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IFAC PapersOnLine 50-1 (2017) 13348–13353

Control of differentially flat linear delay systems with constraints

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Abstract: We consider the control of differentially flat linear delay systems with constraints. The constraints can be given on the state and/or on the control. Linear delay systems are here envisioned as modules over a ring of differential and distributed delay operators. Due to the nice Bezout property that this ring enjoys, the controllability notions of freeness, projectivity and torsion freeness coincide. Thanks to the flatness (corresponding to freeness for linear systems) property, all constraints are reported through the flat output (the basis of the corresponding module). We then make use of polynomial B-splines as specialisations for the flat output; the constraints are then expressed as inequalities in these B-splines control points. Some examples illustrate the effectiveness of the approach.

Keywords: input/state constraints, π-freeness, linear delay system, B-spline trajectory

1. INTRODUCTION

The presence of delays in the state or the input characterizes many natural as well as artificial systems. A large part of the control literature is thus devoted to the study of linear and non linear delay systems (Richard, 2003), but few results are also able to handle constraints. Constraints in the input or the state variables are usually tackled with numerical methods entailing the computation of trajectory invariant intervals (Moussaoui et al., 2014) and positively invariant sets (Dambrine et al., 1995; Hennet and Tarbouriech, 1998), or more complex, but similar in spirit, Model Predictive Control approaches (see for instance (Marcus et al., 2010; Olaru and Niculescu, 2008)). A systematic approach capable of embedding the constraint satisfaction directly in the control formulation is still lacking for delay systems.

In the present work we propose a control design technique based on the π-freeness property for infinite linear systems (delay systems (Mounier, 1995; Mounier et al., 1997; Fliess and Mounier, 2001), partial differential equations (Gehring et al., 2013; Mounier and Greco, 2016)) which is an extension of the differential flatness originally developed for finite dimensional systems. Other algebraic related approaches include (Sename et al., 1995; Picard et al., 1998; Conte and Perdon, 2005).

According to this method, all the states and the control input of the linear delay system can be parametrized through a so-called π-flat output by using differentiations, delays and advances.

In other words, the nominal input and the states can be expressed as a linear combinations of the delayed and advanced reference trajectory and its derivatives. We here propose to embed a priori the constraints in the trajectory design and to exploit the differential flatness to ensure precise tracking.

The framework of the differential flatness, usually called *two degree of freedom controller*, can be decomposed into two steps: 1. Design of the reference trajectory of the flat outputs; offline computation of the open loop controls *(feedforward part)*. 2. The second stage is online computation of the complementary closed loop controls in order to stabilize the system around the reference trajectories *(feedback part)*.

Our goal is to write the *state/input constraints* in terms of the flat output and its derivatives. The idea is to translate the beauty and simplicity of the freeness property of a delay system (analogue of flatness for finite dimensional system) into constrained control via B-spline procedures for the reference trajectories. The adoption of B-spline curves is motivated by their peculiar properties, which allow a natural remapping of the constraints from the input/state to the flat output, while leaving sufficient flexibility to express a rich class of reference trajectories. We establish explicit relations between the control points of the B-spline describing the reference trajectory and those of the B-spline expressing the control input and states. This way, the constraints on input and states is directly translated in a set of inequalities for the control points of the reference trajectory. Such inequalities can then be efficiently solved by means of the *cylindrical algebraic decomposition (CAD)* (Strzebonski, 2006) to find admissible regions for the control points.

Once the constrained open-loop trajectories are generated offline, and in order to guarantee the stability and a certain robustness of the approach, we need a feedback control (second step). There are many different linear and nonlinear feedback controls that can be used to ensure convergence to zero of the tracking error. We obtain a

10.1016/j.ifacol.2017.08.1901

stable trajectory tracking with prescribed tracking error dynamics if distributed delays are admitted in the feedback law. This is a model based prediction.

The outline of the paper is as follows. In section 2, we recall the definition of π-freeness for delay linear systems and in the section 3, we give a stabilizing feedback law for a class of delay systems that makes use of predictor forms elaborated with distributed delays. In section 4, we give an overview of the B-splines curves and its properties. In section 5, we detail the procedure in establishing reference trajectories for constrained open-loop control. In section 6, we illustrate an example on car-following with human memory effects.

2. π-FREENESS FOR DELAY LINEAR SYSTEMS

We shall use a module theoretic approach framework developed, among others, in (Mounier, 1995; Fliess and Mounier, 1999). The adopted framework emphasizes on equations (rather than solutions) in order to study a given system. When dealing with linear equations, a system is associated with a module over a ring, this notion playing for differential equations the role played by vector spaces for linear algebraic equations. The basic definitions we shall use in this paper, can be found in (Mounier, 1995; Fliess, 1990).

2.1 Algebraic setting and preliminaries

Spectral controllability We shall consider linear delay systems as modules over the polynomial ring $\mathbb{R}[\frac{d}{dt}, \delta_1, \ldots, \delta_r]$ where the δ_i's play the role of localized delay operators. This ring is isomorphic to the ring $\mathbb{R}[s, e^{-h_1 s}, \ldots, e^{-h_r s}]$ (the variable s plays the role of $\frac{d}{dt}$, the h_i's being the amplitudes of the corresponding delays). In order to involve distributed delays, we use an extended ring: $\mathfrak{S}_r = \mathbb{R}(s)[e^{-h_1 s}, \ldots, e^{-h_r s}] \cap \mathfrak{E}$, where $\mathfrak{E}$ denote the ring of entire functions. This ring is a Bézout domain *i.e.* any finitely generated ideal in this domain is principal. A typical element of $\mathfrak{S}_r$ is $(1 - e^{-h_i s})/s$ (it is an entire function, since $1 - e^{-h_i s} = h_i s - h_i^2 s^2 + h_i^3 s^3 + \cdots$ is zero when $s = 0$), which corresponds to a distributed delay operator in the time domain. Another slightly larger ring is $\mathbb{R}[s, s^{-1}, e^{-h_1 s}, \ldots, e^{-h_r s}]$ which contains the integration (through application of the s^{-1} operator).
Given a ring R (commutative, with unity and no zero divisors, such as one of the above), an *R-system* is a module over R.
We shall consider three controllability notions, corresponding to algebraic properties of the corresponding module.
An R-system Λ is called *R-torsion free* (resp. *projective, free*) controllable if the corresponding module is torsion free (resp. projective, free).
Let us recall that, on a Bezout ring (as well as on a principal ideal domain such as $\mathbb{R}[\frac{d}{dt}]$), the three notions coincide.

In the next sections, by using this π-freeness formalism, we obtain all the system open-loop trajectories z_r (the states and the inputs) as functions of the π-flat output y_r, a finite number of its derivatives, time delays, and advances.

In the case of a π-free delay system, we embed *constraints* $K_l, K_h \in \mathbb{R}$ on a system open-loop trajectory by imposing:

$K_l \leqslant z_r \leqslant K_h$ with
$z_r = R(y_r, \dot{y}_r, \ldots, \delta_i^{\pm j} y_r^{(q)}, \theta y_r^{(q)}, \ldots, \delta_i^{\pm j} y_r^{(\gamma)}, \theta y_r^{(\gamma)})$,

where $\delta_i^{\pm j} y_r(t) = y_r(t \mp j\tau_i)$, are delays and advances respectively, and $(\theta y_r)(t) = \int_{t-h}^{t} e^{(t-\tau)} y_r(\tau) d\tau$ represents a distributed delay.

3. STABILIZATION OF THE SYSTEM

Here by the means of distributed delays in the feedback law, we avoid pure predictions (torsion-free controllable) *i.e.* the delay is compensated by the controller (Mounier and Rudolph, 1998). The control law achieves asymptotic tracking compensating the effects of the input delay. With this, we want to overcome the delay in the closed-loop which may be a source of poor system performance and instability.

Let us first consider one of the simplest system, *i.e.* a linear system with commensurate delay in the input

$$\dot{y}(t) = u(t - h) \tag{1}$$

for which the open-loop control yields

$$u_r(t) = \dot{y}_r(t + h).$$

For the closed-loop control, setting

$$u(t) = \dot{y}_r(t + h) - K_p e(t + h), \quad e(t) = y(t) - y_r(t) \tag{2}$$

$$e(t + h) = \int_t^{t+h} \dot{e}(\tau) d\tau + e(t)$$

we obtain

$$u(t) = \dot{y}_r(t + h) - K_p \int_t^{t+h} \dot{e}(\tau) d\tau - K_p e(t)$$

$$= K_p \big(- \int_t^{t+h} \dot{y}(\tau) d\tau + \int_t^{t+h} \dot{y}_r(\tau) d\tau - e(t) \big) + \dot{y}_r(t + h)$$

Finally, we obtain a closed-loop control

$$u(t) = K_p \big(- \int_{t-h}^{t} u(\tau) d\tau + \int_t^{t+h} \dot{y}_r(\tau) d\tau - e(t) \big) + \dot{y}_r(t+h)$$

which involves only distributed delays of finite support but no pure predictions.

4. B-SPLINES PRELIMINARIES

Using B-spline curve as reference trajectory is a simple way to reduce the problem of infinite unspecified function $f(t)$ into a finite dimensional one determined by control points c_j associated to a basis functions $B_{j,d}$.

4.1 B-splines

The B-spline $B_{j,d}$ depends on the knots $t_j, \ldots, t_{j+1+d}$. This means that if the knot vector is given by $\boldsymbol{t} = (t_j)_{j=1}^{n+d+1}$ for some positive integer n, we can form n B-splines $\{B_{j,d}\}_{j=1}^{n}$ of degree d associated with this knot vector. A B-spline curve (or a linear combination of B-splines) is a combination of B-splines of the form

$$f = \sum_{j=1}^{n} c_j B_{j,d} \tag{3}$$

where $\boldsymbol{c} = (c_j)_{j=1}^{n}$ are n real numbers. We formalise this in a definition.

Definition 1. (B-spline curves). Let

$$\boldsymbol{t} = (t_j)_{j=1}^{m=n+d+1} = [\underbrace{0,\dots,0}_{d+1}, t_{d+1}, \dots, t_{m-d-1}, \underbrace{1,\dots,1}_{d+1}]$$

be a non-decreasing sequence of real numbers, *i.e.* , a knot vector for a total of n B-splines. The linear space of all linear combinations of these B-splines is the spline space $\mathbb{S}_{d,\boldsymbol{t}}$ defined by

$$\mathbb{S}_{d,\boldsymbol{t}} = \left\{ \sum_{j=1}^{n} c_j B_{j,d} \mid c_j \in \mathbb{R} \text{ for } 1 \leqslant j \leqslant n \right\}$$

An element $f = \sum_{j=1}^{n} c_j B_{j,d}$ of $\mathbb{S}_{d,\boldsymbol{t}}$ is called a B-spline curve or a spline function, of degree d with knots $\boldsymbol{t}$, and $(c_j)_{j=1}^{n}$ are called the control points of the B-spline curve (see Fig.1).

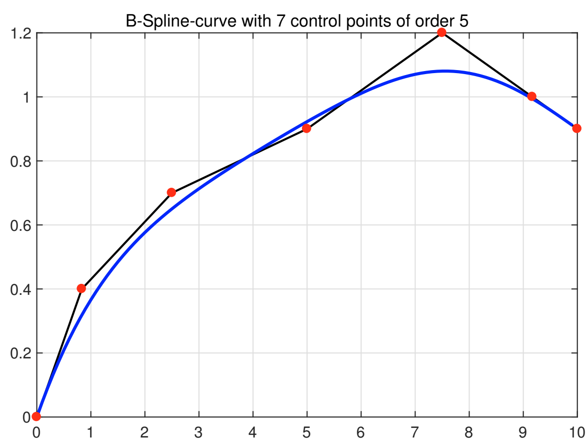

Fig. 1. The B-spline curve and its control polygon

Definition 2. (Cox-DeBoor recursion formula) Let d be a nonnegative integer and let $\boldsymbol{t} = (t_j)$, the knot vector or knot sequence, be a non-decreasing sequence of real numbers of at least $d+2$. The jth B-spline of degree d (order k) with knots $\boldsymbol{t}$ is defined by:

$$B_{j,k,\boldsymbol{t}}(x) = \frac{x - t_j}{t_{j+d} - t_j} B_{j,d-1,\boldsymbol{t}}(x) + \frac{t_{j+k} - x}{t_{j+1+d} - t_{j+1}} B_{j+1,d-1,\boldsymbol{t}}(x)$$

for all real numbers x, with

$$B_{j,0,\boldsymbol{t}}(x) = \begin{cases} 1, & \text{if } t_j \leqslant x < t_{j+1} \\ 0, & \text{otherwise} \end{cases}$$

Remark 1.
- Choosing the knot vector in this way guarantees the start and end tangent property.
- Each control point movement only has local effects.

Definition 3. (Control polygon for B-spline curve (see (Lyche and Morken, 2002))). Let $f = \sum_{j+1}^{n} c_j B_{j,d}$ be a spline in $\mathbb{S}_{d,\boldsymbol{t}}$. the control points of f are the points with coordinates (t_j^*, c_j) for $j = 1, \dots, n$ where

$$t_j^* = \frac{t_{j+1} + \dots + t_{j+d}}{d} \tag{4}$$

are the knot averages of $\boldsymbol{t}$. The control polygon of f is the piecewise linear function obtained by connecting neighbouring points by straight lines.

4.2 B-spline properties

B-splines play a central role in the representation of B-spline curves. For that purpose, we report here the most important properties.

Lemma 1. ((Lyche and Morken, 2002) page 40) Let d be a nonnegative polynomial degree and let $\boldsymbol{t} = (t_j)$ be a knot sequence. The B-splines on $\boldsymbol{t}$ have the following properties:

(1) Local knots. The jth B-splines $B_{j,d}$ depends only on the knots $t_j, t_{j+1}, \dots, t_{j+d+1}$.
(2) Local support
- If x is outside the interval $[t_j, t_{j+d+1})$ then $B_{j,d}(x) = 0$. In particular, if $t_j = t_{j+d+1}$ then $B_{j,d}$ is identically zero.
- If x lies in the interval $[t_\mu, t_{\mu+1})$ then $B_{j,d}(x) = 0$ if $j < \mu - d$ or $j > \mu$.

(3) Positivity. If $x \in (t_j, t_{j+d+1})$ then $B_{j,d}(x) > 0$. the closed interval $[t_j, t_{j+d+1}]$ is called the support of $B_{j,d}$.
(4) Piecewise polynomial. The B-spline $B_{j,d}$ can be written

$$B_{j,d}(x) = \sum_{k=j}^{j+d} B_{j,d}^{k}(x) B_{k,0}(x) \tag{5}$$

where each $B_{j,d}^{k}(x)$ is a polynomial of degree d.
(5) Special values. If $z = t_{j+1} = \dots = t_{j+d} < t_{j+d+1}$ then $B_{j,d}(z) = 1$ and $B_{i,d}(z) = 0$ for $i \neq j$.
(6) Smoothness. If the number z occurs m times among $t_j, t_{j+1}, \dots, t_{j+d+1}$ then the derivatives of $B_{j,d}$ of order $0, 1, \dots, d-m$ are all continuous at z.

5. CONSTRAINED TRAJECTORY GENERATION PROCEDURE

In this section we present the design of the desired B-spline trajectory $y_r(t)$. The initial equilibrium point is its first control point $y_{initial} = c_0$ and the final equilibrium point is its last control point $y_{final} = c_n$.

For the sake of completeness, we state a few necessary B-spline ingredients (derivative, integral, and degree elevation) that are crucial in establishing our result.

5.1 Derivative property of the B-spline curve

Theorem 1. (see (Lyche and Morken, 2002)) The derivative of the jth B-spline of degree d on $\boldsymbol{t}$ is given by

$$DB_{j,d}(x) = d\left(\frac{B_{j,d-1}(x)}{t_{j+d} - t_j} - \frac{B_{j+1,d-1}(x)}{t_{j+1+d} - t_{j+1}}\right) \tag{6}$$

for $d \geqslant 1$ and for any real number x.

According to the previous theorem, if the flat output or the reference trajectory $\boldsymbol{y_r}$ is a B-spline curve, its derivative is still a B-spline curve and we can explicitly compute its control points.

Let $y^{(\nu)}(x)$ denote the νth derivative of $y(x)$. If x is fixed, we can obtain $y^{(\nu)}(x)$ by computing the νth derivatives of the basis functions:

$$y^{(\nu)}(x) = \sum_{j=1}^{n} c_j B_{j,d}^{(\nu)}(x) \tag{7}$$

Letting $c_j^{(0)} = c_j$, we write

$$y(x) = y^{(0)}(x) = \sum_{j=1}^{n} c_j^{(0)} B_{j,d}(x) \tag{8}$$

Then,

$$y^{(\nu)}(x) = \sum_{j=1}^{n-\nu} c_j^{(\nu)} B_{j,d-\nu}(x) \tag{9}$$

with derivative control points such that

$$c_j^{(\nu)} = \begin{cases} c_j, & \nu = 0 \\ \dfrac{d-\nu+1}{t_{j+d+1} - t_{j+\nu}} \left(c_{j+1}^{(\nu-1)} - c_j^{(\nu-1)} \right), & \nu > 0 \end{cases} \tag{10}$$

and a vector knot

$$\boldsymbol{t}^{(\nu)} = \underbrace{0, \ldots, 0}_{d-\nu+1}, t_{d+1}, \ldots, t_{m-d-1}, \underbrace{1, \ldots, 1}_{d-\nu+1}$$

Remark 2. We should take a B-spline curve of degree $d > \nu$, where ν is the derivation order of the flat output, to avoid to introduce discontinuities.

5.2 Integral property of B-spline curve

Similar to the derivation operation, an integral of a B-spline is a B-spline and we are able to find the control points of the integral of the B-spline curve in terms of the control points of the initial B-spline curve.
The indefinite integral of a B-spline function $f(x)$ (see (de Boor, 2001))

$$f(x) = \sum_{j=1}^{n} c_j B_{j,d}(x) \tag{11}$$

on the knot vector $(t_j)_{j=1}^{n+d+1}$ is given by the B-spline function $g(x)$ where

$$\begin{aligned} g(x) &= \int_{t_1}^{x} \sum_{j=1}^{n} c_j B_{j,k}(u) du \\ &= \sum_{j=1}^{n} \left(\frac{t_{j+k+1} - t_j}{k} \sum_{i=1}^{i} c_i \right) B_{j,k+1}(x), \\ & t_k \leqslant x \leqslant t_{n+1}. \end{aligned} \tag{12}$$

Hence the integral of a B-spline is presented as:

$$g(x) = \int_{t_1}^{x} f(u) du = \sum_{j=1}^{n+1} e_j B_{j,k+1}(x), \tag{13}$$

where

$$e_1 = 0, \quad e_{j+1} = \frac{1}{d+1} \sum_{i=1}^{j} c_i \left(t_{i+d+1} - t_i \right) \qquad 1 \leqslant j \leqslant n. \tag{14}$$

The knot vector for $g(x)$ matches that of the original curve except for the extra knot at both ends due to the increased degree. For a definite integral of a B-spline we have:

$$\int_{x_1}^{x_2} f(u) du = \int_{t_1}^{x_2} f(u) du - \int_{t_1}^{x_1} f(u) du = g(x_2) - g(x_1),$$

$t_k \leqslant x_1, \quad x_2 \leqslant t_{n+1}$.

Remark 3. Notice that the input constraints in the presence of commensurable and/or distributed known delay in the state $x(t-h)$ or $\int_{t-h}^{t} x(\tau) d\tau$ can be given in straightforward algebraic manner.

Thanks to the integral property, we can easily deal with distributed delay in the state. For instance, consider a system in the form:

$$\dot{y} = \int_{t-h}^{t} y(\tau) d\tau + y(t-h) + u \tag{15}$$

where y is the flat output. An open loop control allowing the tracking of y_r by y is:

$$u_r = \dot{y}_r(t+h) - \int_{t-h}^{t} y_r(\tau) d\tau - y_r(t+h). \tag{16}$$

5.3 Degree elevation and knot insertion

To accomplish an addition and/or subtraction of two B-spline curves $f(x)$ and $g(x)$ with different degrees d_f and d_g respectively s.t. $d_f < d_g$, first, we need to increase $(d_g - d_f)$ times the degree of $f(x)$. A good visual algorithm of the degree elevation and knot insertion of the B-spline reference trajectory can be found in (Piegl and Tiller, 1994).

5.4 Reference trajectory design procedure

The simple-minded idea on the reference trajectory design is based on the following steps:

- Assign a B-spline reference trajectory to each flat output
- Find the analytical B-spline expressions of the states and the inputs
- Express the input/state constraints as inequalities in terms of the B-spline control points and find the suitable region for each control point of the B-spline reference trajectory by using the B-spline properties (see the previous sections 5.1, 5.2, 5.3).

We find the relationship between the input control points U_i and the control points of the reference trajectory c_j such that $U_i = \Phi_i(c_1, \ldots, c_n)$. Our aim is to constraint $K_l \leqslant |U_i| \leqslant K_h$ by choosing suitable regions for the reference trajectory.
Remark 4. To solve the system of inequalities, we use symbolic computation of the Cylindrical Algebraic Decomposition (CAD) algorithm which is the best currently known algorithms for solving many classes of problems related to systems of real polynomial equations and inequalities (Strzebonski, 2006). By using Cylindrical Algebraic Decomposition, we compute the regions in which one chooses the values for c_j's of the reference trajectory.

6. EXAMPLE: CAR-FOLLOWING MODEL

We investigate a car-following model including human drivers memory effects from (Sipahi and Niculescu, 2010). For the sake of clarity, we consider a simplified example. The dynamics of two vehicles, when the second vehicle follows the first vehicle is represented by the following equations:

$$\ddot{y}_1 = \alpha \int_{h}^{h+\delta} f(\tau) H(t-\tau) d\tau - K_p(y_2 - y_1) \tag{17a}$$

$$\ddot{y}_2 = u \tag{17b}$$

where y_1 and y_2 are the positions of the first and the second vehicle respectively, $H(t) = y_2(t) - y_1(t)$ is the headway perturbation between the vehicles, u is the motor torque, taken as control input and the constants: $\alpha > 0$ is the measure of the driver's aggressiveness per unit vehicle mass and K_p is the human regulation parameter. The delayed action/decision of human drivers is represented using distributed delays. As distribution function $f(t)$, as stated in (Sipahi and Niculescu, 2010), we take the uniform distribution, which is a good fit for modelling the short-term memory of drivers:

$$f(\tau) = \begin{cases} \frac{1}{\delta}, & h \leqslant \tau \leqslant h + \delta \\ 0, & \text{otherwise,} \end{cases} \tag{18}$$

where h is the memory dead-time and δ is the memory window.

The model (17a)-(17b) is π-free, with basis (or flat output) given by H, *i.e.* all system variables can be differentially parametrized by H.

With the notation $\hat{y}_1$, $\hat{y}_2$ and $\hat{u}$ for the Laplace transform, (17a)-(17b) is given by:

$$s^2\hat{y}_1 = \alpha e^{-hs}\frac{1-e^{-\delta s}}{\delta s}(\hat{y}_2 - \hat{y}_1) - K_p(\hat{y}_2 - \hat{y}_1), \tag{19}$$

$$s^2\hat{y}_2 = \hat{u} \tag{20}$$

Let

$$a(s) = s^2 - K_p + \alpha e^{-hs}\frac{1-e^{-\delta s}}{\delta s}.$$

We get a differential parametrization of the system as

$$\hat{y}_1 = \left(\frac{a(s)}{s^2} - 1\right)\hat{H},$$
$$\hat{y}_2 = \frac{a(s)}{s^2}\hat{H},$$
$$\hat{u} = a(s)\hat{H}.$$

From where, we obtain the time domain expression for the open loop control:

$$u_r(t) = \ddot{H}_r(t) - K_p H_r(t) + \frac{\alpha}{\delta}\int_{t-\delta}^{t} H_r(\tau - h)d\tau \tag{21}$$

For the sake of simplicity, we take $\delta = (k-1)h$.

$$u_r(t) = \underbrace{\ddot{H}_r(t)}_{\text{first term}} - K_p \underbrace{H_r(t)}_{\text{second term}} + \underbrace{\frac{\alpha}{(k-1)h}\int_{t-kh}^{t-h} H_r(\tau)d\tau}_{\text{third term}} \tag{22}$$

We take as symbolic reference trajectory H_r a B-spline curve with degree $d = 4$, knot vector

$$T = \{0, 0, 0, 0, 0, 10/3, 20/3, 10, 10, 10, 10, 10\}$$

and control points vector $\boldsymbol{A} = (a_j)_{j=1}^7$ as

$$H_r = \sum_{j=1}^{7} a_j B_{j,d}. \tag{23}$$

The constraints we consider are the following:

(1) Distance constraint: $H_{min} \leqslant H \leqslant H_{max}$
(2) Actuator limit: $U_{min} \leqslant u \leqslant U_{max}$.

The first constraint will be respected by choosing the control points for the reference trajectory H_r such that $H_{min} \leqslant a_j \leqslant H_{max}$.

For the second constraint, using the properties of the B-spline curve, we can find the control points of the open-loop control u_r in terms of the a_j's by following these steps:

(1) First, we find the control points $a_j^{(2)}$ for the second derivative $\ddot{H}_{xr}$ by using the formula (10):

$$\ddot{H}_{xr} = \sum_{j=1}^{5} a_j^{(2)} B_{j,1}(t)$$

(2) We obtain the third term $\int_{t-kh}^{t-h} H(\tau)d\tau$ by

$$\int_{t-kh}^{t-h} H(\tau)d\tau = \sum_{j=1}^{8} e_j B_{j,5}(t)$$

which is a B-spline curve of degree 5 and where the control points e_i are calculated by the integral operation (14).

(3) We elevate the degree of the first term and the second term up to 5 and then, we add additional knots in order to end up with the same number of control points in the three terms. After, we can find the sum of these terms. We end up with u_r as a B-spline curve of degree 8 with control points U_i:

$$u_r(t) = \sum_{i=1}^{14} U_i B_{i,5}(t)$$

We want all the input control points to respect the actuator limits $U_{min} \leqslant u \leqslant U_{max}$. The latter form a system of inequalities that can be used as a prior study to the sensibility of the control inputs with respect to the flat outputs. To solve this system, we use the Mathematica function *CylindricalDecomposition* for the symbolic computation of the Cylindrical Algebraic Decomposition. We compute the regions in which to choose the values for a_i's of the reference trajectory. For the sake of clarity, instead of keeping U_{min}, U_{max} symbolically, we give a value for the constraints $U_{min} = 0.2$ and $U_{max} = 10$. The initial and final trajectory points are defined as $H_r(t_0) = a_1$ and $H_r(t_f) = a_7$ respectively. The condition under which the reference trajectory H_r will respect the input constraint is

$$a_2 \in \mathbb{R}$$
$$a_i < \frac{1}{20}(1 - 20a_{i-2} + 40a_{i-1}), \text{for} \quad 3 \leqslant i \leqslant 6.$$

The reference 4th degree B-spline trajectory is specified with the control points $a_1 = 0.5$; $a_2 = 2$; $a_3 = 2$; $a_4 = 5$; $a_5 = 6$; $a_6 = 5$; $a_7 = 4$ chosen in the constrained region.

Figure 2 depicts the performance of the closed-loop control.

7. CONCLUSION

This work seeks to find an explicit constraint on the control input and/or the state of a linear delay system. Thus it provides a useful tool that can be implemented on various applications using the B-spline curves and flat system theory. By expression of the flat outputs in the form of

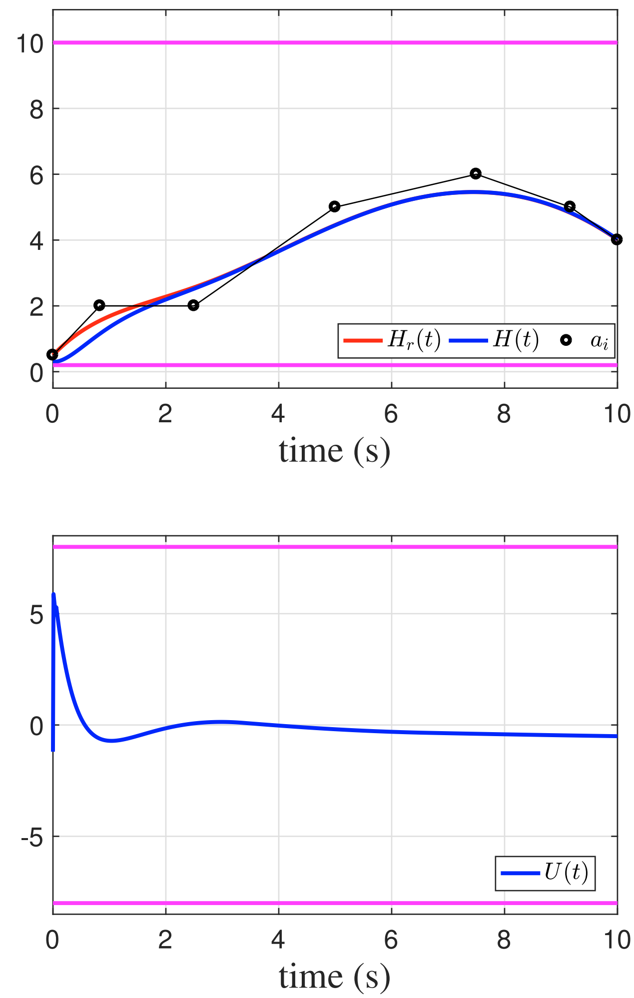

Fig. 2. Closed-loop performance

B-spline curves, the input controls depend on the control points and the degree of the B-spline curves (flat outputs). In our future works, we shall develop our approach further for systems represented by partial differential equations.

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