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Tarek Ahmed-Ali [a], Fouad Giri [a], Miroslav Krstic [b], Laurent Burlion [c], Françoise Lamnabhi-Lagarrigue [d]

[a] *Université de Caen Normandie, UNICAEN, ENSICAEN, France*
[b] *Department of Mechanical and Aerospace Engineering, University of California at San Diego, La Jolla, CA 92093-0411, USA*
[c] *ONERA, the French Aerospace Lab, 31055 Toulouse, France*
[d] *CNRS L2S, CentraleSupelec, EECI, 91192 Gif-sur-Yvette, France*

The problem of state and parameter estimation is addressed for systems with cascade structure including finite-dimensional dynamics followed in series with infinite-dimensional dynamics. The formers are captured through an ODE that is state- and parameter-affine. The latter, referred to as sensor dynamics, are represented by a diffusion parabolic PDE. Both equations are subject to parameter uncertainty. Furthermore, the connection point between the ODE and the PDE blocs is not accessible to measurements. The aim is to get online estimates of all inaccessible states and unknown parameters of both the ODE and the PDE subsystems. This observation problem is dealt with by combining the backstepping design method and the extended Kalman observer approach. The obtained adaptive observer is shown to be exponentially convergent under an ad-hoc persistent excitation condition.

1. Introduction

An intensive research activity has been devoted to observer design over the past three decades. This activity has mainly been focused on (finite-dimensional) nonlinear systems described by ordinary differential equations (ODEs) leading to numerous observer design techniques and related stability and convergence results. So far, much less research effort has been devoted to the problem of state observation for systems involving partial differential equations (PDEs). In this respect, the emphasis has been put on the problem of designing exponential boundary observers for various types of systems including linear wave equation (Guo & Xu, 2007), semilinear diffusion equation (Fridman & Blighovsky, 2012). Attention has also been paid to designing finite-time convergent observers, see e.g. Miranda, Moreno, Chairez, and Fridman (2012) and sampled-data exponentially convergent observers for parabolic-type PDEs (Ahmed-Ali, Fridman, Giri, Burlion, & Lamnabhi-Lagarrigue, 2015, 2016; Ahmed-Ali, Giri, Krstic, & Lamnabhi-Lagarrigue, 2016).

In Smyshlyaev and Krstic (2005), a quite different approach to boundary observer design has been developed for a class of parabolic partial integro-differential equations. The resulting observers involve gain kernels that are constructed using the continuum version of the backstepping design method. This approach has been applied to observer design for various systems described by PDEs e.g. magnetohydrodynamic system (Vazquez, Schuster, & Krstic, 2008), linear hyperbolic system (Vazquez, Krstic, & Coron, 2011), and drilling system (Hauge, Aamo, & Godhavn, 2013). Later on, the backstepping-like approach has proved to be applicable to various other classes of systems including cascades of ODEs and PDEs. For these cascade systems, the aim is to recover the (finite-dimensional) state of the system ODE part and the (infinite-dimensional) state of the PDE part. A major difficulty lies in the fact that the connection point between the two parts is not accessible to measurements. In Krstic and Smyshlyaev (2008), the observation problem has been addressed for linear ODE–PDE cascade systems where the PDE part can be seen as representing sensor dynamics. The latter have been captured by a (purely convection) first-order hyperbolic PDE which models transport delays. Using the

E-mail addresses: Tarek.Ahmed-Ali@ensicaen.fr (T. Ahmed-Ali), fouad.giri@unicaen.fr (F. Giri), krstic@ucsd.edu (M. Krstic), Laurent.Burlion@onera.fr (L. Burlion), Francoise.Lamnabhi-Lagarrigue@lss.supelec.fr (F. Lamnabhi-Lagarrigue).

backstepping design approach, new state transformations and Lyapunov–Krasovskii functional have been employed in observer design resulting in a new family of boundary predictor observers. A similar design approach has been developed in Krstic (2009) to get state observers for ODE–PDE cascades where the PDE part is a (diffusion) parabolic equation (describing e.g. heat diffusion phenomena). Like the system, the observer structure is an ODE–PDE cascade augmented by innovation terms with observer gains. The design PDEs yielding observer gains, which were hyperbolic of first order in the delay problem considered in Krstic and Smyshlyaev (2008), have turned out to be hyperbolic of second-order in the diffusion problem studied in Krstic (2009). Extensions of this observer design to sampled-output ODE–PDE systems have been proposed in Ahmed-Ali, Karafyllis, Giri, Krstic, and Lamnabhi-Lagarrigue (2016) and Karafyllis, Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2016).

In this paper, the problem of state observation for ODE–PDE systems, where the PDE is a diffusion parabolic equation, is addressed in presence of parametric uncertainty. The unknown parameter vectors come linearly in the finite- and infinite-dimensional subsystem, but the associated regression vectors are allowed to be nonlinear in the system output adding thus extra nonlinear dynamics. Clearly, this class of systems is much wider than that of Krstic and Smyshlyaev (2008) which was limited to parameter-uncertainty-free linear ODE–PDE systems. We seek an observer that provides online estimates of the ODE and PDE states and their unknown parameters. This observation problem is dealt with by designing an adaptive observer making use of the (infinite-dimensional) backstepping design technique and the (finite-dimensional) Kalman observer approach. Interestingly, the resulting observer turns out to be a nonlinear adaptive version of the generalized-predictor observer of Krstic (2009). It involves design PDEs yielding observer gains similar to those of their nonadaptive counterparts. Besides, the adaptive observer includes additional filters, defined by ODEs and PDEs, that define the (time-varying) direction along which the parameter adaptive laws evolve in the parameter space.

Compared to Ahmed-Ali, Giri, Krstic, Burlion, and Lamnabhi-Lagarrigue (2015), Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2015) and Ahmed-Ali, Giri, Krstic, Burlion, and Lamnabhi-Lagarrigue (2016), the observation problem that is presently studied is quite different. Indeed, the class of systems considered in Ahmed-Ali, Giri, Krstic, Burlion, and Lamnabhi-Lagarrigue (2016, 2015) is described using solely a semilinear heat PDE, i.e. no ODE subsystem was involved in those systems. Furthermore, the adaptive observers in Ahmed-Ali, Giri, Krstic, Burlion, and Lamnabhi-Lagarrigue (2015) and Ahmed-Ali, Giri, Krstic, Lamnabhi-Lagarrigue, and Burlion (2016) necessitated as many sensors as the number of unknown parameters. In Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2015), an ODE–PDE system structure was considered but the system was not subject to parameter uncertainty and so the proposed observer was not adaptive. In Ahmed-Ali, Fridman et al. (2016); Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2016), the problem of observer design is considered for delayed ODE systems subject to parameter uncertainty. The constant delay is described by first-order hyperbolic PDE leading to an ODE–PDE representation of the system. As the parameter uncertainty only enters in the ODE, the adaptive observer design problem proves to be much simpler than the present problem where parameter uncertainty also affects the PDE.

The paper is organized as follows: in Section 2, the observation problem under study is formulated and an adaptive observer is proposed; the observer convergence is analyzed in Section 3 and numerical simulation results are provided in Section 4; a conclusion and reference list end the paper. To alleviate the presentation, some technical proofs are appended.

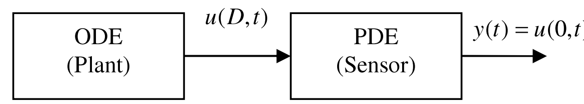

Fig. 1. System structure.

Notations. Throughout the paper, $\mathbf{R}^n$ denotes the n dimensional real space and the corresponding Euclidean norm is denoted $\|\cdot\|$. This also denotes the induced matrix norm in $\mathbf{R}^{n\times m}$, the set of all $n\times m$ real matrices. Functions that are continuously differentiable with respect to all their arguments are denoted C^1. $L_2[0,\ D]$ is the Hilbert space of square integrable functions and the corresponding L_2 norm is denoted $\|\cdot\|_2$. Accordingly, $\|w\|_2 = \left(\int_0^D w^2(\varsigma)d\varsigma\right)^{1/2}$ for all $w \in L_2[0,\ D]$. $H^1(0,D)$ is the Sobolev space of absolutely continuous functions $w : [0,D] \to \mathbf{R};\ x \to w(x)$ with $dw/dx \in L_2[0,D]$. $H^2(0,D)$ is the Sobolev space of scalar functions $w : [0,D] \to \mathbf{R};\ x \to w(x)$ with absolutely continuous $dw/dx \in L_2[0,D]$ and $d^2w/dx^2 \in L_2[0,D]$. For any $w \in H^1(0,D)$ such that $w(0) = 0$ or $w(D) = 0$, the following Wirtinger's inequalities hold:

$$\int_0^D w^2(x)dx \le \frac{4D}{\pi^2}\int_0^D w_x^2(x)dx \tag{1a}$$

$$\max_{0\le x\le D} w^2(x) \le D\int_0^D w_x^2(x)dx. \tag{1b}$$

Finally, given a two-dimensional function $w : [0,D]\times\mathbf{R}_+ \to \mathbf{R};\ (x,t) \to w(x,t)$, the notation $w[t]$ and $w_x[t]$ refer to the functions defined on $0 \le x \le D$ by $(w[t])(x) = w(x,t)$ and $(w_x[t])(x) = \partial w(x,t)/\partial x$.

2. Problem formulation and adaptive observer statement

The system under study is composed of a finite-dimensional nonlinear subsystem connected in series with an infinite-dimensional subsystem as depicted in Fig. 1. The former assumes the following state-space representation:

$$\dot{X}(t) = AX(t) + \phi_1(t)\theta_1, \quad t \ge 0, \tag{2}$$

with $X(0) = X_0 \in \mathbf{R}^n$ is arbitrary. The infinite-dimensional subsystem is modeled by a parabolic PDE of the form,

$$u_t(x,t) = u_{xx}(x,t) + \phi_2(x,t)\theta_2, \quad 0 \le x \le D \tag{3a}$$

$$u_x(0,t) = 0, \qquad u(D,t) = CX(t) \tag{3b}$$

where $A \in \mathbf{R}^{n\times n}$ and $C \in \mathbf{R}^{1\times n}$ are known constant matrices and the pair (A,C) is observable; $\phi_1 \in C^1([0,\infty) : \mathbf{R}^{n\times m_1})$ and $\phi_2 \in C^1([0,D]\times[0,\infty) : \mathbf{R}^{1\times m_2})$ are known bounded functions; the domain length $D > 0$ is a known scalar. The state vector $X(t) \in \mathbf{R}^n$ and the distributed state $u(x,t) \in \mathbf{R}$ are not accessible to measurement, except for the boundary state $u(0,t)$ which stands as the output of the whole system. The system parameter vectors $\theta_1 \in \mathbf{R}^{m_1}$ and $\theta_2 \in \mathbf{R}^{m_2}$ are not known but their dimensions (m_1, m_2) are.

The aim is to design an observer able to provide accurate online estimates of the system state functions $X(t)$ and $u(x,t)$ ($t > 0, \le x \le D$), on the one hand, and the unknown parameter vectors θ_1 and θ_2, on the other. The observer must only make use of the system output $y(t) = u(0,t)$. The uncertain quantities in (2) and (3a) constitute a new feature of this study compared to Krstic (2009) where $\phi_1(t)\theta_1 = \phi_2(x,t)\theta_2 = 0$. On the other hand, the functions $\phi_1(t)$ and $\phi_2(x,t)$ are presently allowed to be output dependent i.e. one might have $\phi_1(t) = \psi_1(t,y(t))$ and $\phi_2(x,t) = \psi_2(x,t,y(t))$ for some functions $\psi_1(\cdot,\cdot)$ and $\psi_2(\cdot,\cdot)$. That is, the present observer design is not limited to linear systems, unlike Krstic (2009).

To cope with the observation problem at hand, the adaptive observer of Table 1 is proposed.

Table 1
Adaptive observer.

State observer:	
$\dot{\hat{X}}(t) = A\hat{X}(t) + \phi_1(t)\hat{\theta}_1(t) - K(\hat{u}(0,t) - u(0,t)) + \lambda_1(t)\dot{\hat{\theta}}_1(t) + \lambda_2(t)\dot{\hat{\theta}}_2(t)$	(4a)
$\hat{u}_t(x,t) = \hat{u}_{xx}(x,t) + \phi_2(x,t)\hat{\theta}_2(t) - CM(x)M^{-1}(D)K(\hat{u}(0,t) - u(0,t))$ $+ \left(\lambda_3(x,t) + CM(x)M^{-1}(D)\lambda_1(t)\right)\dot{\hat{\theta}}_1(t)$ $+ \left(\lambda_4(x,t) + CM(x)M^{-1}(D)\lambda_2(t)\right)\dot{\hat{\theta}}_2(t)$	(4b)
$\hat{u}_x(0,t) = 0,\ \hat{u}(D,t) = C\hat{X}(t)$ for all $t \geq 0$ and all $x \in [0,\ D]$, where $K \in \mathbf{R}^n$ is such that $A - KCM^{-1}(D)$ is Hurwitz.	(4c)
Parameter adaptive law	
$\dot{\hat{\theta}}(t) = -\rho R(t)\Lambda(t)\tilde{u}(0,t)$	(4d)
$\dot{R}(t) = R(t) - R(t)\Lambda(t)\Lambda^T(t)R(t)$	(4e)
with $\hat{\theta}(t) = \left[\hat{\theta}_1^T(t)\ \ \hat{\theta}_2^T(t)\right]^T \in \mathbf{R}^m$, $R(t) \in \mathbf{R}^{m\times m}$, $m = m_1 + m_2$, and $\Lambda(t) = \left[CM^{-1}(D)\lambda_1(t) + \lambda_3(0,t)\ CM^{-1}(D)\lambda_2(t) + \lambda_4(0,t)\right]^T \in \mathbf{R}^m$, where $\hat{\theta}(0) \in \mathbf{R}^m$, $R(0) = R^T(0) > 0$ and $\rho > 0$ are arbitrary.	(4f)
Filters	
$\dot{\lambda}_1(t) = (A - KCM^{-1}(D))\lambda_1(t) + \phi_1(t) - K\lambda_3(0,t)$	(4g)
$\dot{\lambda}_2(t) = (A - KCM^{-1}(D))\lambda_2(t) - K\lambda_4(0,t)$	(4h)
$\lambda_{3,t}(x,t) = \lambda_{3,xx}(x,t) - CM(x)M^{-1}(D)\phi_1(t)$	(4i)
$\lambda_{4,t}(x,t) = \lambda_{4,xx}(x,t) + \phi_2(x,t)$	(4j)
$\lambda_1(0) = 0 \in \mathbf{R}^{n\times m_1}$, $\lambda_3(D,t) = 0 \in \mathbf{R}^{1\times m_1}$, $\lambda_{3,x}(0,t) = 0,\ \lambda_2(0) = 0 \in \mathbf{R}^{n\times m_2}$	(4k)
$\lambda_4(D,t) = 0 \in \mathbf{R}^{1\times m_2},\ \lambda_{4,x}(0,t) = 0$	(4l)
$M(x) = (\mathbf{I}\ \ 0)\exp\left(\begin{pmatrix} 0 & A \\ \mathbf{I} & 0 \end{pmatrix} x\right)\begin{pmatrix} \mathbf{I} \\ 0 \end{pmatrix} \in \mathbf{R}^{n\times n}$	(4m)

Remark 1. (a) Relevant properties of the matrix function $M(x)$ defined by (4m) are described in Appendix B. Accordingly, one has $A = M(D)AM^{-1}(D)$ which implies

$$\begin{aligned} A - KCM^{-1}(D) &= M(D)AM^{-1}(D) - KCM^{-1}(D) \\ &= M(D)\left(A - M^{-1}(D)KC\right)M^{-1}(D). \end{aligned} \tag{5}$$

This shows a similarity between the matrices $A - KCM^{-1}(D)$ and $A - M^{-1}(D)KC$ which then have identical eigenvalues. On the other hand, we know that the pair (A, C) is observable (by assumption). Therefore, $A - M^{-1}(D)KC$ can be made Hurwitz by appropriately choosing the gain K. The same conclusion applies to the similar matrix $A - KCM^{-1}(D)$. It is thus demonstrated that the requirement on the gain K (to be selected so that $A - KCM^{-1}(D)$ is Hurwitz) is not an issue.

(b) A well posedness analysis of the system and the adaptive observer is outlined in Appendix A.

(c) Introduce the variable change $L = M^{-1}(D)K$. Then, $A - LC = M^{-1}(D)\left(A - KCM^{-1}(D)\right)M(D)$ is clearly Hurwitz and the adaptive observer (4a)–(4c) rewrites in terms of L as follows:

$$\begin{aligned} \dot{\hat{X}}(t) &= A\hat{X}(t) + \phi_1(t)\hat{\theta}_1(t) - M(D)L\tilde{u}(0,t) \\ &\quad + \lambda_1(t)\dot{\hat{\theta}}_1(t) + \lambda_2(t)\dot{\hat{\theta}}_2(t) \end{aligned} \tag{6a}$$

$$\begin{aligned} \hat{u}_t(x,t) &= \hat{u}_{xx}(x,t) - CM(x)L\tilde{u}(0,t) + \phi_2(x,t)\hat{\theta}_2(t) \\ &\quad + \left(\lambda_3(x,t) + CM(x)M^{-1}(D)\lambda_1(t)\right)\dot{\hat{\theta}}_1(t) \\ &\quad + \left(\lambda_4(x,t) + CM(x)M^{-1}(D)\lambda_2(t)\right)\dot{\hat{\theta}}_2(t) \end{aligned} \tag{6b}$$

$$\hat{u}_x(0,t) = 0, \qquad \hat{u}(D,t) = C\hat{X}(t). \tag{6c}$$

Clearly, if $\phi_1(t)$, $\phi_2(x,t)$ and $\dot{\hat{\theta}}_i(t)$ $(i = 1, 2)$ are all set to zero then, the adaptive observer (6a)–(6c) boils down to the (nonadaptive) observer (84)–(87) of Krstic (2009).

(d) The observer proposed in Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2015) concerned a class of ODE–PDE systems involving no parameter uncertainty. That non-adaptive high-gain type observer is quite different from the present adaptive observer of Table 1.

(e) In Ahmed-Ali, Fridman et al. (2016); Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2016), ODEs with output delay and parameter uncertainty were considered. Following Krstic and Smyshlyaev (2008), the observer design was performed on the basis of an ODE–PDE representation where the PDE, a first-order hyperbolic subject to no parameter uncertainty, represents the constant well-known delay. The resulting adaptive observer turned out to be a much simpler version of the present adaptive observer, e.g. while the parameter adaptive law in the former involved a single filter, the present adaptive law involves four interconnected filters, i.e. (4g)–(4l). In this respect, note also that the present adaptive observer is not an extension of the one in Ahmed-Ali, Giri, Krstic, Burlion, and Lamnabhi-Lagarrigue (2016). Indeed, the former does not boil down to the latter in the simpler case where $\theta_2 = 0$ because the PDEs are different in the two papers. That is, the adaptive observers in both papers are quite different from each other.

3. Adaptive observer analysis

Introduce the state and parameter estimation errors:

$$\begin{aligned} &\tilde{X} = \hat{X} - X, \qquad \tilde{u} = \hat{u} - u, \\ &\tilde{\theta}_1 = \hat{\theta}_1 - \theta_1, \qquad \tilde{\theta}_2 = \hat{\theta}_2 - \theta_2. \end{aligned} \tag{7a}$$

Also, for convenience, the following notations are introduced:

$$\theta = \left[\theta_1^T\ \theta_2^T\right]^T \in \mathbf{R}^m, \quad \tilde{\theta} = \hat{\theta} - \theta. \tag{7b}$$

Using (1)–(2) and (3a)–(3b), it is readily seen that these errors undergo the following equations:

$$\dot{\tilde{X}} = A\tilde{X} + \phi_1(t)\tilde{\theta}_1 - K\tilde{u}(0,t) + \lambda_1(t)\dot{\tilde{\theta}}_1(t) + \lambda_2(t)\dot{\tilde{\theta}}_2(t) \tag{8a}$$

$$\begin{aligned} \tilde{u}_t(x,t) &= \tilde{u}_{xx}(x,t) + \phi_2(x,t)\tilde{\theta}_2(t) - CM(x)M^{-1}(D)K\tilde{u}(0,t) \\ &\quad + \left(\lambda_3(x,t) + CM(x)M^{-1}(D)\lambda_1(t)\right)\dot{\tilde{\theta}}_1(t) \\ &\quad + \left(\lambda_4(x,t) + CM(x)M^{-1}(D)\lambda_2(t)\right)\dot{\tilde{\theta}}_2(t) \end{aligned} \tag{8b}$$

$$\tilde{u}(D,t) = C\tilde{X}(t) \tag{8c}$$

$$\tilde{u}_x(0,t) = 0 \tag{8d}$$

$$\dot{\tilde{\theta}}(t) = -\rho R(t)\Lambda(t)\tilde{u}(0,t) \tag{8e}$$

$$\dot{R}^{-1} = -R^{-1} + \Lambda\Lambda^T \tag{8f}$$

where the penultimate equation is obtained noticing that $\dot{\tilde{\theta}}(t) = \dot{\hat{\theta}}(t), \dot{\tilde{\theta}}_i(t) = \dot{\hat{\theta}}_i(t)\ (i = 1, 2)$ and the argument t is omitted (when clear) to alleviate expressions. In (8f), $\dot{R}^{-1}$ refers to the derivative dR^{-1}/dt. Eq. (8f) is equivalent to (4e) provided that $R(t)$ is uniformly invertible. To meet this requirement, a persistent excitation (PE) property is needed. To this end, the following notation is introduced:

$$\Lambda(t) = \left[CM^{-1}(D)\lambda_1(t) + \lambda_3(0,t) \quad CM^{-1}(D)\lambda_2(t) + \lambda_4(0,t)\right]^T.$$

Then, the PE assumption is stated as follows:

PE Assumption. The vector signal $\Lambda(t)$ is supposed to be persistently exciting in the sense that,

$$\exists\delta, \varepsilon_0 > 0,\ \forall t > 0 : \quad \int_t^{t+\delta} \Lambda(s)\Lambda^T(s)ds > \varepsilon_0 I \tag{9}$$

where $I \in \mathbf{R}^{m\times m}$ denotes the identity matrix.

Intuitively, the PE assumption means that the vector subset $\{\Lambda(s);\ t \le s \le t+\delta\}$ spans the parameter vector space $\mathbf{R}^m$, whatever t. It is readily seen from (4g)–(4j) that $\Lambda(t)$ only depends on the known input signals $\phi_1(t)$ and $\phi_2(0,t)$ and the auxiliary function $M(x)$ (i.e. $\Lambda(t)$ does not depend on the state estimates generated by the observer). Therefore, it is quite possible to check whether the PE condition (9) is satisfied. Given that $\phi_1(t)$ and $\phi_2(x,t)$ linearly enter (4g)–(4j), the PE requirement is met if the power spectra of $\phi_1(t)$ and $\phi_2(0,t)$ are rich enough (e.g. Ioannou & Sun, 2006). Now, it is shown in many places (e.g. Ioannou & Sun, 2006; Zhang, 2002) that if (9) holds then the time-varying matrix gain inverse $R^{-1}(t)$ exists, is positive definite, and stays bounded away from 0. Specifically, one then has

$$r_0 \le R^{-1}(t) \le r_1, \quad \text{for all } t \ge 0 \tag{10}$$

for some couple (r_0, r_1) of positive scalars. In the sequel, it is supposed that property (9) holds so that one can make use of (9)–(10).

Remark 2. Conditions (9)–(10) will prove to be crucial in making the parameter estimation error $\tilde{\theta}$ exponentially convergent to zero. It is worth noting that the structure of the parameter adaptive law (4d)–(4f) is quite similar to the so-called "forgetting-factor least-squares" estimator (see e.g. Ioannou & Sun, 2006, p. 198). Except for the structural similarity, the adaptive law (4d)–(4f) is novel due to the way the regressor $\Lambda(t)$ is generated. □

The exponential convergence of the adaptive observer is now stated in the following theorem:

Theorem 1. *Consider the adaptive observer of Table 1 and let there the gain ρ of the parameter adaptive law be such that $\rho > 1/2$. Then, when applied to the system (2)–(3), the observer is globally exponentially convergent in the sense that the errors, $\tilde{X}(t)$, $\tilde{\theta}(t)$, and the norm $\int_0^D \tilde{u}^2(x,t)dx$, are exponentially vanishing (as $t \to \infty$), whatever the initial conditions are $\hat{X}(0) \in \mathbf{R}^n$, $\hat{u}[0] \in L_2(0,D)$, $\hat{\theta}(0) \in \mathbf{R}^m$.* □

Proof. The proof of the theorem is divided in four parts. In the first part, the four auxiliary state vectors, $\lambda_1(t)$ to $\lambda_4(x,t)$, generated by (4g)–(4l) are shown to be bounded. The second part of the proof is devoted to introducing a transformation of the error system (8a)–(8f). The transformed system is shown in the third part to be exponentially stable. Finally, the results of the theorem are established in the final part.

Part 1. Boundedness of $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(x,t)$ and $\lambda_4(x,t)$.

It will be shown that there exist real numbers, $\lambda_{i,\max} > 0$ $(i = 1, 2, 3, 4)$, such that the following properties hold:

$$\|\lambda_i(t)\| \le \lambda_{i,M} \quad (i = 1,2) \quad \text{and} \quad \|\lambda_i(x,t)\| \le \lambda_{1,M}$$
$$(i = 3, 4), \quad \text{for all } t \ge 0 \text{ and all } x \in [0,\ D]. \tag{11}$$

To alleviate the paper body, the proof of this result is placed in Appendix C.

Part 2. Transformation of the error system (8a)–(8f).

Consider the following backstepping transformations, partly inspired by Krstic (2009):

$$Z(t) = \tilde{X}(t) - \lambda_1(t)\tilde{\theta}_1(t) - \lambda_2(t)\tilde{\theta}_2(t), \tag{12a}$$

$$\begin{aligned}\varepsilon(x,t) = {} & \tilde{u}(x,t) - CM(x)M^{-1}(D)\tilde{X}(t) \\ & - \lambda_3(x,t)\tilde{\theta}_1(t) - \lambda_4(x,t)\tilde{\theta}_2(t).\end{aligned} \tag{12b}$$

Differentiating $Z(t)$, one gets using (8a)–(8b), (12a)–(12b) and (4m):

$$\begin{aligned}
\dot{Z} &= \dot{\tilde{X}} - \dot{\lambda}_1\tilde{\theta}_1 - \lambda_1\dot{\tilde{\theta}}_1 - \dot{\lambda}_2\tilde{\theta}_2 - \lambda_2\dot{\tilde{\theta}}_2 \\
&= A\tilde{X} + \phi_1(t)\tilde{\theta}_1 - K\tilde{u}(0,t) - \dot{\lambda}_1\tilde{\theta}_1 - \dot{\lambda}_2\tilde{\theta}_2 \quad \text{(using (8a))} \\
&= AZ + A\lambda_1\tilde{\theta}_1 + A\lambda_2\tilde{\theta}_2 + \phi_1(t)\tilde{\theta}_1 - K\,\tilde{u}(0,t) \\
&\quad - \dot{\lambda}_1\tilde{\theta}_1 - \dot{\lambda}_2\tilde{\theta}_2 \quad \text{(using (12a))} \\
&= AZ - K\varepsilon(0,t) - KCM^{-1}(D)\tilde{X}(t) \\
&\quad - [K\lambda_3(0,t) + \dot{\lambda}_1 - A\lambda_1 - \phi_1(t)]\tilde{\theta}_1 \\
&\quad - [K\lambda_4(0,t) + \dot{\lambda}_2 - A\lambda_2]\tilde{\theta}_2 \quad \text{(using (12b))} \\
&= \left(A - KCM^{-1}(D)\right)Z - K\varepsilon(0,t) \\
&\quad - [K\lambda_3(0,t) + \dot{\lambda}_1 - A\lambda_1 - \phi_1(t) + KCM^{-1}(D)\lambda_1]\tilde{\theta}_1 \\
&\quad - [K\lambda_4(0,t) + \dot{\lambda}_2 - A\lambda_2 + KCM^{-1}(D)\lambda_2]\tilde{\theta}_2
\end{aligned}$$

using (12a), where the arguments (x,t) have been omitted to alleviate expressions. Using (4g) and (4j), the last equality further simplifies to:

$$\dot{Z} = \left(A - KCM(0)M^{-1}(D)\right)Z - K\varepsilon(0,t). \tag{13a}$$

Similarly, differentiating $\varepsilon(x,t)$ with respect to t, one obtains using (8a)–(8b) and (12b):

$$\begin{aligned}
\varepsilon_t &= \tilde{u}_t - CM(x)M^{-1}(D)\dot{\tilde{X}} - \lambda_{3,t}\tilde{\theta}_1 - \lambda_3\dot{\tilde{\theta}}_1 - \lambda_{4,t}\tilde{\theta}_2 - \lambda_4\dot{\tilde{\theta}}_2 \\
&= \tilde{u}_{xx} - CM(x)M^{-1}(D)A\tilde{X} \\
&\quad - \left(CM(x)M^{-1}(D)\phi_1 + \lambda_{3,t}\right)\tilde{\theta}_1 + \left(\phi_2 - \lambda_{4,t}\right)\tilde{\theta}_2 \\
&\qquad \text{(using (8a)–(8b))} \\
&= \varepsilon_{xx} + C\left(\frac{d^2M}{dx^2}(x)M^{-1}(D) - M(x)M^{-1}(D)A\right)\tilde{X} \\
&\quad - \left(CM(x)M^{-1}(D)\phi_1 + \lambda_{3,t} - \lambda_{3,xx}\right)\tilde{\theta}_1 \\
&\quad + \left(\phi_2 - \lambda_{4,t} + \lambda_{4,xx}\right)\tilde{\theta}_2 \quad \text{(using (12b))}.
\end{aligned}$$

Using (4i)–(4j) and Appendix B (Part 2), the last equality boils down to:

$$\varepsilon_t(x,t) = \varepsilon_{xx}(x,t). \tag{13b}$$

This is completed by the following two boundary conditions:

$$\varepsilon(D,t) = 0 \tag{13c}$$

$$\varepsilon_x(0,t) = 0 \tag{13d}$$

where (13c) is obtained from (12b) using (8c)–(8d); Eq. (13d) is also obtained from (12b) using Appendix B (Part 1) and (4k)–(4l).

In addition to (13a)–(13d), Eqs. (8e)–(8f) which govern $\tilde{\theta}$ must be retained but the term $\tilde{u}(0,t)$ on the right side of (8e) has to be expressed in terms of $\varepsilon(0,t)$, $Z(t)$ and $\tilde{\theta}(t)$, using (12a)–(12b). The resulting equation together with (13a)–(13d) constitutes the new error system expressed in terms of the new coordinates, Z and ε. For convenience, this new system is recapitulated here:

$$\dot{Z} = \left(A - KCM^{-1}(D)\right) Z - K\varepsilon(0,t) \tag{14a}$$

$$\varepsilon_t(x,t) = \varepsilon_{xx}(x,t) \tag{14b}$$

$$\varepsilon(D,t) = 0 \tag{14c}$$

$$\varepsilon_x(0,t) = 0 \tag{14d}$$

$$\dot{\tilde{\theta}} = -\rho R\Lambda\Lambda^T\tilde{\theta} - \rho R\Lambda\left(\varepsilon(0,t) + CM^{-1}(D)Z\right) \tag{14e}$$

$$\dot{R}^{-1} = -R^{-1} + \Lambda\Lambda^T. \tag{14f}$$

Part 3. Stability analysis of the transformed system (14a)–(14f).

To analyze the stability of this new error system, consider the following Lyapunov functional candidate:

$$V = Z^TPZ + a\int_0^D \varepsilon^2(x,t)dx + \tilde{\theta}^TR^{-1}\tilde{\theta} \tag{15}$$

with P any symmetric positive definite matrix satisfying the algebraic equation,

$$P(A - KCM^{-1}(D)) + (A - KCM^{-1}(D))^TP \leq -\mu\,\mathbf{I}. \tag{16}$$

The matrix P exists because $(A - KCM^{-1}(D))$ is Hurwitz. At this stage, the scalars $a > 0$ are arbitrary. Differentiating (15) yields, using (16) and (14a)–(14f):

$$\begin{aligned}
\dot{V} &= \dot{Z}^TPZ + Z^TP\dot{Z} + 2a\int_0^D \varepsilon(x,t)\varepsilon_t(x,t)dx \\
&\quad + \tilde{\theta}^T\dot{R}^{-1}\tilde{\theta} + 2\tilde{\theta}^TR^{-1}\dot{\tilde{\theta}} \\
&= \left([A - KCM^{-1}(D)]Z - K\varepsilon(0,t)\right)^T PZ \\
&\quad + Z^TP\left([A - KCM^{-1}(D)]Z - K\varepsilon(0,t)\right) \\
&\quad + 2a\int_0^D \varepsilon(x,t)\varepsilon_{xx}(x,t)dx + \tilde{\theta}^T[-R^{-1} + \Lambda\Lambda^T]\tilde{\theta} \\
&\quad + 2\tilde{\theta}^TR^{-1}\left(-\rho R\Lambda\Lambda^T\tilde{\theta} - \rho R\Lambda[\varepsilon(0,t) + CM^{-1}(D)Z]\right) \\
&\leq -\mu\left\|\tilde{Z}\right\|^2 - 2\tilde{Z}^TPK\varepsilon(0,t) - 2a\int_0^D \varepsilon_x^2(x,t)dx \\
&\quad - \tilde{\theta}^TR^{-1}\tilde{\theta} + (\Lambda\tilde{\theta})^2 - 2\rho(\Lambda^T\tilde{\theta})^2 \\
&\quad - 2\rho\tilde{\theta}^T\Lambda\varepsilon(0,t) - 2\rho\tilde{\theta}^T\Lambda CM^{-1}(D)Z
\end{aligned} \tag{17}$$

where an integration by parts and the boundary conditions (14c–d) have been used in the last inequality. Applying Young's inequality to the cross products on the right side of (17), one gets:

$$\begin{aligned}
\dot{V} &\leq -\mu\,\|Z\|^2 - 2a\int_0^D \varepsilon_x^2(x,t)dx - \tilde{\theta}^TR^{-1}\tilde{\theta} - (2\rho - 1)(\tilde{\theta}^T\Lambda)^2 \\
&\quad + \xi\,\|Z\|^2 + \frac{\|PK\|^2}{\xi}(\varepsilon(0,t))^2 \\
&\quad + \frac{\rho}{\upsilon}\varepsilon^2(0,t) + \rho\upsilon(\tilde{\theta}^T\Lambda)^2 + \rho\omega\,\|Z\|^2 \\
&\quad + \frac{\rho\left\|CM^{-1}(D)\right\|^2}{\omega}(\tilde{\theta}^T\Lambda)^2 \\
&\leq -(\mu - \xi - \rho\omega)\,\|Z\|^2 - \tilde{\theta}^TR^{-1}\tilde{\theta} \\
&\quad - \left(2a - \frac{D\,\|PK\|^2}{\xi} - \frac{D\rho}{\upsilon}\right)\int_0^D \varepsilon_x^2(x,t)dx \\
&\quad - \left(2\rho - 1 - \rho(\upsilon + \frac{\left\|CM^{-1}(D)\right\|^2}{\omega})\right)(\tilde{\theta}^T\Lambda)^2
\end{aligned} \tag{18}$$

where the last inequality is obtained applying Wirtinger's inequality (1b) and the scalars $\xi > 0$, $\upsilon > 0$ and $\omega > 0$ are arbitrary. Let these free parameters be selected so that:

$$\mu - \xi - \rho\omega > 0 \tag{19a}$$

$$2a - \frac{D\,\|PK\|^2}{\xi} - \frac{D\rho}{\upsilon} > 0 \tag{19b}$$

$$2\rho - 1 - \rho(\upsilon + \frac{\left\|CM^{-1}(D)\right\|^2}{\omega}) > 0. \tag{19c}$$

The last inequality is satisfied provided the pair (υ, ρ) is set such that $\upsilon + \frac{\left\|CM^{-1}(D)\right\|^2}{\omega} < \frac{2\rho-1}{\rho}$. This is possible because $\rho > 1/2$. Furthermore, letting $\xi = 1$ and noting that a is independent on the rest of the parameters (especially D, K, P, ρ), it follows that (19b) is satisfied provided that a is sufficiently large. In turn, (19a) is satisfied by letting μ be sufficiently large. Using (19a)–(19c), it follows from (18) and (15) that:

$$\begin{aligned}
\dot{V} &\leq -(\mu - 1 - \rho\omega)\,\|Z\|^2 - \tilde{\theta}^TR^{-1}\tilde{\theta} \\
&\quad - \left(2a - D\,\|PK\|^2 - \frac{D\rho}{\upsilon}\right)\frac{\pi^2}{4D}\int_0^D \varepsilon^2(x,t)dx \\
&\leq -\min\left(1, \frac{\mu - 1 - \rho\omega}{\lambda_{\max}(P)}, \frac{\pi^2}{4D}(2 - \frac{D\,\|PK\|^2}{a} - \frac{D\rho}{a\upsilon})\right) V
\end{aligned} \tag{20}$$

with $\lambda_{\max}(P)$ being the largest eigenvalue of P, where Wirtinger's inequality (1a) has been used to get the first inequality. Clearly, this implies that V is exponentially vanishing (as $t \to \infty$). Due to (15), so are $Z(t)$, $\tilde{\theta}(t)$ and $\int_0^D \varepsilon^2(x,t)dx$.

Part 4. Exponential convergence of the original errors.

Using the fact that $M(x)$, $\lambda_1(t)$ and $\lambda_3(x,t)$ are bounded and $\tilde{Z}(t), \tilde{\theta}(t)$ are exponentially convergent, it follows from (12a)–(12b) that $\tilde{X}(t)$ and $\tilde{u}(x,t)$ are also exponentially vanishing (as $t \to \infty$). Theorem 1 is proved. ■

Remark 3. The transformation defined by (12a)–(12b) and (4g)–(4l) is a key feature of the observer design and analysis. It is partly inspired by, but is also a generalization of, previous finite- or infinite-dimensional transformations. Indeed, the finite-dimensional transformation in Zhang (2002), which applies to adaptive observer design for ODEs, is obtained letting $\lambda_2(t)$, $\lambda_3(x,t)$ and $\lambda_4(x,t)$ be zero and deleting (12b). Also, the infinite-dimensional transformation in Krstic (2009), that applies to non-adaptive observer design for ODE–PDEs, is obtained by letting all of $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(x,t)$ and $\lambda_4(x,t)$ be zero and deleting (12a). The infinite-dimensional transformation in Ahmed-Ali, Giri, Krstic, Burlion, and Lamnabhi-Lagarrigue (2015) and Ahmed-Ali, Giri, Krstic, Lamnabhi-Lagarrigue, and Burlion (2016), used in adaptive observer design for parabolic PDEs, is obtained letting $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(x,t)$ be zero and deleting (12a). Leaving aside the difference between the PDEs in this paper and in Ahmed-Ali, Fridman et al. (2016); Ahmed-Ali, Giri, Krstic, and Lamnabhi-Lagarrigue (2016), the transformation in the latter can be viewed as a particular case of (12a)–(12b), obtained letting $\lambda_2(t)$ and $\lambda_4(x,t)$ be zero. Finally, note that only the new transformation (12a)–(12b) features a coupling between the (finite-dimensional) transformation, defined by (12a) and (4g)–(4h), on one hand, and the transformation (12b)

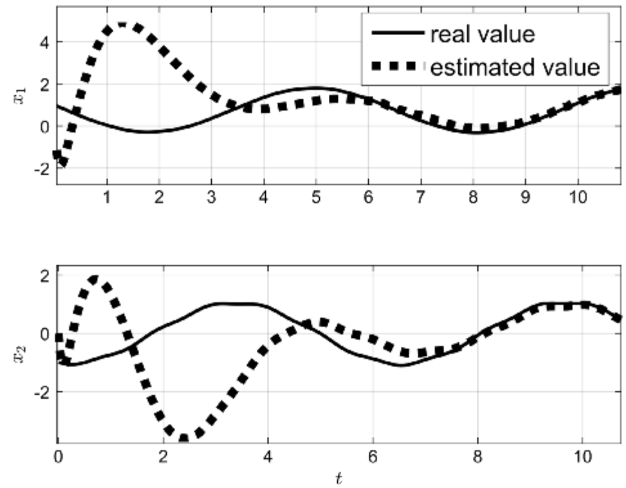

Fig. 2. System state variables $X(t) = [x_1(t)\ \ x_2(t)]^T$ (solid) and their estimates (dashed).

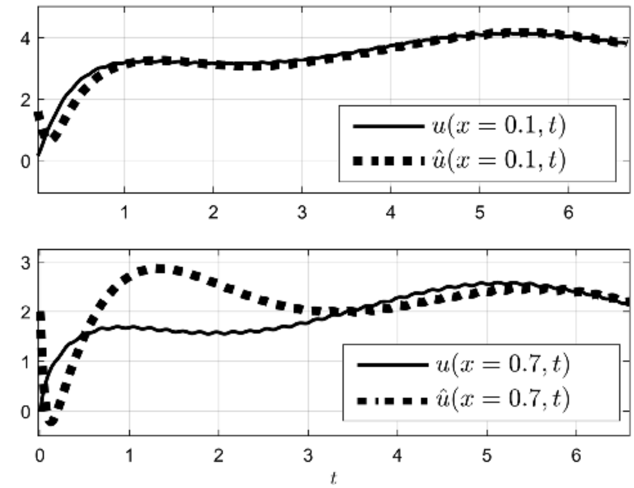

Fig. 4. PDE states $u(0.1, t)$ (top) and $u(0.7, t)$ (bottom). Solid: true system signals. Dashed: signal estimates.

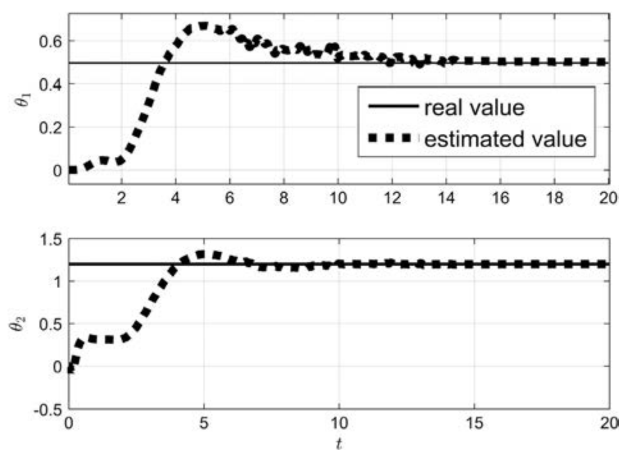

Fig. 3. System parameters $(\theta_1,\ \theta_2)$ (solid) and their estimates (dashed).

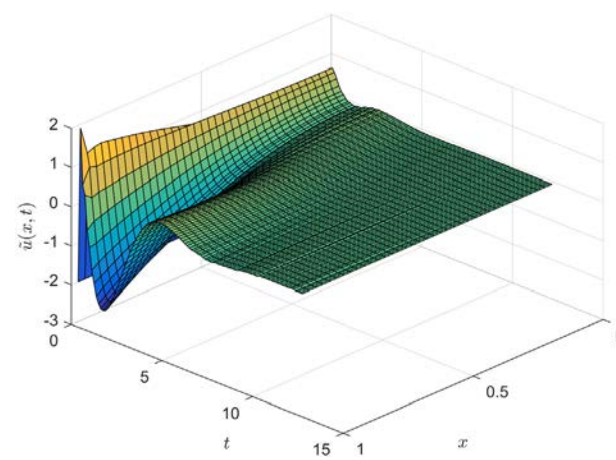

Fig. 5. Estimation error $\tilde{u}(x, t)$ vs $0 \le x \le 1$ and $t \ge 0$.

and (4i)–(4j) (involving infinite-dimensional signals), on the other hand. This coupling is part of the current transformation novelty that makes the observer analysis harder.

4. Simulation

To illustrate the performances of the adaptive observer of Table 1, a system of the form (2)–(3) is considered with the following parameters and functions:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix};$$
$$D = 1; \qquad \theta_1 = 0.5; \qquad \theta_2 = 1.2$$
$$\phi_1(t) = \begin{bmatrix} 0 \\ 1 + (\sin(3t))^2 \end{bmatrix}; \qquad \phi_2(x, t) = e^{0.2x}\left(5 + (\sin(3t))^2\right);$$

with the initial conditions $X(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$ and $u(x, 0) = 0$ ($0 \le x < D$). The parameters of the adaptive observer of Table 1 are set to $K = [17.1\ 11.8]^T$ (leading to the eigenvalues $(-3, -6)$ of the matrix $A - KCM^{-1}(D)$), $\rho = 1$, and $R(0) = 5I$. The observer is run with the initial conditions $\hat{X}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $\hat{u}(x, 0) = 2$ ($0 \le x < D$), $\hat{\theta}_1(0) = 0, \hat{\theta}_2(0) = 0$. The resulting observer performances are illustrated by Figs. 2–5. Clearly, all state and parameter estimates converge to their true values confirming the theoretical result of Theorem 1.

5. Conclusion

The problem of state observation is addressed for ODE–PDE cascades modeled by (2)–(3). The aim is to get online estimates of the states $X(t)$ and $u(x, t)$ ($0 \le x \le D$), on the one hand, and of the parameter vectors θ_1 and θ_2, on the other. To this end, the adaptive observer of Table 1 is designed and shown to be exponentially convergent. The gain matrix $M(x)$, the filters (4g)–(4l), and the forgetting-factor least-squares estimators (4d)–(4f) are instrumental components of this observer. Also, the presence of the parameter estimate speed $\dot{\hat{\theta}}$ in the state observer (4a)–(4b) is an appealing feature. The observer analysis shows that the backstepping transformation defined by (12a)–(12b) plays a crucial role in the proof of exponential convergence.

Appendix A. Existence analysis

The well-posedness analysis of the system (2)–(3) can be performed in many ways. One simple way is to introduce the following backstepping transformation, for $(x, t) \in [0, 1] \times [0, +\infty)$:

$$p(x, t) = u(x, t) - CM(x)M^{-1}(D)X(t). \tag{A.1}$$

Then, it is readily checked using (A.1) and Appendix B (Parts 1 and 2) that, the system (2)–(3) can be rewritten in the coordinates

(X, p) as follows:

$$\dot{X}(t) = AX(t) + \phi_1(t)\theta_1, \quad t \geq 0, \tag{A.2}$$

$$p_t(x,t) = p_{xx}(x,t) + \phi_2(x,t)\theta_2 - CM(x)M^{-1}(D)\phi_1(t)\theta_1 \tag{A.3}$$

$$p_x(0,t) = p(D,t) = 0, \quad \text{for all } t \geq 0 \tag{A.4}$$

$$u(x,t) = p(x,t) + CM(x)M^{-1}(D)X(t). \tag{A.5}$$

One feature of the new system representation is that the infinite-dimensional subsystem, here defined by (A.3)–(A.4), is decoupled from the finite-dimensional subsystem described by (A.2) (while a coupling existed in the initial model (2)–(3)). Then, the existence of solutions of each parts can be analyzed separately. The existence and uniqueness of the solution $X \in C^1([0,\infty) : \mathbf{R}^n)$ of (A.2) is not an issue, due to the usual existence theorem of ODEs. Owing to (A.3)–(A.4), this is a particular case of the more general semi-linear PDE (21) in Fridman and Blighovsky (2012). It has been proved that a unique strong solution $p[t]$ exists in the Hilbert space $H_{1/2} = D((-\Gamma)^{1/2}) = \left\{w \in H^1(0,D) : w_x(0) = w(D) = 0\right\}$ with $\Gamma = \frac{\partial^2}{\partial x^2}$, where $(-\Gamma)^{1/2}$ is the square root of $-\Gamma$. Then, a similar result holds with $u[t]$ due to (A.5).

To analyze the well posedness of the observer of Table 1, it is more judicious to start analyzing the (transformed) error system (14a)–(14f). First, the parabolic equation (14b)–(14d) has a well known solution in $H^2(0,D)$ that can be found in many places (see e.g. Smyshlyaev & Krstic, 2005, p. 315). Then, by the usual existence theorem of ODEs, (14a) has a unique solution $Z \in C^1([0,\infty) : \mathbf{R}^n)$. For the same reason, the ODEs (14e)–(14f) have unique solutions $\tilde{\theta} \in C^1([0,\infty) : \mathbf{R}^m)$ and $R^{-1} \in C^1([0,\infty) : \mathbf{R}^m)$ and, by (10) and (7b), one gets the unique existence of $\hat{\theta} \in C^1([0,\infty) : \mathbf{R}^m)$ and $R \in C^1([0,\infty) : \mathbf{R}^m)$. Well posedness of the parabolic PDEs (4i)–(4j) can also be established applying the analysis of Fridman and Blighovsky (2012). Accordingly, one gets that those equations have strong solutions $\lambda_3[t]$ and $\lambda_4[t]$ in the Hilbert space $H_{1/2}$ defined above. Then, by the usual existence theorem of ODEs, (4g)–(4h) have unique solutions $\lambda_1 \in C^1([0,\infty) : \mathbf{R}^{n\times m_1})$ and $\lambda_2 \in C^1([0,\infty) : \mathbf{R}^{n\times m_2})$. The results obtained so far, together with (12a)–(12b), imply the existence of a unique solution $\tilde{X} \in C^1([0,\infty) : \mathbf{R}^n)$ and unique strong solution $\tilde{u}[t]$ in the space $H_{1/2}$. Similar existence results hold with the states $\hat{X}$ and $\hat{u}[t]$, due to (7a). ■

Appendix B. Properties of the matrix function $M(x)$

The matrix function $M(x)$ satisfies the following properties (see Lemma 1 in Ahmed-Ali, Giri, Krstic, & Lamnabhi-Lagarrigue, 2015):

(1) $\frac{d^2M}{dx^2}(x) = M(x)A$, with $M(0) = I$, $\frac{dM}{dx}(0) = 0$.

(2) $M(x) = \sum_{i=0}^{\infty} \frac{x^{2i}}{(2i)!}A^i$.

(3) $AM(x) = M(x)A$ and $M^{-1}(x)A = AM^{-1}(x)$.

The first equality in (3) is obtained by pre- and post-multiplication of both sides of the development of Part 2 by A. The second equality is obtained by pre- and post-multiplication of both sides of the first equality by $M^{-1}(x)$. ■

Appendix C. Proof of boundedness of $\lambda_i(t)$ $(i = 1, 2)$ and $\lambda_i(x,t)$ $(i = 3, 4)$

As the vector signals $\phi_1(t)$ are bounded and $A - KCM^{-1}(D)$ is Hurwitz it follows from (4g)–(4f) that, $\lambda_1(t)$ and $\lambda_2(t)$ are bounded provided $\lambda_3(0,t)$ and $\lambda_4(0,t)$ are so. Then, one just needs to show that the latter are bounded. The proof will only be performed for $\lambda_3(0,t)$ (the proof for $\lambda_4(0,t)$ matches similar arguments). Consider the following Lyapunov functional candidate:

$$W_1(\lambda_3) = \frac{1}{2}\int_0^D \lambda_3(x,t)\lambda_3^T(x,t)dx + \frac{1}{2}\int_0^D \lambda_{3,x}(x,t)\lambda_{3,x}^T(x,t)dx. \tag{C.1}$$

Differentiating (C.1) gives, using (4i)–(4k) and integrating by parts:

$$\begin{aligned}
\dot{W}_1(\lambda_3) &= \int_0^D \lambda_3(x,t)\lambda_{3,t}^T(x,t)dx + \int_0^D \lambda_{3,x}(x,t)\lambda_{3,xt}^T(x,t)dx \\
&= \int_0^D \lambda_3(x,t)\lambda_{3,xx}^T(x,t)dx - \int_0^D \lambda_3(x,t)[CM(x)M^{-1}(D)\phi_1(t)]^T dx - \int_0^D \lambda_{3,x}(x,t)\lambda_{3,tx}^T(x,t)dx \\
&= \int_0^D \lambda_3(x,t)\lambda_{3,xx}^T(x,t)dx - \int_0^D \lambda_3(x,t)[CM(x)M^{-1}(D)\phi_1(t)]^T dx - \int_0^D \lambda_{3,xx}(x,t)\lambda_{3,t}^T(x,t)dx
\end{aligned} \tag{C.2}$$

where the last term is obtained using an integration by parts and the boundary conditions (4k). Again using integration by parts and (4k), equality (C.2) further develops as follows:

$$\begin{aligned}
\dot{W}_1(\lambda_3) &= -\int_0^D \left\|\lambda_{3,x}(x,t)\right\|^2 dx - \int_0^D \lambda_3(x,t)[CM(x)M^{-1}(D)\phi_1(t)]^T dx \\
&\quad - \int_0^D \lambda_{3,xx}(x,t)(\lambda_{3,xx}^T(x,t) - [CM(x)M^{-1}(D)\phi_1(t)]^T)dx \\
&\leq -\int_0^D \left\|\lambda_{3,x}(x,t)\right\|^2 dx + \frac{\xi}{2}\int_0^D \|\lambda_3(x,t)\|^2 dx + \frac{1}{2\xi}\int_0^D \left\|CM(x)M^{-1}(D)\phi_1(t)\right\|^2 dx \\
&\quad - \int_0^D \left\|\lambda_{3,xx}(x,t)\right\|^2 dx + \frac{\varsigma}{2}\int_0^D \left\|\lambda_{3,xx}(x,t)\right\|^2 dx + \frac{1}{2\varsigma}\int_0^D \left\|CM(x)M^{-1}(D)\phi_1(t)\right\|^2 dx
\end{aligned} \tag{C.3}$$

using Young's inequality twice, where $\xi > 0$ and $\varsigma > 0$ are arbitrary scalars. Applying Wirtinger's inequality (1a) to the second term on the right side of (C.3) yields:

$$\begin{aligned}
\dot{W}_1(\lambda_3) &\leq -\left(1 - \frac{2D\xi}{\pi^2}\right)\int_0^D \left\|\lambda_{3,x}(x,t)\right\|^2 dx - \left(1 - \frac{\varsigma}{2}\right)\int_0^D \left\|\lambda_{3,xx}(x,t)\right\|^2 dx \\
&\quad + \left(\frac{1}{2\xi} + \frac{1}{2\varsigma}\right)\int_0^D \left\|CM(x)M^{-1}(D)\phi_1(t)\right\|^2 dx.
\end{aligned} \tag{C.4}$$

Let the still free scalars be selected as follows:

$$1 - \frac{2D\xi}{\pi^2} < 0; \qquad 1 - \frac{\varsigma}{2} < 0. \tag{C.5}$$

Then, by applying Wirtinger's inequality (1a) to the two first terms on the right side of (C.4), it follows that:

$$\begin{aligned}
\dot{W}_1(\lambda_3) \le & -\left(1 - \frac{2D\xi}{\pi^2}\right)\frac{\pi^2}{4D}\int_0^D \|\lambda_3(x,t)\|^2\,dx \\
& -\left(1 - \frac{\varsigma}{2}\right)\frac{\pi^2}{4D}\int_0^D \left\|\lambda_{3,x}(x,t)\right\|^2 dx \\
& +\left(\frac{1}{2\xi} + \frac{1}{2\varsigma}\right)\int_0^D \left\|CM(x)M^{-1}(D)\phi_1(t)\right\|^2 dx \\
\le & -\frac{\pi^2}{2D}\min\left(1 - \frac{2D\xi}{\pi^2}, 1 - \frac{\varsigma}{2}\right) W_1(\lambda_3) \\
& +\left(\frac{1}{2\xi} + \frac{1}{2\varsigma}\right)\int_0^D \left\|CM(x)M^{-1}(D)\phi_1(t)\right\|^2 dx
\end{aligned} \tag{C.6}$$

where the last inequality is obtained using (C.1). Note that $\phi_1(t)$ is also bounded by assumption and $M(x)$ is bounded due to (4m). Then, it follows from (C.6) that $W_1(\lambda_3)$ is bounded and, from (C.1), so is $\int_0^D \left\|\lambda_{3,x}(x,t)\right\|^2 dx$. Applying Wirtinger's inequality (1b), it follows that $\|\lambda_3(x,t)\|$ is bounded. The proof is complete. ■

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