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Abstract: In engineering practice delay-difference is often used to approximate the derivatives of output signals for feedback control, which leads to a closed-loop system with delay both in the states and in the system coefficients. Our objective is to find all the delay values below some upper bound that guarantee the exponential stability of the closed-loop system subject to the delay-difference approximation. A method for stability analysis of systems with delay-dependent coefficients developed in our previous work is extended and applied for the systems considered in this paper. The proposed stability analysis procedure is demonstrated through the design of a mobile-robot path-following controller.

Keywords: Time-Delay Systems, Stability Analysis, Delay-dependent Coefficients, τ-decomposition Approach

1. INTRODUCTION

Control design based on output measurement is very common in engineering practice, often due to the difficulty in measuring all the state variables. In some occasions a static feedback of the output is not sufficient for stabilizing the system or to ensure a satisfactory performance. There are several strategies to deal with this problem. For instance, when dynamic output feedback is considered, one may design an observer to reconstruct the whole state based on the information of the output. Another strategy commonly taken in practice is to use the time derivative of the output signal $y(t)$ for stabilizing the system, which leads to simpler controllers in comparison with observer-based control design. Since the derivative of the output usually can not be measured directly, it is usually realized through delay-difference:

$$\dot{y}(t) \approx \frac{y(t) - y(t-\tau)}{\tau} \tag{1}$$

where τ is some positive delay value. In consequence of delay-difference approximation (1) used in the feedback, the closed loop system is a delay system with delay-dependent parameters.

The idea of using delay for stabilization is not new. For instance, a multiple delay framework is developed in Niculescu & Michiels (2004) for stabilizing a chain of integrators, while in Yamanaka & Shimemura (1993) multiple delays are used for analysing some internal model control scheme. Bounded control for global stabilization has also been addressed in Mazenc, Mondie & Niculescu (2003), where only a single delay is used. Our research differs from the previous ones in that we fix the other parameter of the controller while looking for the range of the delay parameter which guarantees that the closed-loop system is exponentially stable with some pre-specified decay rate.

Systems with delay-dependent coefficients appears can be found in biological systems Fabien (2005), physical systems Wilmot-Smith et al. (2006) as well general control system as discussed in this paper. A large amount of research effort has been dedicated to stability analysis of delay systems, see Gu, Kharitonov & Chen (2003); Niculescu (2001); Michiels & Niculescu (2014) for comprehensive discussion of the related results. However, research pertaining to systems with delay-dependent coefficients is not common in the literature. Berreta and Kuang provided an efficient method for analysing stability of such type of systems with a single delay in Beretta & Kuang (2002). Gu et al. (2016) relaxed some of their restrictive conditions and extended their approach for more general delay systems. Given $\mathscr{I}$, a closed delay interval of interest, the method presented in Gu et al. (2016) can be used to find all the sub-sets in $\mathscr{I}$ that guarantees asymptotical stability of the system. It can be considered as a generalization of the classical τ-decomposition approach (see for instance Michiels & Niculescu (2014); Lee & Hsu (1969)), which was proposed originally for delay-systems with fixed coefficients.

We will first specify the form of control law considered in this paper and the characteristic equation of the linearized closed-loop system resulting from the control design. Then we will show that by shifting the variable in the characterize equation, the condition for exponential stability with decay rate α is equivalent to a condition for just asymptotical stability. After that we will make some further extension for the method developed in Gu et al. (2016) so that it can be used for the stability analysis of the control system considered in this paper.

⋆ Sponsor and financial support acknowledgment goes here. Paper titles should be written in uppercase and lowercase letters, not all uppercase.

2. MOTIVATING EXAMPLE

Consider a robot path following problem. A unicycle travelling at a constant speed V is required to follow a straight line. The robot is non-holonomic, so the direction of its translational velocity is always along its heading. The control input u is the derivative of its yaw rate, which reflects the yaw moment applied to the robot. It is easy to see the linearized dynamics of the system is described by

$$\dot{e} = V\theta$$
$$\dot{\theta} = \omega$$
$$\dot{\omega} = u$$

where e stands for the lateral tracking error, θ is the heading angle of the robot and ω is the yaw rate. Variable u denotes the control input. [Put a figure here to illustrate the model]

To stabilize the system, one may first render the subsystem consisting only the variable ω stable. Therefore we define the control input as

$$u = -k_0\omega + v$$

where v is the new control input to be designed. Then it is easy to see that there exits real number k_1, k_2 such that by choosing

$$v = -k_1 e - k_2\theta$$

one can stabilize the system. However in practice θ is not convenient to measure. Noticing $\theta = V^{-1}\dot{e}$ and $\dot{e} \approx \frac{e(t)-e(t-\tau)}{\tau}$, we choose instead the following control law that uses delayed signal:

$$u = -k_0\omega - k_1 y - k_2\frac{y(t)-y(t-\tau)}{\tau} \qquad (2)$$

It can be shown that if System (2) can be stabilized by the following control law for some fixed number k_0, k_1, k_2

$$u = -k_0\omega - k_1 y - k_2\dot{y} \qquad (3)$$

then it can also be stabilized using (2) for sufficiently small delay. Unfortunately if the delay value is too small, the noise contained in the measurement of y will be greatly amplified and injected into the closed-loop system and thus severely undermine the performance. On the other hand too large value of τ may cause slow convergence, strong oscillation, or even instability. Therefore for practical consideration it is useful to find a set of delay value for (2) such that the closed loop system is exponentially stable with some guaranteed decay rate and then one can choose an appropriate delay value in this set.

3. PROBLEM STATEMENT

Consider a linear system of the form

$$\dot{x} = Ax + Bu \qquad (4)$$

where $x \in \mathbb{R}^n$ is the system state and $u \in \mathbb{R}$ is the control input. Suppose that the system admits a unique equilibrium and there exits a set of outputs $y = Cx$ that can be measured for control feedback and the following two conditions holds: **Condition 1**. The relative degree of y w.r.t u is larger than 1, in other words $CB = 0$.
Condition 2. There exits feedback law of the form

$$u(t) = K_0 y(t) + K_1\dot{y}(t) \qquad (5)$$

such that System (4) with (5) is asymptotically stable.
In practice, it is very often that signal $\dot{y}$ can not be measured. In this case, delay-difference of $y(t)$ can be used to approximate $\dot{y}(t)$ and (5) thus becomes

$$u(t) = K_0 y(t) + K_1\frac{y(t)-y(t-\tau)}{\tau} \qquad (6)$$

where $\tau > 0$ is a constant number. For the closed loop system consisting of (4) and (6), if the system trajectory x_t converges to zero exponentially fast with decay rate α, then we say the system is α-stable. Otherwise the system is α-unstable. For fixed K_0, K_1 and a given delay interval $\mathscr{I} = (0, \tau^u)$ as well as some positive decay rate α, we are interested in finding all the subintervals contained in $\mathscr{I}$ such that the closed loop system is α-stable for all τ in these subintervals. We require that no eigenvalue of the open loop system has a real part equal to $-\alpha$.

3.1 Discussion

Although in the framework described above the feedback that only utilizes $y(t)$ and $\dot{y}(t)$, however in practice a variety of stabilization problems can be converted into the form described here by first introducing some extra feedback terms in the control law. The basic idea is as follows. Suppose $z = C_z x$ are all the outputs that can be measured. One may first define the control law as

$$u = K_z z + v \qquad (7)$$

and the system dynamics becomes

$$\dot{x} = (A + BK_z C_z)x + Bv \qquad (8)$$

Then System (8) with v as the new control input may satisfy our conditions and thus fits our framework. This procedure has already been illustrated by our motivating example. As a matter of fact, the classical PID controller can be analysed in this Framework using the same idea. To generate the integral term, one first introduces a state σ that satisfies

$$\dot{\sigma} = y$$

Further define $u = K_I\sigma + v$, then by setting $v = K_P y + K_D\dot{y}$ we realize a *PID* control.

As noted in Niculescu & Michiels (2004), when the open loop system possesses more than a pair of imaginary roots, then it is necessary to introduce multiple delays in order to stabilize. We will leave this issue to our future work and restrict ourselves to control system with a single delay in this paper.

4. STABILITY ANALYSIS

4.1 Characteristic Equation and Stability

Let $G(\lambda) = 0$ be characteristic equation of the open loop system (4) with $u \equiv 0$, then $G(\lambda)$ is a polynomial of λ of order n. The characteristic equation of the delay-free system consisting of (4) and (5) takes the form

$$G(\lambda) + G_u(\lambda)(k_1 + k_2\lambda) = 0 \qquad (9)$$

where k_1, k_2 are two constants depending on K_1 as well as K_2, and $G_u(\lambda)$ is also a polynomial of λ and its order is smaller than $n-1$ due to Condition I. When the delayed feedback (6) is applied, it is easy to see that the corresponding characteristic equation of the closed loop system becomes

$$G(\lambda) + G_u(\lambda)(k_1 + k_2\frac{1-e^{-\lambda\tau}}{\tau}) = 0 \qquad (10)$$

It is known that the closed loop system with control law (6) is α-stable if and only if all the roots of (10) in λ admit a real part smaller than $-\alpha$. Define

$$P_\alpha(\lambda,\tau) = G(\lambda-\alpha) + (k_1 + k_2\tau^{-1})G_u(\lambda-\alpha)$$

$$Q_\alpha(\lambda,\tau) = -\tau^{-1}k_2e^{\alpha\tau}G_u(\lambda-\alpha)$$

It is easy to see that α-stability of the closed loop system with control law (6) is now equivalent to the asymptotical stability of a delay system with the following characteristic equation

$$P_\alpha(\lambda,\tau) + Q_\alpha(\lambda,\tau)e^{-\lambda\tau} = 0 \tag{11}$$

Our objective can be stated as finding all the delay subintervals contained in $\mathscr{I}$ that guarantee that all the the roots of (11) in λ are located on L.H.P.

4.2 A General τ-decomposition Approach

Characteristic equation (11) corresponds to a delay system with delay-dependent coefficients. Stability analysis of systems with this special feature is not common in the literature. Inspired by the earlier work of Beretta and Kuang [BK], the authors proposed a systematic method for analysing the stability of systems with delay-dependent coefficients, which can be viewed as a generalized τ-decomposition approach. In this subsection we will present some related results.

To begin with, define

$$F(\omega,\tau) = P_\alpha(j\omega,\tau)P_\alpha(-j\omega,\tau) - Q_\alpha(j\omega,\tau)Q_\alpha(-j\omega,\tau) \tag{12}$$

Consider the equation consisting of

$$F(\omega,\tau) = 0 \tag{13}$$

$$\partial_\omega F(\omega,\tau) = 0 \tag{14}$$

Define $\Phi_F = \{\tau | \tau \in \mathscr{I}, \exists\omega, \text{ s.t } (\omega,\tau) \text{ satisfies (13) and (14) }\}$. Let $\mathscr{T}$ be exactly the set of all critical delay values in $\mathscr{I}$ such that (11) admits imaginary roots. We assume that both Φ_F and $\mathscr{T}$ are finite sets and $\Phi_F \cap \mathscr{T} = \{\phi\}$. The elements of these two sets are arranged in the ascending order:

$$\Phi_F = \{\tau^{(i)}, i = 1,...,N_F\} \text{ and } \tau^{(1)} < ... < \tau^{(N_F)}$$

$$\mathscr{T} = \{\tau_i, i = 1,...,N_T\} \text{ and } \tau_1 < ... < \tau_{N_T}$$

Notice that both sets can be empty.

$\mathscr{I}$ is thus decomposed into subintervals $\mathscr{I}^{(i)}$, $i = 1,...,T$ and $\mathscr{I}^{(1)} = (0,\tau^{(1)}]$, $\mathscr{I}^{(i)} = (\tau^{(i-1)},\tau^{(i)}]$ for $i \neq 1$.

It follows from Gu et al. (2016) that within each $\mathscr{I}^{(i)}$, there exits a fixed number $m(i)$ ($m(i)$ can be zero), such that equation (13) with (14) admits exactly $m(i)$ positive solutions in ω. Each of these solutions can be viewed as an analytical function of τ within $\mathscr{I}^{(i)}$, denoted as $\omega_k^{(i)}(\tau)$, $1 \le \tau \le m(i)$. These functions are referred to as the *critical frequency functions*. Now we can introduce the phase angle function $\theta_k^{(i)}$ defined in interval $\mathscr{I}^{(i)}$ for $\omega_k^{(i)}$ as

$$\theta_k^{(i)}(\tau) = \angle P_\alpha(j\omega_k^{(i)}(\tau),\tau) - \angle Q_\alpha(j\omega_k^{(i)}(\tau),\tau) + \omega_k^{(i)}(\tau)\tau + \pi \tag{15}$$

where $\angle$ can be any differentiable function in each $\mathscr{I}^{(i)}$ that measures the phase angle of a non-zero complex number. The function $\theta_k^{(i)}$s are well defined because Condition 1 and Condition 2 guarantee that

$$|P_\alpha(j\omega_k^{(i)}(\tau),\tau)| + |Q_\alpha(j\omega_k^{(i)}(\tau),\tau)| \neq 0 \tag{16}$$

To see the statement above is correct, one only needs to notice that if the L.H.S of the last inequality can become zero for some τ^*, then we must have $G_u(j\omega_k^i(\tau^*) - \alpha) = 0$, which further implies that $\lambda^* = j\omega_k^i(\tau^*) - \alpha$ is an imaginary root of $G(\lambda) = 0$. In other words, the open loop system admits an eigenvalue with real part $-\alpha$, a contradiction to our requirement on α.

A necessary and sufficient for $\lambda = j\omega^*$ to be an imaginary root for $\tau = \tau^* \in \mathscr{I}$ is that the following equation holds

$$\theta_k^{(i)}(\tau) = 2l\pi \tag{17}$$

for some integer l, k, and i is the number that satisfies $\tau^* \in \mathscr{I}^{(i)}$. By finding the solutions of (17) for all possible integer i, k and l, one is able to obtain the set $\mathscr{T}$.

It can be shown that for some delay value $\tau^* \in \mathscr{I}^i$ and for some corresponding imaginary root $\lambda = j\omega^*$, λ is a differentiable function of τ in a neighbourhood of $\tau = \tau^*$. We denote it as $\lambda(\tau)$. To indicate whether this root will be come stable or unstable, we define

$$\text{Inc}(\tau^*,\omega^*) = \text{sign}\big(\Re\big(\frac{d\lambda(\tau)}{d\tau}\big)\big)_{\tau=\tau^*} \tag{18}$$

$$\text{Inc}(\tau^*) = \sum_{k=1}^{m(i)} Inc(\tau^*,\omega_k^i(\tau^*)) \tag{19}$$

In this work we assume that for any critical delay τ^* the right hand side of the last equation is not zero. Then it is easy to see that if $\text{Inc}(\tau^*,\omega^*) = 1$, then this imaginary root moves to R.H.P as τ increases. It moves to L.H.P if the indicator is -1. Consequently $\text{Inc}(\tau^*)$ is just the increased number of unstable roots. It is clear that if this assumption is violated, or in other words at some critical delay the derivative of the *root* w.r.t delay has zero real part, then more involved definition as well as more advanced root crossing criterion than that used in this paper will be needed. More comprehensive discussion of this issue and the related results have recently been developed in Gu et al. (2016). Once the indicating functions are know for each critical delay, the total number of unstable roots for any $\tau \in \mathscr{I} - \mathscr{T}$, denoted as $N^u(\tau)$, can be computed as

$$N^u(\tau) = N^u(\tau^l) + \sum_{i=1}^{i=L(\tau)} \text{Inc}(\tau_i) \tag{20}$$

where $L(i)$ is the largest integer such that $\tau_{L(\tau)} < \tau$. The following theorem provides a convenient way to compute the indicating function in (18).

Theorem 1. Suppose for (11) τ^* is a critical delay value and $\lambda = j\omega^*$ is a corresponding imaginary root. Then there exits integer i, k such that $\tau^* \in \mathscr{I}^{(i)}$ and $\omega^* = \omega_k^{(i)}(\tau^*)$. Assume also 13 and 14 do not hold together for (ω^*,τ^*). Then λ as a local function of τ denoted as $\lambda(\tau)$, which satisfies

$$\text{sgn}\left(\Re\left(\frac{d}{d\tau}\lambda(\tau^*)\right)\right) = \text{sgn}\left(\frac{\partial}{\partial\omega}F(\omega^*,\tau^*)\right) \times\text{sgn}\left(\frac{d}{d\tau}\theta_k^{(i)}(\tau^*)\right) \tag{21}$$

The last theorem indicates an interesting correlation between the imaginary root cross direction and how the corresponding phase function crosses $2l\pi$ for some integer l. One can easily determine the imaginary root cross direction based on the graph of $\theta_k^{(i)}(\cdot)$ for some integer i and k.

We now summarize the analysis procedure in several steps:

Step 1. Solve (13) together with (14) subject to $\omega \geq 0$, $\tau \in \mathscr{I}$ to obtain Φ_F as well as $\tau^{(i)}$, $i = 0,...N_F$. $\mathscr{I}$ is thus decomposed into each $\mathscr{I}^{(i)} = [r^{(i-1)}, r^{(i)}]$.

Step 2. In each $\mathscr{I}^{(i)}$, solve (17) to find all the critical delay value τ_i, $i = 1,...,N_T$ and thus the set $\mathscr{T}$.

Step 3. Compute $\mathrm{Inc}(\tau_i)$ for each $\tau_i > 0$ by plugging (21) into (18) and further uses (19).

Step 4. Now for any interval $\mathscr{I}_i^* = [\tau_{i-1}, \tau_i]$, we can arbitrarily pick a delay value r' in its interior and compute $N^u(r')$ via (20), then it follows that $N^u(\mathscr{I}_{io}^*) = N^u(r')$, where $\mathscr{I}_{io}^* = (\tau_{i-1}, \tau_i)$.

4.3 Lower-bounding the Delay Interval

The stability analysis we have presented requires the system coefficients to be continuous on the delay interval $\mathscr{I}$. However for the system considered in this paper the coefficients is not bounded in $\mathscr{I}$ as τ appears in the denominator. Fortunately it follows from [cite] that for any real c, all the roots of (11) with real part larger than c converge to the roots of (5) with real parts also larger than c. The direct implication is the existence of some positive number τ_ε such that $N^u(\tau) = N^u(0)$ for all $\tau \in (0, \tau_\varepsilon]$ and therefore we only need to carry out the analysis on the interval $I = [\tau_\varepsilon, \tau^u]$. The issue remaining is how to find such a lower bound delay.

For fixed α, define

$$P_1(s,\tau) = G(s-\alpha) + G_u(s-\alpha)(k_1 + k_2(s-\alpha))$$

$$P_2(s,\tau) = \left(\frac{1-e^{j(s-\alpha)\tau}}{\tau} - s + \alpha\right)$$

then (11) can be written as

$$P_1(s,\tau) + Q(s)P_2(s,\tau) = 0 \quad (22)$$

where we denote $Q(s) = k_2 G_u(s-\alpha)$

We limit τ_ε below some positive constant τ_{up}, and denote

$$c_0 = e^{\alpha\tau} - 1,\ c_1 = \frac{\tau_{up} - e^{\tau_{up}} + 1}{\tau_{up}} \quad (23)$$

Regarding term P_2, we have

$$\begin{aligned} P_2(j\omega,\tau) &= \frac{1-e^{j\omega\tau} - j\omega\tau}{\tau} + \frac{\tau - e^{\tau} + 1}{\tau} \\ &\quad + \frac{j\sin(\omega\tau)(e^{\alpha\tau}-1) + 1 - \cos(\omega\tau)}{\tau} \\ &= P_{20}(j\omega,\tau,c_0) + c_1 \end{aligned}$$

where the expression for P_{20} can be determined according to the last equation and the definition of c_1.

To show there exits $\tau_\varepsilon > 0$ such that (11) admits no imaginary solution, it is clear that we only need to show that for $\tau \in [0, \tau_\varepsilon]$, the following equation does not hold for real ω

$$G(\omega) = |P_2(j\omega,\tau)| \quad (24)$$

where

$$G(\omega) = \frac{|P_1(j\omega,\tau)|}{\sqrt{|Q(j\omega)|^2 + c_2}} \quad (25)$$

and constant c_2 can be any positive number to ensure that $G(\omega)$ is well-defined. For computing a larger τ_ε, it is desirable to choose a very small c_2. In the case where $Q(j\omega) \neq 0$ for $\omega \geq 0$, c_2 can be set to 0.

Condition 1 indicates that $\mathrm{Ord}(P_1) - \mathrm{Ord}(Q) \geq 2$. We can also pick small τ_{up} to make c_0 as small as possible, which also renders $|\partial_\omega P_{20}(0,\tau,c_0)|$ as small as we wish. Then it is easy to see that there exit positive numbers k_1, $k_2 \geq k_1$ such that $G(\omega) - c_0 > k_1\omega$ for $\omega \geq 0$ and $|P_{20}(j\omega,\tau,c_0)| \leq k_2\omega$ for $\omega \geq 0$. Let $\omega_1(\tau)$ be the minimal positive solution of

$$|P_{20}(j\omega,\tau,c_0)|^2 = (k_1\omega)^2 \quad (26)$$

if there exists any, otherwise let $\omega_1 = 0$. Let ω_2 be the maximal solution of

$$G^2(\omega) = (k_2\omega)^2 \quad (27)$$

if there exits any real solutions in ω, otherwise let $\omega_2 = 0$. We note that both (26) and (27) are polynomial equations and thus easy to solve using numerical methods. Define the piecewise linear function

$$L_\tau(\omega) = \begin{cases} k_1\omega & 0 \leq \omega \leq \omega_1(\tau) \\ k_2\omega & \omega_1(\tau) < \omega \end{cases} \quad (28)$$

it is easy to see that $\tau_1\omega_1(\tau_1) = \tau_0\omega_1(\tau_0)$ for any τ_1, τ_0.

Proposition 2. There exits positive numbers τ_{ub} and $\tau_\varepsilon \leq \tau_{ub}$ such that

$$G(\omega) > |P_2(j\omega,\tau)| \quad (29)$$

For any $\tau \in (0, \tau_\varepsilon]$, (11) has no real solution in ω. Furthermore, if (29) holds for $\omega \in [0, \omega_2]$, then it also holds for all non-negative ω.

For any given delay value τ^*, the last proposition allows us to check whether τ^* can be taken as τ_ε by only checking a bounded interval of ω. If (29) holds for $\tau = \tau^*$ $\omega \in [0, \omega_2]$, then τ_ε can be set as τ^*.

Proof. Throughout the proof ω only assumes non-negative value and τ is always positive. We first point out two facts that can check straightforwardly

$$|P_{20}(j\omega,\tau,c_0)| < L_{(\tau)}(\omega),\ \tau > 0 \quad (30)$$

and for $0 < \tau_0 < \tau$

$$L_{(\tau_0)}(\omega) < L_{(\tau)}(\omega) \quad (31)$$

We first consider the case $\omega_1(1) = 0$. This implies that for any $\tau > 0$

$$G(\omega) - c_1 > L_1(\omega) = L_\tau(\omega) \geq |P_2(\omega,\tau)|$$

and thus $G(\omega) > |P_2(\omega,\tau)|$ which indicates that (24) can not hold and therefore (11) does not admit imaginary roots for any delay value.

Now consider the case $\omega_1(1) > 0$, we pick $\tau_\varepsilon = \frac{\omega_1(1)}{\omega_2}$. Then L_{τ_ε} takes the form

$$L_\tau(\omega) = \begin{cases} k_1\omega & 0 \leq \omega \leq \omega_2 \\ k_2\omega & \omega < \omega_2 \end{cases}$$

then it is straightforward to verify that $G(\omega) - c_1 > L_{\tau_\varepsilon}(\omega)$. Since we have as well $L_{\tau_\varepsilon}(\omega) \geq |P_2(j\omega,\tau)|$, it follows that $G(\omega) - c_1 > |P_{20}(j\omega,\tau,c_0)|^2$ and thus (24) can not hold. As a result (11) does not have real solutions in ω for $\tau \in (0, \tau_\varepsilon]$.

5. APPLICATION

Now we apply the stability analysis method developed so for for the unicycle model (2) and assume the possible delay value ranges in $\mathscr{I} = (0, 0.5]$. We start with designing the control law of the form (5). Various tools for controlling LTI systems can be employed. For instance, to minimize a quadratic cost function in the time domain, one can use LQR to optimally determine the control parameters. Since the specific way to determine control parameters is irrelevant to our stability analysis procedure, we simple set the eigenvalues of the closed loop system as -2

and $-1.5 \pm 4j$. Accordingly we obtain the following control parameters:

$$k_0 = 5,\ k_1 = 97/4,\ k_2 = 73/2$$

and the characteristic equation of the closed loop system is

$$\lambda^3 + k_0\lambda^2 + k_1\lambda + k_2 = 0 \tag{32}$$

to shift the eigenvalues of the closed loop system to

$$\lambda_1 = -1.5+4j,\ \lambda_2 = -1.5-4j,\ \lambda = -2$$

From the real part of the eigenvalues we deduce that the decay rate of the control system without delay is 1.5. Suppose we require that when the delay-difference approximation is used for feed-back control, the decay rate is no less than 1, hence we pick $\alpha = 1$.

Now replace λ in (32) with $s-\alpha$, we arrive at the explicit expression of (11) for this particular system

$$s^3 + a_0 s^2 + a_1 s + a_2(\tau) + a_3(\tau)e^{-s\tau} = 0 \tag{33}$$

where $a_0 = k_0 - 3$, $a_1 = 3 - 2k_0$, $a_2 = k_1/\tau - 1$ and $a_3 = -e^{\tau}k_1/\tau$ from which the expression of $P_\alpha(j\omega,\tau)$ and $Q_\alpha(j\omega,\tau)$ is derived as

$$P_\alpha = (-\omega^3 + a_1\omega)j + a_2 - a_0\omega^2$$
$$Q_\alpha = a3$$

We pick $\tau_{up} = 0.05$ and define

$$G(\omega) = \sqrt{\frac{(-\omega^3 + k_1\omega)^2 + (k_2 - k_0\omega^2)^2}{k_2}} \tag{34}$$

It can be verified that one can pick $L_1 = 1.8$. Solving the polynomial equation

$$G(\omega)^2 = (L_1\omega + c_1(0.05))^2 \tag{35}$$

we find the maximal real solution in ω is approximately 7.5752. Therefore we pick $\omega_c = 7.5752$ and we only need to ensure the graph of $G(\omega)$ has no intersection with that of $|P_2(j\omega,\tau_\varepsilon)|$ for $\omega \in [0,\omega_c]$. The graphs of G and P_2 for $\tau = 0.05$ is plotted in Fig.1. Hence we will pick $\tau_\varepsilon = 0.05$ and will only be concerned about the delay interval $\mathscr{I}_\varepsilon = [0.05, 0.5]$.

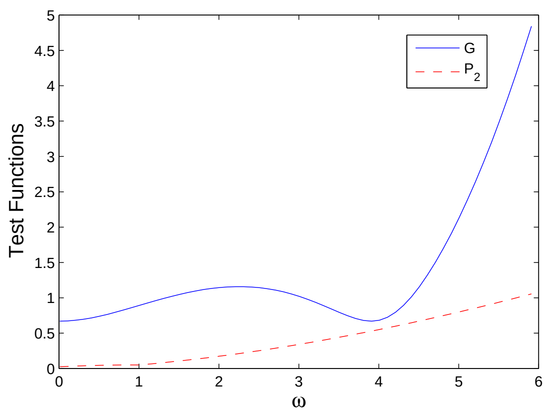

Fig. 1. The Graphs of function G and $|P_2|$ have no intersection for $\tau_\varepsilon = 0.05$

It follows that

$$F(\omega,\tau) = \omega^6 + (a_0^2 - 2*a_1)\omega^4 + (a_1^2 - 2a_0a_2)\omega^2 + a_2^2 - a_3^2 \tag{36}$$

$$\partial_\omega F(\omega,\tau) = 6\omega^5 + 4(a_0^2 - 2a_1)\omega^3 + 2(a_1^2 - 2a_0a_2)\omega \tag{37}$$

Solving the previous two equations together numerically using MATLAB for τ with the constraint $\omega \in \mathbb{R}$ and $\tau \in \mathscr{I}$, we find only one solution $\tau^{(1)} \approx 0.1143$. Hence $\mathscr{I}$ is decomposed as

$$\mathscr{I} = \mathscr{I}^{(1)} \cup \mathscr{I}^{(2)}$$

and $\mathscr{I}^{(1)} = [\tau_\varepsilon, \tau^{(1)}]$ and $\mathscr{I}^{(2)} = [\tau^{(1)}, \tau^u]$. It can be verified that $m(1) = 2$ and $m(2) = 0$. The value of $\omega_1^{(1)}$ and $\omega^{(2)}$ for various $\tau \in \mathscr{I}^{(1)}$ numerically based on (36) and the graph of these two functions are depicted in Fig.2.

Subsequently we obtain the function $\theta_1^{(1)}(\tau)$ and $\theta_2^{(1)}(\tau)$, the graph of which is presented in Fig.3.

We see that the only intersection between the phase curves corresponding to $\theta_i^{(1)}$, $i = 1,2$ and the horizontal lines whose value equals $2l\pi$ for integer l takes place only at $\tau_1 \approx 0.0915$ when $l = 0$. Therefore two imaginary roots appear when $\tau = \tau_1$. Since $\partial_\omega F(\omega_2^{(1)}(\tau_1),\tau_1) > 0$ and the derivative of $\theta_2^{(1)}(\tau_1)$ is positive, we conclude that (33) the two imaginary roots crosses toward the imaginary axis toward R.H.P as τ goes through τ_1 increasingly. No roots of (33) crosses the imaginary roots in Interval $\mathscr{I}^{(2)}$ as $m(2) = 0$. Now it follows from (20) $N^u(\tau) = 0$ for $\tau \in [\tau_\varepsilon, \tau_1)$ and $N^u(\tau) = 2$ for $\tau \in (\tau_1, \tau^u]$. Consequently, we can claim that for $\alpha = 1$ the system is α-stable for $\tau \in (\tau_1, \tau^u]$ and not α-stable for $\tau \in (\tau_1, \tau^u]$.

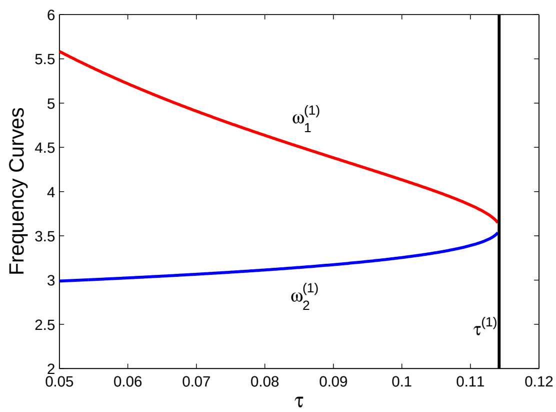

Fig. 2. Frequency functions.

Simulation is carried out using Simulink. We investigate the evolution of the signal $e^{\alpha t}e(t)$ over time by setting $\tau = 0.09$ and 0.11 respectively. It is clear that if the system is α-stable, then $e^{\alpha t}e(t)$ must be convergent, otherwise it diverges. Fig corresponds to $\tau =$ and we observe the convergence of $e^{\alpha t}e(t)$ converges and Fig corresponds to $\tau =$ and the divergence of $e^{\alpha t}e(t)$ is observes. These results are all consistent with our previous analysis.

6. CONCLUSION

We have addressed the stability analysis for control scheme that uses delay-deference for approximating the derivative of output signals. We regard the delay as a design parameter and for any given bounded delay interval of interest, we are concerned with finding all the sub-sets of delay-values contained in the interval

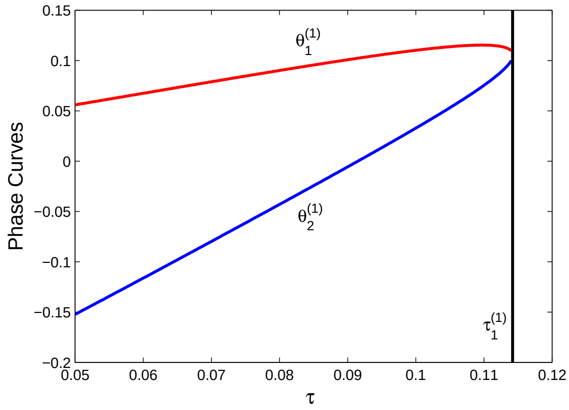

Fig. 3. Phase functions.

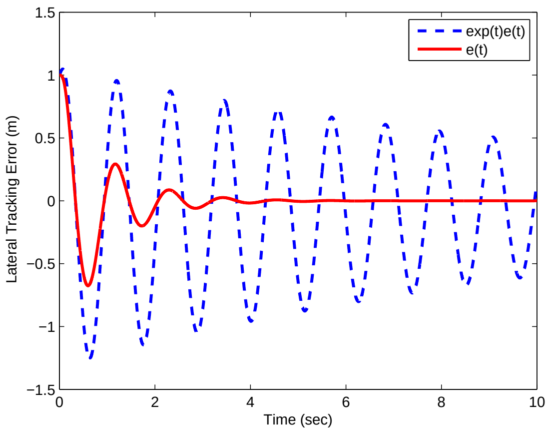

Fig. 4. Original and exponentially weighted lateral tracking error for $\tau = 0.09s$.

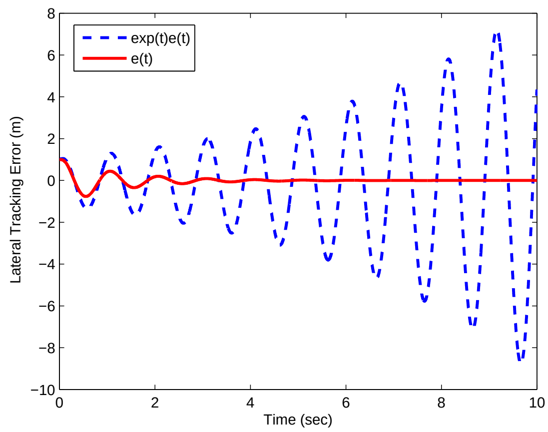

Fig. 5. Original and exponentially weighted lateral tracking error for $\tau = 0.11s$.

such that the system is exponentially stable with guaranteed convergence rate. An approach developed in our previous work for systems with delay-dependent coefficients is modified and applied to solve this problem. The stability analysis procedure is illustrated with a mobile robot path tracking problem. Simulation shows that our conclusion drawn from frequency-domain analysis is consistent with the time-domain response of the robot trajectory.

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