \tag{7}$$
$$S_i\Big(\sum_{j=1}^{n} \int_{\Omega} w_{ij}(r,r') z_j(r', t - d_j(r,r'))dr' + I_i^{\text{ext}}(r,t)\Big),$$

for each $i \in \{1, \ldots, n\}$. Ω is a compact set of $\mathbb{R}^q$, $q \in \{1, 2, 3\}$, representing the physical support of the populations. $r, r' \in \Omega$ are the space variables, whereas $t \in \mathbb{R}_{\geq 0}$ is the time variable. $z_i(r,t) \in \mathbb{R}$ represents the neuronal activity of population i at position r and at time t. $\tau_i > 0$ is the time decay constant of the activity of population i. $w_{ij}(r,r')$ describes the synaptic strength between location r' in population j and location r in population i; it is assumed that $w_{ij} \in L_2(\Omega \times \Omega, \mathbb{R})$. $I_i^{\text{ext}} : \Omega \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a continuous function describing the external input of population i, arising either from the influence of exogenous cerebral structures or from an artificial stimulation device. The function $d_j : \Omega \times \Omega \to [0; \bar{d}]$, $\bar{d} \geq 0$, is continuous; $d_j(r,r')$ represents the axonal, dendritic and synaptic delays between a pre-synaptic neuron at position r' in population j and a post-synaptic neuron at position r. It typically depends on the distance separating the two considered positions, namely $|r - r'|$. $S_i : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function, known as the activation function of the neural population i.

Neural fields have been introduced in [1], originally without considering delays. Since then, they have been employed in a number of computational neuroscience studies, ranging from the understanding of brain functions to the development of neuromorphic architectures [5]. A reason explaining this success probably stands in the good compromise they offer between richness of possible behaviors and mathematical tractability. In particular, a Lyapunov framework has been developed to study the stability of stationary patterns [14], [15]. Qualitative behavior can also be assessed through bifurcation theory [4], [39]. Delayed neural fields have also been the subject of stability and robustness studies, using linearization techniques [3] and a spatiotemporal extension of the Lyapunov-Krasovskii approach [16], [40], [7]. This mathematical background constitutes a fertile ground for the analysis of incremental stability of delayed neural fields, and particularly their ability to be entrained by a periodic input.

B. Delay-independent condition for δGAS

Let $[x(t)](\cdot) := z(\cdot, t)$, $[u(t)](\cdot) := I^{\text{ext}}(\cdot, t)$, and $f = (f_1, \ldots, f_n)^T$ where each f_i, $i \in \{1, \ldots, n\}$, is defined for all all $\phi \in \mathcal{C}$ and all $v \in \mathcal{F}$ as

$$f_i(\phi, v) := \frac{1}{\tau_i}\Big[-\phi(0) \tag{8}$$
$$+ S_i\Big(\sum_{j=1}^{n} \int_{\Omega} w_{ij}(\cdot, r)[\phi_j(-d_j(\cdot, r))](r)dr + v_i\Big)\Big].$$

Then the delayed neural fields (7) take the form (1). Applying Theorem 1 to this particular dynamics leads to the following explicit condition for the δGAS of delayed neural fields.

Theorem 3 (δGAS of delayed neural fields) *For each $i \in \{1, \ldots, n\}$, let S_i be globally Lipschitz with Lipschitz constant ℓ_i. Then, under the condition*

$$\sum_{i,j=1}^{n} \ell_i^2 \int_{\Omega}\int_{\Omega} w_{ij}(r,r')^2 dr'dr < 1, \tag{9}$$

the delayed neural fields (7) *are δGAS.*

To the best of our knowledge, this result is new even in the non-delayed case ($\bar{d} = 0$). Roughly speaking, condition (9) can be interpreted as follows: δGAS holds if the L_2 norm of the synaptic gains distributions is below the inverse of the Lipschitz constant of the activation functions. It therefore imposes a trade-off between synaptic strength (as measured through w_{ij}) and excitability (as measured through ℓ_i).

The exact same condition was proposed in [16] to establish the existence of a globally asymptotically stable equilibrium configuration for any given constant input. In view of Theorem 2, the above statement allows to go beyond that conclusion by considering periodic inputs.

Corollary 1 (Entrainment of neural fields) *Assume that the assumptions Theorem 3 hold and that the activation functions S_i, $i \in \{1, \ldots, n\}$, are bounded. Then the delayed neural fields* (7) *are entrained by their input I^{ext}.*

The proof of this result, provided in Section VI-E, relies on Theorems 2 and 3 by observing that the boundedness of S_i ensures ultimate boundedness of solutions.

IV. EXAMPLE

In order to illustrate our theoretical findings, we make use of the model introduced in [11]. This model involves two brain structures: the subthalamic nucleus (STN), mostly excitatory, and the external part of the globus pallidus (GPe), mostly inhibitory. These two structures are believed to be involved in the generation of beta oscillations (13-30Hz), the intensity of which correlates with the severity of parkinsonian motor symptoms [21]. More precisely, [28] and [29] showed that too strong coupling between STN and GPe may yield beta oscillations through an instability process. Here, we advocate that another mechanism might be at work that relies on entrainment: the preferential amplification of exogenous oscillations in the beta band. The spatiotemporal dynamics of these two structures can be grasped by:

$$\tau_1 \frac{\partial z_1}{\partial t}(r,t) = -z_1(r,t) + S_1\Big(\int_\Omega w_{12}(r,r')z_2(r',t-d_2(r,r'))dr' + I_1^{\text{ext}}(r,t)\Big), \tag{10a}$$

$$\tau_2 \frac{\partial z_2}{\partial t}(r,t) = -z_2(r,t) + S_2\Big(\int_\Omega w_{21}(r,r')z_1(r',t-d_1(r,r'))dr' + \int_\Omega w_{22}(r,r')z_2(r',t-d_2(r,r'))dr' + I_2^{\text{ext}}(r,t)\Big), \tag{10b}$$

where I_1^{ext} denotes the cortical input, whereas I_2^{ext} represents inputs from the striatum. The physical space $\Omega = [0, 15]$ mm is divided in two regions: $\Omega_1 = [0, 2.5]$ mm corresponding to STN, and $\Omega_2 = [12.5, 15]$ mm corresponding to GPe. The time constants are picked as $\tau_1 = 6$ ms and $\tau_2 = 14$ ms. The activation functions S_i, $i \in \{1, 2\}$, are taken as sigmoids, namely $S_i(x) = m_i b_i/(b_i + (m_i - b_i)e^{-4x/m_i})$, where $m_1 = 300$, $b_1 = 17$, $m_2 = 400$ and $b_2 = 75$. These functions are bounded and globally Lipschitz with Lipschitz constants $\ell_i = 1$. The delays d_i are given by $d_i(r, r') = \frac{|r-r'|}{c_i}$, for all $r, r' \in \Omega$, where c_i denotes the conductance velocity for neurons projecting from population i ($c_1 = 2.5$ m/s and $c_2 = 1.4$ m/s). The synaptic weights are taken as

$$\begin{aligned} w_{12}(r,r') &= -g_{12}(|r-r'-\mu_2|), \quad \forall r \in \Omega_1, r' \in \Omega_2 \\ w_{21}(r,r') &= g_{21}(|r-r'-\mu_1|), \quad \forall r \in \Omega_2, r' \in \Omega_1, \\ w_{22}(r,r') &= -|r-r'|g_{22}(|r-r'|), \quad \forall r, r' \in \Omega_2, \end{aligned}$$

and zero anywhere else, where $\mu_1 = 1.25$ mm and $\mu_2 = 13.25$ mm correspond the centers of STN and GPe respectively, and the functions g_{ij} are Gaussian functions: $g_{ij}(x) := k_{ij}\exp(-x^2/2\sigma_{ij})$ with $\sigma_{12} = \sigma_{21} = 0.03$, and $\sigma_{22} = 0.015$. All these parameters are spatiotemporal extensions of those in [28], that were derived based on experimental evidence: please refer to [11] for more details.

By picking $I_1^{\text{ext}}(r,t) = I_1^*(r)$ and $I_2^{\text{ext}}(r,t) = I_2^*(r) + v(r,t)$, where I_1^* and I_2^* are chosen in such a way that the system exhibits an equilibrium at the point where the slopes of S_1 and S_2 are maximum, this dynamics can be reformulated, after a change of variables, as

$$\tau_1 \frac{\partial z_1}{\partial t} = -z_1 + \mathcal{S}_1\Big(\int_\Omega w_{12}(r,r')z_2(r',t-d_2)dr'\Big), \tag{11a}$$

$$\tau_2 \frac{\partial z_2}{\partial t} = -z_2 + \mathcal{S}_2\Big(\int_\Omega w_{21}(r,r')z_1(r',t-d_1)dr' + \int_\Omega w_{22}(r,r')z_2(r',t-d_2)dr' + v(r,t)\Big), \tag{11b}$$

where some variables have been omitted for the sake of notation compactness. For each $i \in \{1, 2\}$, $\mathcal{S}_i(x) := S_i(x - a_i) - S_i(a_i)$, where a_i is the point at which the slope of S_i is maximum. For simulation purposes, we discretize the space Ω in 60 segments of equal length. By picking $k_{12} = 7$, $k_{21} = 10.5$ and $k_{22} = 3.0$, the quantity

$$\bar{w} := \int_\Omega \int_\Omega \Big(w_{12}(r,r')^2 + w_{21}(r,r')^2 + w_{22}(r,r')^2\Big)dr'dr,$$

equals 0.97, thus making (9) fulfilled. Theorem 3 then ensures that (11) is δGAS and Corollary 1 guarantees that the dynamics are entrained by the striatal input v. We let $v(r,t) = U\sin(\omega t)$ and we compute the ratio $\bar{z}_2(\omega)/U$ in dB, for various frequencies ω between 1 and 1000 rad/s, where $\bar{z}_2(\omega)$ denotes the steady-state magnitude of the oscillations of the spatial average of the GPe activity $\sqrt{\frac{1}{\#_{\Omega_2}}\int_{\Omega_2} z_2(r,t)^2 dr}$, where $\#_{\Omega_2} = 2.5$ denotes the measure of Ω_2. We obtain the frequency profiles depicted in Figure 1 (solid lines). Varying the amplitude U of the striatal input, different frequency profiles are obtained due to the nonlinear nature of the dynamics. For $U = 10$, a slight resonance appears at around 150 rad/s, corresponding to the beta band. Increasing the synaptic coupling ($k_{12} = 21.5$, $k_{21} = 22.5$, and $k_{22} = 20.5$) simulations indicate that (11) remains entrained by its input although $\bar{w} = 6 > 1$ thus violating condition (9). The corresponding frequency profiles appear in dashed lines, for the same input magnitudes as before. The beta resonance is much more pronounced.

When synaptic coupling is increased even more ($k_{12} = 23.5$, $k_{21} = 26$, and $k_{22} = 20.5$), corresponding to $\bar{w} = 7.5$, the spatiotemporal response of Figure 2 shows that endogenous oscillations take place (still in the beta band) even though the applied input is constant ($v(r,t) = 50$). Consequently, the system is no longer entrained by its input and, in view of Theorem 2, δGAS does not hold anymore.

V. CONCLUSION

Incremental stability is thus a powerful tool to ensure that a system is entrained by its input, and to derive frequency profiles, even for spatiotemporal delayed dynamics. The sufficient condition proposed for δGAS of delayed neural fields can easily be tested based on the system parameters. However, simulations indicate that this condition is not tight. In the future, deriving delay-dependent condition for δGAS of spatiotemporal dynamics may help reducing this conservatism. The results of this paper also plead for the

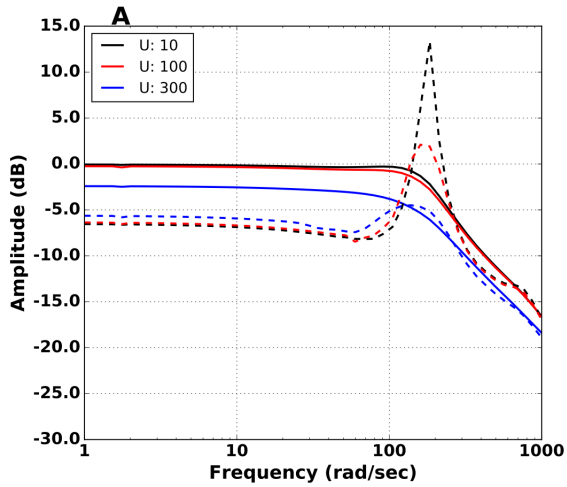

Fig. 1. Frequency responses of (11) for $\bar{w} = 0.97$ (solid lines) and $\bar{w} = 6$ (dashed lines), and for different input amplitudes U.

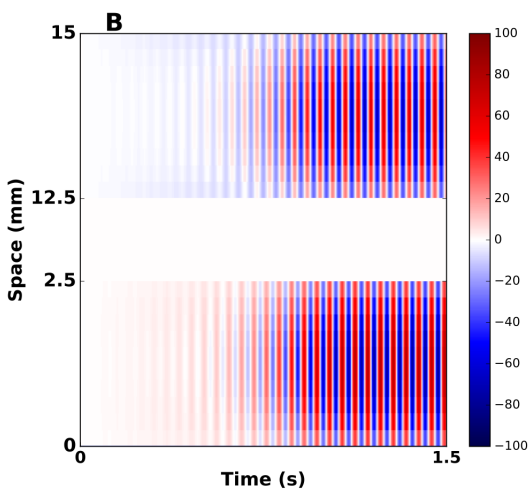

Fig. 2. spatiotemporal evolution of (11) with constant input for $\bar{w} = 7.5$.

development of control tools to tune the frequency profiles of the system, for instance to reduce amplification in a targeted frequency band.

VI. PROOFS

A. Proof of Theorem 1

Given any $x_0, y_0 \in \mathcal{C}$ and $u \in \mathcal{U}$, let $x(\cdot)$ and $y(\cdot)$ denote the corresponding solutions of (1). By assumptions, these solutions exist at all times $t \in [-\bar{d}; +\infty)$. First observe that, in the non-delayed case ($\bar{d} = 0$), it holds that $\|x(t) - y(t)\|_{\mathcal{F}} = \|x_t - y_t\|_{\mathcal{C}}$. It then holds from (3)-(4) that $\dot{V}^{(1)} \leq -\alpha \circ \overline{\alpha}(V)$ at almost all $t \in \mathbb{R}_{\geq 0}$ and uniform incremental stability follows by recalling that $t \mapsto V(x_t, y_t)$ is absolutely continuous[2] and invoking standard comparison lemmas (see [23, Lemmas 3.4 and 4.4]). So, from now on, we consider $\bar{d} > 0$.

[2] $x(\cdot)$ and $y(\cdot)$ are absolutely continuous by the definition of solutions.

Note that (4) implies in particular that $\dot{V}^{(1)} \leq 0$ at almost all times. Due to absolute continuity, it follows in particular that $V(x_t, y_t) \leq V(x_0, y_0)$ for all $t \geq 0$. Consequently, in view of (3),

$$\|x(t) - y(t)\|_{\mathcal{F}} \leq \underline{\alpha}^{-1} \circ \overline{\alpha}(\|x_0 - y_0\|_{\mathcal{C}}), \quad \forall t \geq 0, \tag{12}$$

thus ensuring global uniform boundedness and uniform stability of the incremental state error. We next proceed to showing that $\|x(\cdot) - y(\cdot)\|_{\mathcal{F}}$ uniformly tends to zero. To that aim, we start by showing that, given any $\Delta, \varepsilon > 0$, there exists a time $T \geq 0$ such that

$$\|x_T - y_T\|_{\mathcal{C}} \leq \varepsilon \tag{13}$$

for all $\|x_0 - y_0\|_{\mathcal{C}} \leq \Delta$. To that end, assume on the contrary that (13) does not hold, meaning that there exists an unbounded time sequence $\{t_k\}_{k \in \mathbb{N}}$ satisfying $t_{k+1} - t_k \in [0; \bar{d}]$, some constants $\Delta, \varepsilon > 0$, some bounded input $u \in \mathcal{U}$, and some initial states $x_0, y_0 \in \mathcal{C}$ satisfying $\|x_0 - y_0\|_{\mathcal{C}} \leq \Delta$ such that

$$\|x(t_k) - y(t_k)\|_{\mathcal{F}} > \varepsilon, \quad \forall k \in \mathbb{N}. \tag{14}$$

From this sequence $\{t_k\}_{k \in \mathbb{N}}$, let us extract a subsequence $\{\tau_k\}_{k \in \mathbb{N}}$ satisfying

$$2\bar{d} \leq \tau_{k+1} - \tau_k \leq 4\bar{d}, \quad \forall k \in \mathbb{N}. \tag{15}$$

In view of Assumption 1 and [20, Lemma 2.1, p. 38] (which can be readily extended to systems as (1)), the functional $t \mapsto f(x_t, u(t))$ is continuous. Consequently, $x(\cdot)$ and $y(\cdot)$ are continuously differentiable over $\mathbb{R}_{\geq 0}$. We may thus apply Leibniz rule to get that $\frac{d}{dt}\|x(t) - y(t)\|_{\mathcal{F}} \leq \|\dot{x}(t) - \dot{y}(t)\|_{\mathcal{F}}$. Indeed, let $e := x - y$. Then, using both Leibniz rule and Cauchy-Schwarz inequality, $\frac{d}{dt}\|x(t) - y(t)\|_{\mathcal{F}} = \frac{d}{dt}\|e(t)\|_{\mathcal{F}} = \frac{d}{dt}\int_{\Omega}[e(t)](r)^2 dr / 2\|e(t)\|_{\mathcal{F}} = \int_{\Omega}[e(t)](r)[\dot{e}(t)](r) dr / \|e(t)\|_{\mathcal{F}} \leq \sqrt{[\dot{e}(t)](r)^2 dr} = \|\dot{e}(t)\|_{\mathcal{F}}$. Moreover, let $\bar{u} := \sup_{t \geq 0} \|u(t)\|_{\mathcal{F}}$. It then follows from Assumption 1 and (12) that, for all $t \geq 0$,

$$\begin{aligned}
\frac{d}{dt}\|x(t) - y(t)\|_{\mathcal{F}} &= \|\dot{x}(t) - \dot{y}(t)\|_{\mathcal{F}} \\
&= \|f(x_t, u(t)) - f(y_t, u(t))\|_{\mathcal{F}} \\
&\leq \rho(\|x_t - y_t\|_{\mathcal{C}}) \\
&\leq \rho \circ \underline{\alpha}^{-1} \circ \overline{\alpha}(\|x_0 - y_0\|_{\mathcal{C}}) \\
&\leq \rho \circ \underline{\alpha}^{-1} \circ \overline{\alpha}(\Delta) \\
&\leq \ell,
\end{aligned} \tag{16}$$

where

$$\ell := \max\left\{\rho \circ \underline{\alpha}^{-1} \circ \overline{\alpha}(\Delta)\,;\, \frac{\varepsilon}{2\bar{d}}\right\}. \tag{17}$$

Combining (14) and (16), we get that, for all $k \in \mathbb{N}$,

$$\|x(\tau_k) - y(\tau_k)\|_{\mathcal{F}} > \frac{\varepsilon}{2}, \quad \forall t \in \left[\tau_k - \frac{\varepsilon}{2\ell}; \tau_k + \frac{\varepsilon}{2\ell}\right].$$

Moreover, (17) ensures that $\bar{d} \geq \varepsilon/2\ell$, which implies by (15) that the intervals $[\tau_k - \varepsilon/2\ell; \tau_k + \varepsilon/2\ell]$, $k \in \mathbb{N}$, do not

overlap. It follows from (3)-(4) that

$$V(x_t, y_t) \leq V(x_0, y_0) - \sum_{k=0}^{K(t)} \alpha(\varepsilon/2)\frac{\varepsilon}{\ell}$$
$$\leq \overline{\alpha}(\Delta) - \alpha(\varepsilon/2)\frac{\varepsilon}{\ell}(K(t)+1),$$

where $K(t) := \max\{k \in \mathbb{N} : \tau_k \leq t\}$. In view of (15), it holds that $\tau_k \geq \tau_0 + 2k\bar{d} \geq 2k\bar{d}$, from which we get that $K(t) + 1 \geq t/2\bar{d}$. Therefore

$$V(x_t, y_t) \leq \overline{\alpha}(\Delta) - \alpha(\varepsilon/2)\frac{\varepsilon t}{2\ell\bar{d}},$$

For $t > \frac{2\overline{\alpha}(\Delta)\ell\bar{d}}{\varepsilon\alpha(\varepsilon/2)}$, this leads to $V(x_t, y_t) < 0$ which is impossible. This establishes that $\|x_t - y_t\|_{\mathcal{C}}$ eventually takes values below ε.

Furthermore, this reasoning shows that, given any $\Delta, \varepsilon > 0$, the time needed for $\|x_t - y_t\|_{\mathcal{C}}$ to reach a value smaller than ε from any initial states satisfying $\|x_0 - y_0\| \leq \Delta$ is at most $T = \frac{2\overline{\alpha}(\Delta)\ell\bar{d}}{\varepsilon\alpha(\varepsilon/2)}$, thus independent of the applied input u. In addition, (12) show that if $\|x_{t_0} - y_{t_0}\|_{\mathcal{C}} \leq \varepsilon$ for some $t_0 \geq 0$, then $\|x(t) - y(t)\|_{\mathcal{F}} \leq \underline{\alpha}^{-1} \circ \overline{\alpha}(\varepsilon)$ for all $t \geq t_0$. Since ε and Δ are arbitrary, this shows that $\|x(\cdot) - y(\cdot)\|_{\mathcal{F}}$ globally tends to zero, uniformly in the input u. Combining this fact with (12) and proceeding[3] as in [23, Appendix C.6], we conclude that there exists $\beta \in \mathcal{KL}$ such that, for all $x_0, y_0 \in \mathcal{C}$ and all $u \in \mathcal{U}$, $\|x(t) - y(t)\|_{\mathcal{F}} \leq \beta(\|x_0 - y_0\|_{\mathcal{C}}, t)$ for all $t \geq 0$, which establishes Theorem 1.

B. Proof of Lemma 1

Given any $t_0 \in \mathbb{R}$ and any $x_0 \in \mathcal{X}$, let $x(\cdot; t_0, x_0)$ denote the solution of (6) such that $x(t_0; t_0, x_0) = x_0$. The regularity assumption on g ensures in view of [13, Theorem 6.10] that this solution is uniquely defined. Moreover, the fact that, by assumption, $x_t(t_0, x_0)$ tends to $\mathcal{A}$ as t tends to infinity guarantees that $x(t; t_0, x_0)$ exists at all times $t \geq t_0$ by [13, Proposition 6.16]. Defining $\xi : \mathbb{R} \times \mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathcal{X}$ as $\xi(t_0, x_0, t) := x_{t+t_0}(t_0, x_0)$, the time-periodicity of g ensures that ξ is a T-periodic process, as defined in [20, Definition 1.1, p. 76]. Moreover, by assumption, $\mathbb{R} \times \mathcal{A}$ attracts all points of $\mathcal{X}$, meaning that ξ is point dissipative in the sense of [20, Definition 5.3, p. 96]. The conclusion then follows from [20, Theorem 6.2, p. 98].

C. Proof of Theorem 2

First consider $T > 0$. Given any T-periodic $u \in \mathcal{U}$, let $g(t, \phi) := f(\phi, u(t))$ for all $\phi \in \mathcal{C}$ and all $t \geq 0$. Then it holds that $g(t + T, \phi) = g(t, \phi)$. Moreover, by Assumption 1, g is locally Lipschitz in its second argument and completely continuous. Moreover, letting $\bar{u} := \sup_{t \geq 0} \|u(t)\|_{\mathcal{F}} = \max_{t \in [0;T]} \|u(t)\|_{\mathcal{F}}$, (5) ensures that the bounded set $\mathcal{A} := \{\phi \in \mathcal{C} : \|\phi\|_{\mathcal{C}} \leq \sigma(\bar{u})\}$ is globally attractive. We can thus apply Lemma 1 with $X = \mathcal{F}$ (hence, $\mathcal{X} = \mathcal{C}$), to ensure the existence of a T-periodic solution of $\dot{x}(t) = g(t, x_t)$, which in turn ensures the existence of a T-periodic solution $\bar{x}^u : \mathbb{R}_{\geq -\bar{d}} \to \mathcal{F}$ for (1). Let $\varphi \in \mathcal{C}$ be defined as $\varphi(s) := \bar{x}^u(s)$ for all $s \in [-\bar{d}; 0]$. Then the solution of (1) defined with φ as initial state coincides with $\bar{x}^u$ at all times. The assumption of δGAS ensures that (2) holds for some $\beta \in \mathcal{KL}$. Applying this bound with this particular φ, we conclude that the solution of (1) starting from any $x_0 \in \mathcal{C}$ satisfies

$$\|x(t; x_0, u) - \bar{x}^u(t)\|_{\mathcal{F}} \leq \beta(\|x_0 - \varphi\|_{\mathcal{C}}, t), \quad \forall t \geq 0,$$

and the conclusion follows. By convention, in the case when $T = 0$, u is constant. Repeating the above reasoning for any arbitrary $T > 0$ shows that $\bar{x}^u$ is then constant too, which ends the proof.

D. Proof of Theorem 3

First observe that Assumption 1 is satisfied since the functions S_i, $i \in \{1, \ldots, n\}$, are globally Lipschitz. We consider the functional V defined, for all $\phi, \varphi \in \mathcal{C}$, as

$$V(\phi, \varphi) = \frac{1}{2}\sum_{i=1}^{n} \tau_i \|\phi_i(0) - \varphi_i(0)\|_{\mathcal{F}}^2$$
$$+ \sum_{i=1}^{n} \int_{\Omega} \gamma(r) \int_{\Omega} \int_{-d_i(r,r')}^{0} \Big([\phi_i(\theta)](r') - [\varphi_i(\theta)](r')\Big)^2 d\theta dr' dr,$$

where γ is a bounded function to be chosen latter. Given any continuous I^{ext}, let $u \in \mathcal{U}$ be defined as $u(t) := I^{ext}(t, \cdot)$. Consider any $x_0, y_0 \in \mathcal{C}$ and let $x(\cdot)$ and $y(\cdot)$ denote the corresponding solutions of (7). Elementary computations show that

$$\underline{c}\|x(t) - y(t)\|_{\mathcal{F}}^2 \leq V(x_t, y_t) \leq \overline{c}\|x_t - y_t\|_{\mathcal{C}}^2,$$

where $\underline{c} := \frac{1}{2}\min_{i=1,\ldots,n} \tau_i$ and $\overline{c} := \max_{i=1,\ldots,n} \tau_i + \#_{\Omega}^2 \max_{r \in \Omega} \gamma(r)$. This establishes (3). In view of [16, Theorem 3.2.1], $x(\cdot)$ and $y(\cdot)$ are uniquely defined and continuously differentiable on $\mathbb{R}_{\geq 0}$. It follows that the map $\mathcal{V} : t \mapsto V(x_t, y_t)$ is continuously differentiable on $\mathbb{R}_{\geq 0}$. Consequently, its upper-right Dini derivative coincides with its classical derivative, which reads

$$\dot{\mathcal{V}}(t) = \sum_{i=1}^{n} \dot{\mathcal{V}}_i(t) + \dot{\mathcal{W}}_i(t), \tag{18}$$

with

$$\mathcal{V}_i(t) := \frac{\tau_i}{2}\|x_i(t) - y_i(t)\|_{\mathcal{F}}^2$$
$$\mathcal{W}_i(t) := \int_{\Omega} \gamma(r) \int_{\Omega} \int_{t-d_i}^{t} \big([x_i(\theta)](r') - [y_i(\theta)](r')\big)^2 d\theta dr' dr,$$

where the spatial dependency of the delays d_i has been omitted in the notation. In order to lighten the notation, we

[3] This $\mathcal{KL}$ function construction is made in a finite dimensional context in [23], but extends readily here.

let $e_i := x_i - y_i$. The derivative of $\mathcal{V}_i$ reads

$$\begin{aligned}\dot{\mathcal{V}}_i =&\frac{d}{dt}\left(\frac{\tau_i}{2}\int_\Omega |[x_i(t)](r)-[y_i(t)](r)|^2 dr\right)\\ =&\tau_i\int_\Omega [e_i(t)](r)\big([\dot{x}_i(t)](r)-[\dot{y}_i(t)](r)|\big)dr\\ =&-\int_\Omega [e_i(t)](r)^2 dr+\int_\Omega [e_i(t)](r)\times\\ &\Big[S_i\Big(\sum_{j=1}^n\int_\Omega w_{ij}[x_j(t-d_j)](r')dr'+[u(t)](r)\Big)\\ &-S_i\Big(\sum_{j=1}^n\int_\Omega w_{ij}[y_j(t-d_j)](r')dr'+[u(t)](r)\Big)\Big]dr.\end{aligned}$$

Using the fact that $|S_i(a-b)|\le \ell_i|a-b|$ for all $a,b\in\mathbb{R}$ and Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\dot{\mathcal{V}}_i \le&-\|e_i(t)\|_{\mathcal{F}}^2\\ &+\ell_i\int_\Omega |[e_i(t)](r)|\left|\sum_{j=1}^n\int_\Omega w_{ij}[e_j(t-d_j)](r')dr'\right|dr\\ \le&-\|e_i(t)\|_{\mathcal{F}}^2\\ &+\ell_i\|e_i(t)\|_{\mathcal{F}}\sqrt{\int_\Omega\Big(\sum_{j=1}^n\int_\Omega w_{ij}[e_j(t-d_j)](r')dr'\Big)^2dr}\\ \le&-\frac{1}{2}\|e_i(t)\|_{\mathcal{F}}^2+\frac{\ell_i^2}{2}\int_\Omega\Big(\sum_{j=1}^n\int_\Omega w_{ij}[e_j(t-d_j)](r')dr'\Big)^2dr,\end{aligned}$$

where we used the fact that $ab\le(a^2+b^2)/2$ to get the last bound. Using again Cauchy-Schwarz inequality, we get that

$$\begin{aligned}\dot{\mathcal{V}}_i \le&-\frac{1}{2}\|e_i(t)\|_{\mathcal{F}}^2\\ &+\frac{\ell_i^2}{2}\int_\Omega\Big(\sum_{j=1}^n\sqrt{\int_\Omega w_{ij}^2dr'\int_\Omega[e_j(t-d_j)](r')^2dr'}\Big)^2dr\\ \le&-\frac{1}{2}\|e_i(t)\|_{\mathcal{F}}^2\\ &+\frac{\ell_i^2}{2}\int_\Omega\Big(\sum_{j=1}^n\int_\Omega w_{ij}^2dr'\Big)\Big(\sum_{j=1}^n\int_\Omega[e_j(t-d_j)](r')^2dr'\Big)dr\\ \le&-\frac{1}{2}\|e_i(t)\|_{\mathcal{F}}^2\\ &+\frac{1}{2}\int_\Omega\bar{w}_i(r)\Big(\sum_{j=1}^n\int_\Omega[e_j(t-d_j)](r')^2dr'\Big)dr,\end{aligned}$$

where $\bar{w}_i(r):=\ell_i^2\sum_{j=1}^n\int_\Omega w_{ij}(r,r')^2dr'$. It follows that

$$\begin{aligned}\sum_{i=1}^n\dot{\mathcal{V}}_i(t)\le&-\frac{1}{2}\|e(t)\|_{\mathcal{F}}^2\\ &+\frac{1}{2}\sum_{i=1}^n\int_\Omega\bar{w}_i(r)\Big(\sum_{j=1}^n\int_\Omega[e_j(t-d_j)](r')^2dr'\Big)dr\\ \le&-\frac{1}{2}\|e(t)\|_{\mathcal{F}}^2 \qquad (19)\\ &+\frac{1}{2}\int_\Omega\Big(\sum_{i=1}^n\bar{w}_i(r)\Big)\Big(\sum_{j=1}^n\int_\Omega[e_j(t-d_j)](r')^2dr'\Big)dr.\end{aligned}$$

Similarly, the derivative of $\mathcal{W}_i$ reads

$$\begin{aligned}\dot{\mathcal{W}}_i(t):=&\frac{d}{dt}\int_\Omega\gamma(r)\int_\Omega\int_{t-d_i}^t[e_i(\theta)](r')^2d\theta dr'dr\\ =&\int_\Omega\gamma(r)\int_\Omega\Big([e_i(t)](r')^2-[e_i(t-d_i)](r')^2\Big)dr'dr\\ =&\|e_i(t)\|_{\mathcal{F}}^2\int_\Omega\gamma(r)dr-\int_\Omega\gamma(r)\int_\Omega[e_i(t-d_i)](r')^2dr'dr.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{i=1}^n\dot{\mathcal{W}}_i(t)=&\|e(t)\|_{\mathcal{F}}^2\int_\Omega\gamma(r)dr \qquad (20)\\ &-\int_\Omega\gamma(r)\Big(\sum_{i=1}^n\int_\Omega[e_i(t-d_i)](r')^2dr'\Big)dr.\end{aligned}$$

Combing (18), (19), and (20), we obtain that

$$\begin{aligned}\dot{\mathcal{V}}(t)\le&-\Big(\frac{1}{2}-\int_\Omega\gamma(r)dr\Big)\|e(t)\|_{\mathcal{F}}^2\\ &-\int_\Omega\Big[\Big(\gamma(r)-\sum_{i=1}^n\frac{\bar{w}_i(r)}{2}\Big)\int_\Omega[e_j(t-d_j)](r')^2dr'\Big]dr.\end{aligned}$$

So the assumptions of Theorem 1 are fulfilled provided that

$$\int_\Omega\gamma(r)<1/2 \quad\text{and}\quad \gamma(r)\ge\sum_{i=1}^n\frac{\bar{w}_i(r)}{2}.$$

By picking

$$\gamma(r)=\sum_{i=1}^n\frac{\bar{w}_i(r)}{2}=\sum_{i,j=1}^n\frac{\ell_i^2}{2}\int_\Omega w_{ij}(r,r')^2dr',$$

the first condition becomes

$$\sum_{i,j=1}^n\ell_i^2\int_\Omega\int_\Omega w_{ij}(r,r')^2dr'dr<1,$$

which corresponds to condition (9).

E. Proof of Corollary 1

Recall that (7) can be written in the form (1) with f defined as (8). Given any $i\in\{1,\dots,n\}$, let $\bar{S}_i:=\sup_{s\in\mathbb{R}}|S_i(s)|$. Then it can easily be checked that, for all $x_0\in\mathcal{C}$ and any $u\in\mathcal{U}$, the solution of the latter system satisfies $\limsup_{t\to\infty}\|x_i(t)\|_{\mathcal{F}}\le\bar{S}_i\sqrt{\#_\Omega}$ where $\#_\Omega:=\int_\Omega dr$. It follows that (5) is satisfied, and the conclusion follows from Corollary 1.

References

[1] S.-I. Amari. Dynamics of pattern formation in lateral-inhibition type neural fields. *Biological Cybernetics*, 27(2):77–87, 1977.

[2] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Trans. Autom. Control*, 47(3):410–421, 2002.

[3] F.M. Atay and A. Hutt. Neural fields with distributed transmission speeds and long-range feedback delays. *SIAM Journal on Applied Dynamical Systems*, 5(4), 2006.

[4] P. beim Graben and A. Hutt. Attractor and saddle node dynamics in heterogeneous neural fields. *EPJ Nonlinear biomedical physics*, 2(4):1–17, 2014.

[5] P.C. Bressloff. Spatiotemporal dynamics of continuum neural fields. *Journal of Physics A: Mathematical and Theoretical*, 45(3):033001, 2012.

[6] G. Buzsaki. *Rhythms of the Brain*. Oxford University Press, 2006.

[7] A. Chaillet, G. Detorakis, S. Palfi, and S. Senova. Robust stabilization of delayed neural fields with partial measurement and actuation. *Automatica*, 83:262–274, Sep. 2017.

[8] A. Chaillet, A.Yu. Pogromsky, and B.S. Rüffer. A Razumikhin approach for the incremental stability of delayed nonlinear systems. In *Proc. IEEE Conf. on Decision and Control*, Florence, Italy, 2013.

[9] S. Coombes, P. beim Graben, R. Potthast, and J. Wright. *Neural Fields: Theory and Applications*. Springer, 2014.

[10] B. Demidovich. *Lectures on Stability Theory*. Moscow, 1967.

[11] G. Detorakis, A. Chaillet, S. Palfi, and S. Senova. Closed-loop stimulation of a delayed neural fields model of parkinsonian STN-GPe network: a theoretical and computational study. *Frontiers in Neuroscience*, 9(237), 2015.

[12] E. Devane and I. Lestas. Delay-independent incremental stability in time-varying monotone systems satisfying a generalized condition of two-sided scalability. *Automatica*, 76:1–9, 2017.

[13] B.K. Driver. *Analysis tools with applications*. Lecture notes. Springer, 2003.

[14] O. Faugeras, F. Grimbert, and J.-J. Slotine. Absolute stability and complete synchronization in a class of neural fields models. *SIAM Journal of Applied Mathematics*, 61(1):205–250, 2008.

[15] O. Faugeras, R. Veltz, and F. Grimbert. Persistent neural states: stationary localized activity patterns in nonlinear continuous n-population, q-dimensional neural networks. *Neural Computation*, 21(1):147–187, 2009.

[16] G. Faye and O. Faugeras. Some theoretical and numerical results for delayed neural field equations. *Physica D: Nonlinear Phenomena*, 239(9):561–578, 2010.

[17] F. Forni and R. Sepulchre. A differential Lyapunov framework for contraction analysis. *IEEE Trans. on Automatic Control*, 59(3):614–628, 2014.

[18] V. Fromion, S. Monaco, and D. Normand-Cyrot. Asymptotic properties of incrementally stable systems. *IEEE Trans. on Automatic Control,*, 41(5):721–723, 2002.

[19] V. Fromion, M.G. Safonov, and G. Scorletti. Necessary and sufficient conditions for lur'e system incremental stability. *Proc. European Control Conference*, pages 71–76, 2003.

[20] J.K. Hale. Theory of functional differential equations. *Applied mathematical sciences*, pages 1–376, 1977.

[21] C. Hammond, H. Bergman, and P. Brown. Pathological synchronization in Parkinson's disease: networks, models and treatments. *Trends in Neurosciences*, 30(7):357 – 364, 2007.

[22] I. Karafyllis. Lyapunov theorems for systems described by retarded functional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 64(3):590–617, 2006.

[23] H. Khalil. *Nonlinear systems*. Macmillan Publishing Co., 2nd ed., New York, 1996.

[24] N. Kopell, G.B. Ermentrout, M.A. Whittington, and R.D. Traub. Gamma rhythms and beta rhythms have different synchronization properties. *Proceedings of the National Academy of Sciences*, 97(4):1867–1872, 2000.

[25] N.N. Krasovskii. *Problems of the theory of stability of motion*. Stanford Univ. Press, 1963.

[26] W. Lohmiller and J.J. Slotine. On contraction analysis for nonlinear systems. *Automatica*, 34(6), 1998.

[27] M. Margaliot, E.D. Sontag, and T. Tuller. Contraction after small transients. *Automatica*, 67:178–184, 2016.

[28] A.J. Nevado-Holgado, J.R. Terry, and R. Bogacz. Conditions for the generation of beta oscillations in the subthalamic nucleus-globus pallidus network. *Journal of Neuroscience*, 30(37):12340–12352, 2010.

[29] W. Pasillas-Lépine. Delay-induced oscillations in Wilson and Cowan's model: An analysis of the subthalamo-pallidal feedback loop in healthy and parkinsonian subjects. *Biological Cybernetics*, 107(3):289–308, 2013.

[30] A.V. Pavlov, N. van de Wouw, and H. Nijmeijer. *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*. Systems & Control: Foundations & Applications, 2006.

[31] A.V. Pavlov, N. van de Wouw, and H. Nijmeijer. Frequency response functions for nonlinear convergent systems. *IEEE Trans. Autom. Control*, 52(6):1159–1165, 2007.

[32] P. Pepe. The problem of the absolute continuity for Lyapunov-Krasovskii functionals. *IEEE Trans. Autom. Control*, 52(5):953–957, 2007.

[33] A.Yu. Pogromsky and A.S. Matveev. A non-quadratic criterion for stability of forced oscillations. *Systems & Control Letters*, 62(5):408–412, 2013.

[34] G. Pola, P. Pepe, M. Di Benedetto, and P. Tabuada. Symbolic models for nonlinear time-delay systems using approximate bisimulations. *Systems & Control Letters*, 59(6):365–373, 2010.

[35] B.S. Rüffer, N. van de Wouw, and M. Mueller. Convergent systems vs. incremental stability. *Systems & Control Letters*, 62:277–285, 2013.

[36] G. Russo, M. di Bernardo, and E.D. Sontag. Global entrainment of transciptional systems to periodic inputs. *PLoS Computational Biology*, 6(4):1–26, 2010.

[37] N. van de Wouw, W. Michiels, and B. Besselink. Model reduction for delay differential equations with guaranteed stability and error bound. *Automatica*, 55:132–139, 2015.

[38] R. Veltz. Interplay between synaptic delays and propagation delays in neural field equations. *SIAM Journal on Applied Dynamical Systems*, 2013.

[39] R. Veltz and O. Faugeras. Local/global analysis of the stationary solutions of some neural field equations. *SIAM Journal on Applied Dynamical Systems*, 9(3):954–998, 2010.

[40] R. Veltz and O. Faugeras. Stability of the stationary solutions of neural field equations with propagation delays. *The Journal of Mathematical Neuroscience*, 1(1):1–28, 2011.