A Distributed Consensus Control Under Disturbances for Wind Farm Power Maximization

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Abstract— **In this paper we address the problem of power sharing among the wind turbines (WTs) belonging to a wind farm. The objective is to maximize the power extraction under the wake effect, and in the presence of wind disturbances. Because of the latter, WTs may fail in respecting the optimal power sharing gains. These are restored by employing a consensus control among the WTs. In particular, under the assumption of discrete-time communication among the WTs, we propose a distributed PID-like consensus approach that enhances the rejection of the wind disturbances by providing the power references to the local WT controllers. The latter are designed by employing a novel feedback linearization control that, acting simultaneously on the WT rotor speed and the pitch angle, guarantees the tracking of general deloaded power references. The obtained results are validated on a 6-WT wind farm example.**

I. Introduction

Nowadays WTs, and wind farms still capture great research attention, as their role in the electric power supply is fairly changing. New grid requirements that has to be met, as well as an increasing know-how regarding the mentioned power systems lead to the interest for different ways to exploit the wind source. Far from being a classic modus operandi of wind farms, the power maximization problem falls within the latter. This is true when considering wind farms composed of several WTs, as they are very likely to experience the so-called *wake effect*. Thus considering the aerodynamic coupling among the turbines, and in turns the wind farm as a whole, proves potential gain when maximizing the power production (see e.g. [1]), and justifies a growing interest in *cooperative* methods to control them. As a result, distributed control approaches are preferable to centralized ones if improvement of performance is seek [3].

Typically the problem of power maximization under wake interaction is handled via a first step of optimization under the assumption of a static system. This approximation is mainly due to the high nonconvexity of the wake model that makes the problem hard to be treated directly under a control perspective. The available approaches mainly deal with either model-free decentralized methods, as in [2], [3], or model-based ones as in [4], and [1]. The aforementioned optimization algorithms rely on the existence of local control strategies for individual WTs that can stabilize around the obtained optimal set points [2]. However, as shown in [5], and [6], the actual attainable power gain can be highly affected by the system dynamics, and it validates the need for the design of efficient controllers to support the optimization step. It is important to point out that wind farm power maximization can be alternatively seen as the problem of finding the optimal *power sharing* of the available wind source among the WTs. Similar power sharing problems for wind farms have been treated in [7], [8], [9]. The latter, based on the common assumption that the available wind power is higher than the demanded one, employ different distributed control approaches to deal with the problem. In particular, in [7], classic consensus algorithm is employed to let knowledge of the total available and demanded powers in order to compute the local WT set points. [8] proposes a static optimal consensus law to enable power sharing, and [9] capitalizes on a leader-follower approach.

In this paper we propose a distributed control approach to let proper power sharing among the WTs in order to maximize the power generation. To the authors' knowledge such distributed control framework was never applied to the problem addressed in this paper, and, despite having some common ideas, it substantially differs from the mentioned references. Our contribution is two-fold. Firstly, a feedback linearization (FL) control is applied at the WT level to let the distributed problem be treated in the linear systems framework. Moreover, this control step has to allow the WT to track a general *deloaded* power reference, which is a necessary condition for wind farm power maximization [1]. The overall approach can be seen as a novel wind farm level maximum power point tracking (MPPT) technique. [10] proposes a FL controller for variable-speed fixed-pitch WTs to track a desired power reference, but it is only applicable at high wind speed. [11] capitalizes on a combined FL, and model predictive control technique enabling general power tracking in the whole WT operating envelope, but it does not particularly simplify the design of a distributed control law. Thus, based on [10] we make use of a new FL application to WTs to let power tracking below the rated wind speed. Secondly, under the assumption of discrete-time communication among the WTs, we employ a distributed PID-like control architecture to force the system to stabilize around the optimal power sharing set points under system disturbances. The PID structure is justified for a simpler

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P-like protocol would not allow a satisfactory disturbance rejection, if the dynamics of the agents, i.e. the WTs, are general. In the literature, generic distributed controllers are proposed for both continuous and discrete-time in e.g. [12], and in continuous time in [13], and [14] to meet respectively limited energy, and H_∞ performance constraints. As far as dynamic controllers in continuous-time with a prescribed structure are concerned, one can cite for instance [15], and [16], where the former proposes a PI-like algorithm for single integrator dynamic agents, and the latter provides a PID-like one for general high-order SISO systems. Our proposed PID-like distributed algorithm applies to general linear discrete-time MIMO systems, and it aims at solving the problem of *leaderless consensus under the presence of disturbances*. Moreover we provide a possible way of tuning the control parameters based on the results of [17] for LMI-based tuning of MIMO centralized PID controllers.

The remainder of the paper is organized as follows. First, some graph theory preliminaries are provided in Section II. The turbine model is presented in Section III. The main control problems, and their objectives are stated in Section IV. We present, and prove our main results concerning the control architecture in Section V, and test its effectiveness on a 6-WT wind farm in Section VI. The paper ends with conclusions, and future perspectives in Section VII.

II. Graph Theory Preliminaries

An undirected graph $\mathscr{G}$ is a pair $(\mathscr{V},\mathscr{E})$, where $\mathscr{V} = \{1,\ldots,N\}$ is the set of nodes, and $\mathscr{E} \subseteq \mathscr{V}\times\mathscr{V}$ is the set of unordered pairs of nodes, named edges. Two nodes i, j are said to be adjacent if $(i,j)\in\mathscr{E}$. Under the assumption of *undirected* graph, the latter implies that $(j,i)\in\mathscr{E}$ too. An undirected graph is connected if there exists a path between every pair of distinct nodes, otherwise is disconnected. The adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ associated with the undirected graph $\mathscr{G}$, considered in this paper, is defined by $a_{ii} = 0$, i.e. self-loops are not allowed, and $a_{ij} = 1$ if $(i,j)\in\mathscr{E}$. The Laplacian matrix $\mathscr{L}\in\mathbb{R}^{N\times N}$ is defined as $\mathscr{L}_{ii} = \sum_{j\neq i} a_{ij}$ and $\mathscr{L}_{ij} = -a_{ij}$, $i \neq j$. Considering an undirected graph we make use of the following

Lemma 1: [18] The Laplacian matrix has the following properties: (i) $\mathscr{L}$ is symmetric and all its eigenvalues are either strictly positive or equal to 0, and $\mathbf{1}$ is the corresponding eigenvector to 0; (ii) 0 is a simple eigenvalue of $\mathscr{L}$ if and only if the graph is connected.

We will also make use of another Laplacian matrix, according to the following

Lemma 2: [19] Let $\bar{\mathscr{L}} = \left[\bar{l}_{ij}\right] \in \mathbb{R}^{N\times N}$ be a Laplacian matrix such that $\bar{l}_{ij} = {}^{N-1}/_{N}$ if $i = j$, and $\bar{l}_{ij} = -{}^{1}/_{N}$ otherwise, then the following hold: (i) the eigenvalues of $\bar{\mathscr{L}}$ are 1 with multiplicity $N-1$, and 0 with multiplicity 1. $\mathbf{1}^\top$ and $\mathbf{1}$ are respectively the left and right eigenvector associated to eigenvalue 0; (ii) there exists an orthogonal matrix $U \in \mathbb{R}^{N\times N}$, i.e. $U : U^\top U = UU^\top = I$, and whose last column is equal to ${}^{1}/_{\sqrt{N}}$, such that for any Laplacian matrix $\mathscr{L}$ associated to any undirected graph we have

$$U^\top \bar{\mathscr{L}} U = \begin{bmatrix} I_{N-1} & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix} \triangleq \bar{\Lambda},$$
$$U^\top \mathscr{L} U = \begin{bmatrix} \mathscr{L}_1 & 0_{(N-1)\times 1} \\ 0_{1\times(N-1)} & 0 \end{bmatrix}$$

where $\mathscr{L}_1 \in \mathbb{R}^{(N-1)\times(N-1)}$ is symmetric and positive definite if the graph is connected.

Moreover we deduce the following extension of Lemma 2.

Lemma 3: Let $\mathscr{L}\in\mathbb{R}^{N\times N}$ be the Laplacian matrix associated to an undirected connected graph, and let $D\in\mathbb{R}^{N\times N}\succ 0$, and symmetric, then the following hold: (i) $\hat{\mathscr{L}} \triangleq D\mathscr{L}$ is a Laplacian generally nonsymmetric matrix, $\hat{\mathscr{L}} \succeq 0$, all its eigenvalues are real, and 0 is a simple eigenvalue with associated eigenvector $\mathbf{1}$; (ii) consider the orthogonal matrix $U\in\mathbb{R}^{N\times N}$ defined in Lemma 2, then

$$U^\top \hat{\mathscr{L}} U = \begin{bmatrix} \hat{\mathscr{L}}_1 & 0_{(N-1)\times 1} \\ * & 0 \end{bmatrix}$$

where $\hat{\mathscr{L}}_1 \in \mathbb{R}^{(N-1)\times(N-1)} \succ 0$, and its eigenvalues are real.

Proof: We have that $D\mathscr{L} = D^{1/2}(D^{1/2}\mathscr{L}D^{1/2})D^{-1/2}$, thus $D\mathscr{L}$ is similar to a symmetric semi-definite positive matrix, so its eigenvalues are positive real. $\hat{\mathscr{L}}$ preserves the 0 eigenvalue, and its associated eigenvector $\mathbf{1}$, as $D\mathscr{L}\mathbf{1} = \mathbf{0}$. 0 is a simple eigenvalue for D is nonsingular, and $\mathscr{L}$ has one simple 0 eigenvalue by hypothesis. The last column of $U^\top\hat{\mathscr{L}}U$ has all its entries equal to 0 because the last column of U is ${}^{1}/_{\sqrt{N}}$. Being $U^\top\hat{\mathscr{L}}U$ block triangular, and similar to $\hat{\mathscr{L}}$, $\hat{\mathscr{L}}_1$ has all real strictly positive eigenvalues. ■

We employ the Kronecker product $\otimes$, for which we have

Lemma 4: [20] Suppose that $U \in \mathbb{R}^{p\times p}$, $V \in \mathbb{R}^{q\times q}$, $X \in \mathbb{R}^{p\times p}$, and $Y \in \mathbb{R}^{q\times q}$. The following hold: (i) $(U\otimes V)(X\otimes Y) = UX \otimes VY$; (ii) suppose U, and V invertible, then $(U\otimes V)^{-1} = U^{-1}\otimes V^{-1}$.

III. Wind Turbine Modeling

The wind turbine model describes the conversion from wind power to electric power. The wind kinetic energy captured by the turbine is turned into mechanical energy of the turbine rotor, turning at an angular speed ω_r and subject to a torque T_r. In terms of extracted power, it can be described by the nonlinear function $P_r = \omega_r T_r = {}^{1}/_{2}\rho\pi R^2 v^3 C_p(\lambda,\vartheta)$, where ρ is the air density, R is the radius of the rotor blades, ϑ is the pitch angle, v is the effective wind speed representing the wind field impact on the turbine, obtained by filtering the time series of wind data as described by [21], λ is the tip speed ratio given by $\lambda = \frac{\omega_r R}{v}$. C_p, nonlinear function of the tip speed ratio and pitch angle, is the power coefficient. This is typically provided in turbine specifications as a look-up table. As far as the turbine parameters are concerned, in this work we make use of the CART (Controls Advanced Research Turbine) power coefficient. This turbine is located at NREL's National Wind Technology Center. Nonetheless, we employ a polynomial approximation of the latter for the purpose of the synthesis of the controller. Referring to a two-mass model as in [10], and as shown in Fig. 1, then, the

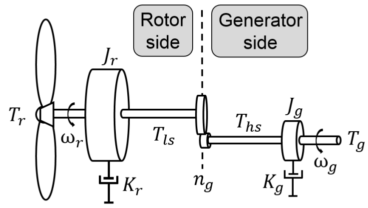


Fig. 1: Two-mass model of the turbine mechanics.

low speed shaft torque T_{ls} acts as a braking torque on the rotor, the generator is driven by the high speed torque T_{hs}, and braked by the generator electromagnetic torque T_g. The drive train turns the slow rotor speed into high speed on the generator side, ω_g. Finally J_r is the rotor inertia, K_r, and K_g damping coefficients, n_g the gear ratio, and J_g the generator inertia. The dynamics of the WT is thus described by $J_r\dot{\omega}_r = T_r - K_r\omega_r - T_{ls}$, and $J_g\dot{\omega}_g = T_{hs} - K_g\omega_g - T_g$. In this paper we also consider a first order system to model the pitch actuator, endowed with a *sigmoid* function $\sigma : \mathbb{R} \to [\vartheta_{min}, \vartheta_{max}]$ to model the pitch constraints. In addition, for ease of further development we can bring the system equations back on the low speed side, obtaining the simplified overall model

$$\begin{cases} \tau_\vartheta \dot{\vartheta}_s = -\vartheta_s + \vartheta_r \\ \vartheta = \sigma(\vartheta_s) \\ J_t\dot{\omega}_r = \dfrac{P_r(\omega_r, \vartheta, v)}{\omega_r} - K_t\omega_r - T_g \end{cases} \tag{1}$$

where $J_t \triangleq J_r + n_g^2 J_g$, $K_t \triangleq K_r + n_g^2 K_g$, and where we used the relation $n_g = \omega_g/\omega_r = T_{ls}/T_{hs}$. Eventually, neglecting the generator loss, the electric power delivered to the grid is $P = T_g\omega_r$. The system inputs are T_g, and ϑ_r, while the wind speed v acts as a disturbance. The feasible domain of the state variable is $X \triangleq \{(\omega_r, \vartheta) \in \mathbb{R}^2 : \omega_r \in [\omega_{r,min}, \omega_{r,max}], \vartheta \in [\vartheta_{min}, \vartheta_{max}]\}$.

IV. Problem Statement

At low wind speed, i.e. $v \in [v_{min}, v_n]$, WTs are usually operated according to the well-known MPPT algorithm. The maximum power that a WT can extract from the wind is thus attained for a constant value of ϑ, named here ϑ^o, depending on the turbine C_p, and by controlling the WT to track the optimal tip speed ratio value $\lambda^o \triangleq \arg\max_\lambda P_r(v, \vartheta^o, \lambda) = \arg\max_\lambda C_p(\lambda, \vartheta^o)$. We name $C_p^o \triangleq C_p(\lambda^o, \vartheta^o)$, and $P^o(v) \triangleq P_r(v, \vartheta^o, \lambda^o)$. For the considered CART turbine $\lambda^o \cong 8$. Nonetheless, as mention in Section I, when considering the wake effect in the optimization step of a farm of N WTs, the optimal value of C_p related to the generic turbine i is such that $C_{p,i}^* \leq C_p^o$. As a matter of fact, this implies that a turbine i should track an optimal power reference $P_i^*(v)$ that satisfies $P_i^*(v) \leq P_i^o(v)$, i.e. it has to be *deloaded* if maximum wind farm power is seek. The reader may refer to the cited works in Section I to see how values $C_{p,i}^*$ can be computed. Note that according to the usually employed wake models, the static optimization step needs to be run only when the wind direction changes, as optimal values $C_{p,i}^*$ do not depend on the wind speed value [1].

Assumption 1: The average wind direction is considered to be slowly varying with respect to the system dynamics. Thus, it is considered to be constant.

In the sequel, for consistency of notation we add the index i to the WT variables described in Section III when referring to turbine i variables, and we drop it when the results hold for any WT. We can formulate the control problem in two subproblems, the first of which being

Problem 1: Consider the system described by (1). Given an effective wind speed signal $v(t)$, and a time-varying reference trajectory $P^{ref}(t)$, verifying $P^{ref}(t) \leq P^o(t)\ \forall t \geq 0$, find the signals $(\vartheta_r(t), T_g(t))\ \forall t \geq 0$ such that $\lim_{t\to\infty} |P^{ref}(t) - P(t)| = 0$ for every initial condition $(\omega_r(0), \vartheta(0)) \in X : P(0) \leq P^o(0)$.

Let us now assume that each local WT controller can measure, or estimate, the effective wind speed $v_{m,i}(t)$ such that $v_i(t) = v_{m,i}(t) + v_{d,i}(t)\ \ \forall t \geq 0$. Thus $v_{d,i}$ represents a nonmeasurable time varying disturbance for turbine i.

Assumption 2: We consider small disturbances $v_{d,i}$ with respect to $v_{m,i}$, and slowly-varying with respect to the dynamics of (1).

In nominal condition, i.e. if $v_{d,i} \equiv 0$, each WT can compute the optimal power reference, as described in [6], from its maximum available power P_i^o, according to

$$P_i = \frac{C_{p,i}^*}{C_p^o} P_i^o \tag{2}$$

We can additionally require the WTs to meet an optimal power sharing condition given by

$$\frac{P_i}{\chi_i} = \frac{P_k}{\chi_k} \quad i,k = 1,\cdots,N \tag{3}$$

Indeed, by naming P_∞^o the maximum power that a WT could extract from the wind if there was no wake effect, from (2) we have that $P_i/C_{p,i}^* = P_i^o/C_p^o = \gamma_i P_\infty^o/C_p^o$, thus $C_p^o/\gamma_i C_{p,i}^* P_i = P_\infty^o$, $i = 1,\cdots,N$, yielding $P_i/\gamma_i C_{p,i}^* = P_k/\gamma_k C_{p,k}^*$, $i,k = 1,\cdots,N$. We name $\chi_i \triangleq \gamma_i C_{p,i}^* \in \mathbb{R}^+$, and where $\gamma_i = P_i^o/P_\infty^o = (v_i/v_\infty)^3$ are constant values for any value of v_∞ according to Assumption 1, being v_∞ the free stream wind speed. Despite being redundant information with respect to (2) in nominal conditions, (3) provides additional signals that can be exploited when the system is subject to disturbances. We can now state the second subproblem.

Problem 2: Given N identical WTs, allowed to communicate on an undirected connected graph $\mathscr{G}$; given optimal values $C_{p,i}^*$, and χ_i, $i = \cdots,N$; find $P_i^{ref}(t)\ \forall t \geq 0$, $i = \cdots,N$ such that every P_i tracks (2), while minimizing the error $|P_i/\chi_i - P_k/\chi_k|$, $i,k = \cdots,N$, under the presence of $v_{d,i}(t)$.

Note that a similar idea of constant weighting factors is used in [9] to deal with wind farm power regulation, and in [22] for reactive power control in microgrids.

V. Control Design

A. FL step for local WT control

According to the optimization step, it turns out that every WT causing a reduction of available wind power of another one, is very likely to be subject to an optimal C_p value such that $C^*_{p,i} < C^o_p$, i.e. *strictly* inferior. Thus, WTs whose $C^*_{p,i}$ verifies $C^*_{p,i} = C^o_p$ should simply perform classic MPPT regardless the disturbances of the system and the other WTs operating points, and they can be controlled with classic local controllers. In the sequel we only consider WTs that have to be *strictly* deloaded with respect to their P^o_i. Following [10], the local control is composed of a first loop to control ω_r. We impose a first order dynamics to the rotor speed tracking error $\varepsilon_\omega \triangleq \omega^{ref} - \omega_r$: $\dot{\varepsilon}_\omega + a_0\varepsilon_\omega = 0$, by choosing $a_0 \in \mathbb{R}^+$. By naming $w \triangleq a_0\omega^{ref} + \dot{\omega}^{ref}$, this is attained via

$$T_g = T_r - (K_t - a_0 J_t)\omega_r - J_t w \tag{4}$$

Differently from [10], we choose to regulate the power output P by acting on the pitch angle. We impose a first order dynamics to the electric power tracking error $\varepsilon_p \triangleq P^{ref} - P$:

$$\dot{\varepsilon}_p + b_0\varepsilon_p = 0 \tag{5}$$

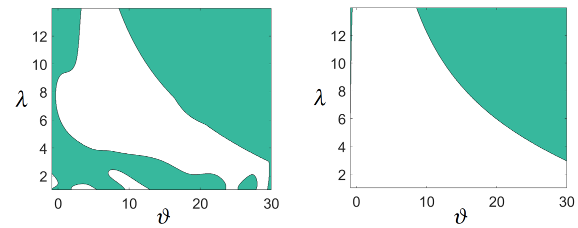
by choosing $b_0 \in \mathbb{R}^+$. This is attained via FL on (1) by choosing the feedback linearizing input

$$\begin{aligned}\vartheta_r = \frac{1}{\beta(\omega_r,\vartheta_s,v)}\Bigg(&\dot{P}^{ref} - \omega_r\frac{\partial T_r}{\partial v}\dot{v} + \frac{\omega_r}{\tau_\vartheta}\frac{\partial T_r}{\partial \vartheta}\frac{d\sigma}{d\vartheta_s}\vartheta_s + J_t\dot{w}\omega_r \\ &+ \left(2(K_t - a_0J_t)\omega_r - T_r + J_t w - \omega_r\frac{\partial T_r}{\partial \omega_r}\right)(-a_0\omega_r + w) \\ &+ b_0\varepsilon_p\Bigg)\end{aligned} \tag{6}$$

where ${\partial T_r}/{\partial \omega_r}$, ${\partial T_r}/{\partial \vartheta}$, and ${\partial T_r}/{\partial v}$ are functions of (ω_r, ϑ, v), and where $\beta(\omega_r,\vartheta_s,v) \triangleq \frac{\omega_r}{\tau_\vartheta}\frac{d\sigma}{d\vartheta_s}\frac{\partial T_r}{\partial \vartheta}$. As pointed out in our previous work [11], there exist points in which $\beta = 0$, called *singular points*, i.e. points in which (6), solution of the FL problem with respect to output P, is not defined. These points are determined by the solution of $\frac{\partial C_q}{\partial \vartheta}(\omega_r,\vartheta,v) = \frac{\partial C_q}{\partial \vartheta}(\lambda,\vartheta) = 0$, being $C_q \triangleq {C_p}/{\lambda}$, for $\beta(\omega_r,\vartheta_s,v) = \frac{\omega_r}{2\tau_\vartheta}\rho\pi R^3 v^2\frac{d\sigma}{d\vartheta_s}\frac{\partial C_q}{\partial \vartheta}(\omega_r,\sigma(\vartheta_s),v) \cong \frac{\omega_r}{2\tau_\vartheta}\rho\pi R^3 v^2\frac{\partial C_q}{\partial \vartheta}(\omega_r,\vartheta,v)$ in the domain of interest of ϑ, and $\omega_r, v > 0$. In Fig. 2 the white area represents $\Lambda = \{(\lambda,\vartheta) : (\omega_r,\vartheta) \in X \wedge \beta < 0\}$. If ω^{ref} is chosen to let λ be in a neighborhood of λ^o, and $\vartheta > 0°$ in order to let the WT to be deloaded, then it is clear that β is negative valued in the points of functioning of interest. In order to ensure that the trajectories of the closed loop system, defined by (1), (4), (6), do not pass through singular points, differently from [11], we consider a modified FL function for ϑ_r, by substituting the β function appearing in (6) with

$$\hat{\beta} \triangleq \frac{\omega_r}{2\tau_\vartheta}\rho\pi R^3 v^2\frac{d\sigma}{d\vartheta_s}\left(\frac{\partial C_q}{\partial \vartheta}(\lambda,\vartheta) - \varepsilon(\lambda,\vartheta)\right)$$
$$\varepsilon(\lambda,\vartheta) \triangleq \begin{cases}\alpha\max\left\{\frac{\partial C_q}{\partial \vartheta}(\lambda,\vartheta),0\right\} & \mathit{if}\ \frac{\partial C_q}{\partial \vartheta}(\lambda,\vartheta) \neq 0 \\ \varepsilon_1 & \mathit{otherwise}\end{cases} \tag{7}$$

where ε_1 is a small positive value, and $\alpha > 1$ is a tunable parameter to let some margin to have $\hat{\beta}$ negative valued in the system trajectories. Thus we obtain $\hat{\Lambda} = \left\{(\lambda,\vartheta) : (\omega_r,\vartheta) \in X \wedge \hat{\beta} < 0\right\}$, shown in Fig. 2. The idea is to perform an approximated FL *only* when the system trajectories come close to a singular point. Clearly, in this case, the chosen ϑ_r no longer guarantees satisfaction of (5). Nonetheless, under proper choice of ω^{ref}, and deloaded mode of functioning, approximation (7) may occur only during transients. We can summarize the results in this subsection by stating the following



(a) Λ: (white area), set of (λ,ϑ) such that $\beta(\lambda,\vartheta) < 0$.
(b) $\hat{\Lambda}$: (white area), set of (λ,ϑ) such that $\hat{\beta}(\lambda,\vartheta) < 0$.

Fig. 2: Singular points with and without β approximation.

Theorem 1: Given system (1), controlled via (4), and (6), where the β function is substituted with (7). For any initial condition $(\omega_r(0),\vartheta(0)) \in \hat{\Lambda}$, the system trajectories are bounded if parameters b_0, ε_1, and α are chosen such that $\varepsilon_1 > 0$ is sufficiently small, $\alpha > 1$, and $b_0 > -{\alpha}/{1-\alpha}$. In addition, if $\exists \bar{t} \geq 0$ such that $\frac{\partial C_q}{\partial \vartheta}(\lambda(t),\vartheta(t)) < 0\ \forall t \geq \bar{t}$, then $\lim_{t\to\infty}|P^{ref}(t) - P(t)| = 0$.

Proof: First of all, initial conditions in $\hat{\Lambda}$ imply $\hat{\beta}(0) < 0$, then $\varepsilon_1 > 0$, $\alpha > 1$ allow $\hat{\beta}(t) < 0\ \forall t \geq 0$. In particular $\hat{\beta}(t) \neq 0\ \forall t \geq 0$, thus (6) is well-defined. Note that initial conditions required in Problem 1 satisfy $\hat{\beta}(0) < 0$, according to $\hat{\Lambda}$ in Fig. 2. The system dynamics in closed loop is given by

$$\dot{\varepsilon}_p = \left(-b_0 + 1 - \frac{\beta}{\beta + \beta_\varepsilon}\right)\varepsilon_p + \left(1 - \frac{\beta}{\beta + \beta_\varepsilon}\right)\varphi(\varsigma) \tag{8}$$

where we named $\beta_\varepsilon \triangleq \hat{\beta} - \beta$, and $\varphi(\varsigma)$ the function composed of all the terms appearing in the right factor of (6) deprived of the term $b_0\varepsilon_p$, and being $\varsigma \triangleq (\omega_r, v, \vartheta_s, \dot{v}, w, \dot{w}, \dot{P}^{ref})$. The term $(1 - {\beta}/{\beta+\beta_\varepsilon})\varphi(\varsigma)$ is bounded in the trajectories thanks to the choice of β_ε, and being φ a continuous function on a compact set. The latter is compact because the wind is limited, w, $\dot{w}$, $\dot{P}^{ref}$ are chosen to be so, ω_r is bounded thanks to (4), and term $\frac{d\sigma}{d\vartheta_s}\vartheta_s$ is bounded. Thus it will be considered as a bounded input of (8) to simplify the analysis. Finally the autonomous system of (8) is stable if, for instance, we choose $b_0 > -{\alpha}/{1-\alpha}$, and ε_1 : $\varepsilon_1^{-1} > 1 - b_0$. This can be proved by choosing a Lyapunov function for the uncertain system $\dot{\varepsilon}_p = (-b_0 + \theta)\varepsilon_p$, where $\theta \in \left[\min\left\{-b_0 + 1 - \varepsilon_1^{-1}, -b_0 - {\alpha}/{1-\alpha}\right\}, 0\right]$. Eventually, if for some $\bar{t} \geq 0 : \frac{\partial C_q}{\partial \vartheta} < 0\ \forall t \geq \bar{t}$, then (8) reduces to $\dot{\varepsilon}_p = -b_0\varepsilon_p$, thus $P \to P^{ref}$ for $t \to \infty$. ∎

Remark 1: Concerning ω^{ref}, we make the choice to use the signal $\omega^o \triangleq \lambda^o v/R$ sufficiently filtered of its high frequency components. There are different motivations to support this choice. First of all, if v varies rapidly so it does ω^o, then if we consider $\omega^{ref} = \omega^o$, its variation would directly effect ϑ_r via (6), and in turns ϑ. This fact risks to make ϑ hit the saturation constraints of the sigmoid function, and more in general, to not let the constraints on $\dot{\vartheta}$ be respected, as in this framework they are only verified a posteriori. Secondly, if ω^{ref} varies too rapidly, by empirical results it turns out that closed-loop system trajectories are more likely to approach singular points, letting the activation of $\varepsilon(\lambda,\vartheta)$ defined in (7), and not allowing satisfaction of (5). On the other hand, filtering ω^o let (6) be defined. The physical explanation of this fact is that there exist infinite pairs $(\omega,\vartheta) \in X$ to deload a WT, i.e. to track P^{ref}, [11].

Remark 2: One of the reasons why the described approach should not be considered for classic MPPT mode of functioning lies in the fact that it is impossible to simultaneously track ω^o, and P^o under any control. This is easily seen considering (1). In addition, for the purpose of this analysis, let us neglect K_t. Then suppose that, thanks to a controller, $\omega_r = \omega^o$, and $P_r = P^o$ $\forall t \geq 0$. If v is not constant, then $|\dot{\omega}_r| > 0$, which implies $|P^o - P| > 0$, and in particular $P \neq P^o$. This basically means that condition $P = P^o$ cannot be forced via (6) because it is not defined. So if (6) is employed, trajectories would pass through singular points.

B. Additional Local Control Settings

From now on we carry out the analysis under the following

Assumption 3: Trajectories of the closed-loop system described by (1), (4), (6) verify $\dfrac{\partial C_q}{\partial \vartheta} < 0$.

As mentioned in Section IV, let us assume that turbine i local controller is able to measure $v_{m,i}$ such that $v_i = v_{m,i} + v_{d,i}$. The effect of $v_{d,i}$ on the closed loop dynamics can be approximated as

$$\dot{\varepsilon}_{p,i} = -b_0\varepsilon_{p,i} + \mu_1(\hat{\zeta}_i)v_{d,i} + \mu_2(\hat{\zeta}_i)v_{d,i}^2 + \mu_3(\hat{\zeta}_i)\dot{v}_{d,i} \qquad (9)$$

obtained via first order Taylor expansion of the functions depending on v_i, in a neighborhood of $v_{m,i}$, e.g. $T_r(v_i) \cong T_r(v_{m,i}) + \dfrac{\partial T_r}{\partial v}(v_{m,i})v_{d,i}$, and where $\hat{\zeta}_i \triangleq (\omega_{r,i}, \vartheta_i, v_{m,i}, \dot{v}_{m,i})$. Functions μ_1, μ_2, μ_3 are not reported in this paper for the sake of brevity. According to Assumption 2 we neglect the last term of (9). Moreover the contribution of term $\mu_2 v_{d,i}^2$ can be neglected with respect to $\mu_1 v_{d,i}$. On the compact set on which μ_1 is defined, the function satisfies $\mu_{1,min} \leq \mu_1 \leq 0$, thus in the sequel we treat μ_1 as a parametric uncertainty, and we drop its dependency on $\hat{\zeta}_i$ for ease of notation. Being interested in a discrete-time communication set-up among the WTs we shall consider the discretized system of (9)

$$\begin{bmatrix} \xi_i^{k+1} \\ P_i^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & (1 - T_s b_0) \end{bmatrix} \begin{bmatrix} \xi_i^k \\ P_i^k \end{bmatrix} + \begin{bmatrix} 1 \\ 1 + T_s b_0 \end{bmatrix} P_i^{ref,k} + \begin{bmatrix} 0 \\ \mu_1 T_s \end{bmatrix} v_{d,i}^k \qquad (10)$$

where we used Euler approximation using sampling time T_s, we approximated $\dot{P}_i^{ref} \cong (P_i^{ref,k} - P_i^{ref,k-1})/T_s$, we named $\xi_i^k \triangleq P_i^{ref,k-1}$, and apex k stands for time kT_s. Before providing the distributed controller, we add an additional *local* PI loop to (10) to be tuned to enhance rejection of $v_{d,i}$. As it will be clear, the latter also has an important role on the wind farm consensus. Naming K_I^l, and K_P^l respectively the integral, and proportional gains of the PI, we can write (10) in closed-loop as $x_i^{k+1} = Ax_i^{k+1} + B_2 u_i + B_{fw} P_i^{fw} + B_w v_{d,i}$, where

$$A \triangleq \begin{bmatrix} 1 & 0 & -K_I^l T_s \\ 1 & 0 & -K_P^l \\ (1 + T_s b_0) & -1 & \begin{matrix}(1 - T_s b_0 \\ -K_P^l(1 + T s_b 0))\end{matrix} \end{bmatrix}, B_w \triangleq \begin{bmatrix} 0 \\ 0 \\ \mu_1 T_s \end{bmatrix}$$
$$B_2 \triangleq \begin{bmatrix} K_I^l T_s \\ 1 \\ (1 + T_s b_0) \end{bmatrix}, B_{fw} \triangleq \begin{bmatrix} K_I^l T_s \\ K_P^l \\ (1 + T_s b_0) K_P^l \end{bmatrix} \qquad (11)$$

and where we named $x_i \triangleq [\delta_i, \xi_i, P_i]^\top$, being δ_i the state of the integral action, P_i^{fw} a forward signal, and u_i is left as a degree of freedom to let distributed control. Note that other choices for the placement of u_i would have been possible.

Remark 3: Forward signal P_i^{fw} is set to be equal to (2), as in steady state it respects the optimal power sharing provided by the optimization step. However, the latter is based on the computation P_i^o, in turns based on the wind measure $v_{m,i}$. Thus if $v_{d,i} \neq 0$ then P_i^{fw} is a set point for (11) that may not respect (3). This motivates next subsection analysis, where optimal power sharing is restored via a distributed algorithm by acting on u_i.

C. Distributed PID-like Consensus

Let us defined the consensus problem addressed in a general formulation. Consider N identical agents governed by general discrete-time linear dynamics, according to

$$x_i^+ = Ax_i + B_2 u_i + B_1 \omega_i, \quad y_i = Cx_i \quad i = 1, \cdots, N \qquad (12)$$

where $A \in \mathbb{R}^{n\times n}$, $B_2 \in \mathbb{R}^{n\times l}$, $B_1 \in \mathbb{R}^{n\times h}$, $C \in \mathbb{R}^{m\times n}$, $x_i \triangleq x_i(k) \in \mathbb{R}^n$ and $x_i^+ \triangleq x_i(k+1) \in \mathbb{R}^n$ are respectively the agent state at the current step k, and at the next step $k+1$, $u_i \triangleq u_i(k) \in \mathbb{R}^l$ is the agent control, $\omega_i \triangleq \omega_i(k) \in \mathbb{R}^h$ its disturbance. Being y_i the measured and the controlled output, we additionally require $l \geq m$. Let A be Schur stable, and let the agent communicate on an undirected graph whose Laplacian is $\mathscr{L}$. Thus we address the problem finding a distributed control law for u_i such that $\|y_i/\chi_i - y_k/\chi_k\|$ is minimized for $i,k = 1, \cdots, N$ with respect to the disturbance $\omega \triangleq [\omega_1^\top, \cdots, \omega_N^\top]^\top$. If the such error is zero, then we say that *weighted consensus* is achieved. We consider $\chi_i \in \mathbb{R}^+$ to simplify the analysis. Results can be extended to case of higher dimensional weights for the general case of $m > 1$. By naming $D \triangleq \mathrm{diag}(1/\chi_i)$, we additionally define matrix $\hat{\mathscr{L}} \triangleq D\mathscr{L}$, which satisfies Lemma 3, and whose positive minimum nonzero and maximum eigenvalues are respectively $\underline{\lambda}_{\hat{\mathscr{L}}}$, and $\bar{\lambda}_{\hat{\mathscr{L}}}$. In this work we focus on local controllers of the form

$$x_{c_i}^+ = A_c x_{c_i} + B_c s_i, \quad u_i = C_c x_{c_i} + D_c s_i \quad i = 1, \cdots, N \qquad (13)$$

where $x_{c_i} \triangleq x_{c_i}(k) \in \mathbb{R}^{2l}$ is the agent controller state, and

$$A_c = \begin{bmatrix} I_l & I_l \\ 0_{l\times l} & 0_{l\times l} \end{bmatrix}_{2l\times 2l} \quad B_c = \begin{bmatrix} (K_i - K_d) \\ K_d \end{bmatrix}_{2l\times m} \tag{14}$$
$$C_c = \begin{bmatrix} I_l & 0_{l\times l} \end{bmatrix}_{l\times 2l} \quad D_c = [(K_p + K_i + K_d)]_{l\times m}$$

where $K_p, K_i, K_d \in \mathbb{R}^{l\times m}$ are gain matrices to be tuned, and where $s_i \triangleq s_i(k) \in \mathbb{R}^m$:

$$s_i \triangleq \sum_{k=1}^{N} a_{ik} \left(\frac{y_i}{\chi_i} - \frac{y_k}{\chi_k} \right) \tag{15}$$

Thus the closed-loop system for agent i has dimension $\bar{n} \triangleq n + 2l$. As shown by [17], (13) is a state representation of the discrete-time PID MIMO controller, whose z-transform is $\frac{u_i(z)}{s_i(z)} = K_p + K_i \frac{z}{z-1} + K_d \frac{z-1}{z}$. The problem can now be restated as the one of finding the matrices B_c, and D_c such that the effect of disturbance ω on the weighted consensus is minimized. Before stating the result we introduce the following

Definition 1: System (12) is said to achieve *fast weighted consensus* with performance index $\tau \in \mathbb{R}^+$ if for any time-constant disturbance ω, and any initial condition, $\lim_{k\to\infty} \|{}^{y_i}/_{\chi_i} - {}^{y_k}/_{\chi_k}\| = 0$ for $i, k = 1, \cdots, N$, and $(1 - e^{-1})\%$ of consensus is achieved with a time constant inferior to τT_s.

Theorem 2: Given the system described by (12), where N agents can communicate on an undirected connected graph; consider the distributed protocol of equations (13),(14),(15); then the systems achieve fast weighted consensus with performance index $\tau = -\frac{1}{\log(R)}$, where $R \in \mathbb{R} : 0 \le R < 1$, if there exist two symmetric positive definite matrices $\underline{P}, \bar{P} \in \mathbb{R}^{\bar{n}\times\bar{n}}$ such that the LMI conditions of Theorem 2 in [17] are simultaneously satisfied for two LTI systems whose matrices are respectively $(A, B_2, \underline{\lambda}_{\mathscr{L}} C)$, and $(A, B_2, \bar{\lambda}_{\mathscr{L}} C)$, and where the real constants (a, b) to be set in Theorem 2 in [17] are chosen to be $(a, b) = (0, R)$.

Proof: The closed-loop dynamics for the generic agent i, by using (12),(13), and by defining the augmented state $\xi_i \triangleq \left[x_i^\top, x_{c_i}^\top\right]^\top \in \mathbb{R}^{\bar{n}}$, and matrices $\bar{C} \triangleq [C \quad 0_{m\times 2l}]$, is given by $\xi_i^+ = \hat{A}\xi_i + \hat{B}\sum_{k=1}^{N} a_{ik}\left({}^{\xi_i}/_{\chi_i} - {}^{\xi_k}/_{\chi_k}\right) + \tilde{B}\omega_i$, $y_i = \bar{C}\xi_i$, where

$$\hat{A} = \begin{bmatrix} A & B_2 C_c \\ 0 & A_c \end{bmatrix}, \hat{B} = \begin{bmatrix} B_2 D_c \bar{C}_2 \\ B_c \bar{C}_2 \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

By naming $\xi \triangleq \left[\xi_1^\top, \cdots, \xi_N^\top\right]^\top$, $y \triangleq \left[y_1^\top, \cdots, y_N^\top\right]^\top$, gathering together the closed-loop agents dynamic, and performing the change of coordinates $\bar{\xi} = (D \otimes I_{\bar{n}})\xi$, it yields

$$\begin{cases} \bar{\xi}^+ = \left(I_N \otimes \hat{A} + D\mathscr{L} \otimes \hat{B}\right)\bar{\xi} + \left(I_N \otimes \tilde{B}\right)\bar{\omega} \\ \bar{y} = \left(I_N \otimes \bar{C}\right)\bar{\xi} \end{cases} \tag{16}$$

where we named $\bar{\omega} \triangleq (D \otimes I_{\bar{h}})\omega$, $\bar{y} \triangleq (D \otimes I_{\bar{m}})y$, and we used point (i) of Lemma 4. Similar to [14], we define $\zeta_i \triangleq \bar{y}_i - \frac{1}{N}\sum_{k=1}^N \bar{y}_k$, and $\delta_i \triangleq \bar{\xi}_i - \frac{1}{N}\sum_{k=1}^N \bar{\xi}_k$, thus $\zeta_i = \bar{C}\delta_i$. Note that if $\zeta_i = 0$ for $i = 1, \cdots, N$ then $\bar{y}_i = \bar{y}_k$, i.e. weighted consensus is achieved. If we now name $\delta \triangleq \left[\delta_1^\top, \cdots, \delta_N^\top\right]^\top$, and $\zeta \triangleq \left[\zeta_1^\top, \cdots, \zeta_N^\top\right]^\top$, we have that $\zeta = \left(I_N \otimes \bar{C}\right)\delta$, and $\delta = \bar{\xi} - \mathbf{1} \otimes \frac{1}{N}\sum_{k=1}^N \bar{\xi}_k = \left(\bar{\mathscr{L}} \otimes I_{\bar{n}}\right)\bar{\xi}$, where $\bar{\mathscr{L}}$ satisfies the conditions of Lemma 2. Thus $\zeta = \left(I_N \otimes \bar{C}\right)\left(\bar{\mathscr{L}} \otimes I_{\bar{n}}\right)\bar{\xi} = \left(\bar{\mathscr{L}} \otimes \bar{C}\right)\bar{\xi}$. Considering the change of coordinates $\delta = \left(\bar{\mathscr{L}} \otimes I_{\bar{n}}\right)\bar{\xi}$, it yields

$$\begin{aligned} \delta^+ &= \left(\bar{\mathscr{L}} \otimes I_{\bar{n}}\right)\left(I_N \otimes \hat{A} + \hat{\mathscr{L}} \otimes \hat{B}\right)\bar{\xi} + \left(\bar{\mathscr{L}} \otimes I_{\bar{n}}\right)\left(I_N \otimes \tilde{B}\right)\bar{\omega} \\ &= \left(\bar{\mathscr{L}} \otimes \hat{A} + \bar{\mathscr{L}}\hat{\mathscr{L}} \otimes \hat{B}\right)\left(\delta + \mathbf{1} \otimes \frac{1}{N}\sum_{k=1}^N \bar{\xi}_k\right) + \left(\bar{\mathscr{L}} \otimes \tilde{B}\right)\bar{\omega} \\ &= \left(\bar{\mathscr{L}} \otimes \hat{A} + \bar{\mathscr{L}}\hat{\mathscr{L}} \otimes \hat{B}\right)\delta + \left(\bar{\mathscr{L}} \otimes \tilde{B}\right)\bar{\omega} \end{aligned}$$

where we used points (i) of Lemma 2, 3, and 4. According to the (ii) point of Lemma 2, we employ the orthogonal matrix $U \in \mathbb{R}^{N\times N}$ to define the change of coordinates: $\hat{\delta} \triangleq \left(U^\top \otimes I_{\bar{n}}\right)\delta$, $\hat{\omega} \triangleq \left(U^\top \otimes I_h\right)\bar{\omega}$, $\hat{\zeta} \triangleq \left(U^\top \otimes I_m\right)\zeta$, so that the system equations in the new coordinates are given by

$$\begin{cases} \hat{\delta}^+ &= \left(U^\top \otimes I_{\bar{n}}\right)\left(\bar{\mathscr{L}} \otimes \hat{A} + \bar{\mathscr{L}}\hat{\mathscr{L}} \otimes \hat{B}\right)\left(U \otimes I_{\bar{n}}\right)\hat{\delta} \\ & \quad + \left(U^\top \otimes I_{\bar{n}}\right)\left(\bar{\mathscr{L}} \otimes \tilde{B}\right)\bar{\omega} \\ &= \left(\bar{\Lambda} \otimes \hat{A} + \bar{\Lambda}U^\top \hat{\mathscr{L}} U \otimes \hat{B}\right)\hat{\delta} + \left(\bar{\Lambda} \otimes \tilde{B}\right)\hat{\omega} \\ \hat{\zeta} &= \left(U^\top \otimes I_m\right)\left(I_N \otimes \bar{C}\right)\left(U \otimes I_{\bar{n}}\right)\hat{\delta} = \left(I_N \otimes \bar{C}\right)\hat{\delta} \end{cases} \tag{17}$$

As shown in Lemma 2, and 3, being the last rows of $\bar{\Lambda}$, and $U^\top \hat{\mathscr{L}} U$ zeros, we can split (17) in two by dividing the system variables as $\hat{\delta} = [\hat{\delta}_1^\top, \hat{\delta}_2^\top]^\top$, $\hat{\omega} = [\hat{\omega}_1^\top, \hat{\omega}_2^\top]^\top$, and $\hat{\zeta} = [\hat{\zeta}_1^\top, \hat{\zeta}_2^\top]^\top$. It follows that, to conclude on system stability, we can study the reduced order system described by $\hat{\delta}_1^+ = \left(I_{N-1} \otimes \hat{A} + \hat{\mathscr{L}}_1 \otimes \hat{B}\right)\hat{\delta}_1 + \left(I_{N-1} \otimes \tilde{B}\right)\hat{\omega}_1$, and $\hat{\zeta}_1 = \left(I_{N-1} \otimes \bar{C}\right)\hat{\delta}_1$. From Lemma 3, it exists an invertible matrix $V \in \mathbb{R}^{(N-1)\times(N-1)} : V^{-1}\hat{\mathscr{L}}_1 V \triangleq \Lambda = diag(\lambda_1, \cdots, \lambda_{N-1})$, where $0 < \underline{\lambda}_{\mathscr{L}} \le \lambda_i \le \bar{\lambda}_{\mathscr{L}}$ for $i = 1, \cdots, N-1$. Thus we can define a further change of coordinates, such that $\tilde{\delta}_1 \triangleq \left(V^{-1} \otimes I_{\bar{n}}\right)\hat{\delta}_1$, $\tilde{\omega}_1 \triangleq \left(V^{-1} \otimes I_h\right)\hat{\omega}_1$, and $\tilde{\zeta}_1 \triangleq \left(V^{-1} \otimes I_m\right)\hat{\zeta}_1$. The latter yields

$$\begin{cases} \tilde{\delta}_1^+ = \left(I_{N-1} \otimes \hat{A} + \Lambda \otimes \hat{B}\right)\tilde{\delta}_1 + \left(I_{N-1} \otimes \tilde{B}\right)\tilde{\omega}_1 \\ \tilde{\zeta}_1 = \left(I_{N-1} \otimes \bar{C}\right)\tilde{\delta}_1 \end{cases} \tag{18}$$

Note that the transfer function matrix of (18) satisfies $\|T_{\tilde{\zeta}_1\tilde{\omega}_1}(z)\|_\infty = \|T_{\hat{\zeta}_1\hat{\omega}_1}(z)\|_\infty = \|T_{\hat{\zeta}\hat{\omega}}(z)\|_\infty = \|T_{\bar{\zeta}\bar{\omega}}(z)\|_\infty$. We can now separate (18) in $N-1$ subsystems, each of them being governed by

$$\begin{cases} \tilde{\delta}_{1_i}^+ = \begin{bmatrix} \tilde{x}_{1_i}^+ \\ \tilde{x}_{1,c_i}^+ \end{bmatrix} = \begin{bmatrix} (A + B_2 D_c(\lambda_i C)) & B_2 C_c \\ B_c(\lambda_i C) & A_c \end{bmatrix} \begin{bmatrix} \tilde{x}_{1_i} \\ \tilde{x}_{1,c_i} \end{bmatrix} \\ \qquad\qquad + \begin{bmatrix} B_1 & 0 \end{bmatrix}^\top \tilde{\omega}_i \\ \tilde{\zeta}_{1_i} = C\tilde{x}_{1_i} \end{cases} \tag{19}$$

System (19) can be equivalently seen as the closed-loop form of the two following systems

$$\begin{cases} \tilde{x}_{1_i}^+ = A\tilde{x}_{1_i} + B_2\tilde{u}_i + B_1\tilde{\omega}_i \\ \tilde{y}_i \triangleq (\lambda_i C)\tilde{x}_{1_i} \end{cases} \qquad \begin{cases} \tilde{x}_{1,c_i}^+ = A_c\tilde{x}_{1,c_i} + B_c\tilde{y}_{1_i} \\ \tilde{u}_i \triangleq C_c\tilde{x}_{1,c_i} + D_c\tilde{y}_{1_i} \end{cases} \tag{20}$$

Thus, we can reformulate the problem as the one finding matrices B_c, and D_c such that for $i = 1, \cdots, N-1$ the closed-loop system of (20) is Schur stable when $\omega_i = 0$. We now invoke Theorem 2 in [17] where it is shown that given two real constants (a, b), if there exists a symmetric positive definite matrix P_i such that a given LMI condition is satisfied, then system (20) is stable with all its eigenvalues λ laying in the complex plane region defined by $\mathscr{F}_D \triangleq \left\{(\Re[\lambda], \Im[\lambda]) : (\Re[\lambda] + a)^2 + \Im[\lambda]^2 < b^2\right\}$. Such LMI condition happens to be *affine* in the system matrices, variables and matrix P_i. We make use of this fact to provide sufficient conditions for which it exists a controller of the considered form such that the mentioned LMI is *simultaneously* verified for $i = 1, \cdots, N-1$. Since the generic eigenvalue of $\hat{\mathscr{L}}_1 : \lambda_i$ is such that $\underline{\lambda}_{\mathscr{L}} \leq \lambda_i \leq \bar{\lambda}_{\mathscr{L}}$, then it always exists $\alpha_i \in \mathbb{R} : 0 \leq \alpha_i \leq 1$ so that $\lambda_i = \alpha_i \underline{\lambda}_{\mathscr{L}} + (1-\alpha_i)\bar{\lambda}_{\mathscr{L}}$. Notice that the systems to be stabilized, appearing in the first set of equations in (20), can be seen as *one* single system with an uncertain measurement matrix, whose parameter is λ_i. In other words, $C_i \triangleq \lambda_i C$, and $\exists \alpha_i : C_i = \alpha_i C_{min} + (1-\alpha_i) C_{max}$, where $C_{min} \triangleq \underline{\lambda}_{\mathscr{L}} C$, and $C_{max} \triangleq \bar{\lambda}_{\mathscr{L}} C$, i.e. it can be written as a convex combination of the extreme matrices C_{min}, and C_{max}. Thus, we make use of classic results of robust linear control, and in particular by introducing an affine parameter dependent Lyapunov matrix $P(\alpha_i) \triangleq \alpha_i \underline{P} + (1-\alpha_i)\bar{P}$, where $\underline{P}$, $\bar{P}$ are Lyapunov matrices solution of simultaneous LMI of Theorem 2 in [17] written for respectively C_{min}, and C_{max}. Eventually, it is easy to show that if $\underline{P}$, $\bar{P}$ exist, then the controller solves the problem $\forall \lambda \in \mathbb{R} : \underline{\lambda}_{\mathscr{L}} \leq \lambda \leq \bar{\lambda}_{\mathscr{L}}$, and in particular for $\lambda = \lambda_i$, for $i = 1, \cdots, N$. Such a controller is easily found from the solution of the aforementioned LMI condition. Indeed among the LMI variables there are matrices B_c, and D_c, from which it is easy to calculate the PID gains K_p, K_i, and K_d by employing relations in (14). If such a solution exists then the eigenvalues of system (16) are guaranteed to lay in $\mathscr{F}_D$. In this framework we are interested in speeding up the system response to ω. For this reason we set $a = 0$, and $b = R$, where $R : 0 \leq R < 1$. Thus, all system eigenvalues are guaranteed to have a module inferior to R. As a result, the system has the slowest time-constant inferior to $-{}^{T_s}\!/\log(R)$. ■

Notice that the tools developed in this subsection can be directly applied to solve the wind farm weighted consensus. This is obtained by choosing as A, B_2 the homonym matrices of (11), $C = [0 \quad 0 \quad 1]$, i.e. measure and control of P_i, and $B_1 = [B_{fw} \quad B_w]$, i.e. P_i^{fw}, and $v_{d,i}$ are both considered as disturbances with respect to the weighted consensus. Eventually, in order not to contrast Assumption 3, we require the additional

Assumption 4: All $v_{d,i}$ are such that the corresponding P_i^{ref}, depending on (2), on the local PI control, and on the distributed PID, is always lower than the real maximum power that turbine i can extract from the wind.

Remark 4: The distributed PID minimizes the influence of all P_i^{fw}, and $v_{d,i}$ on the power sharing error among P_i. This can be seen as forcing given *relative* distances among

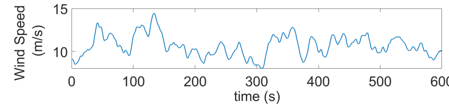
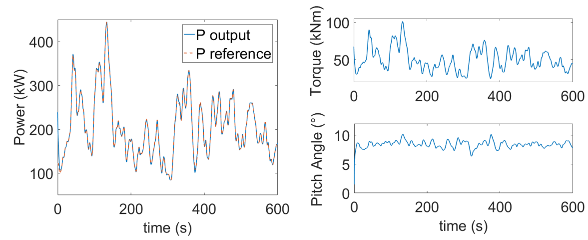


Fig. 3: Wind speed signal.



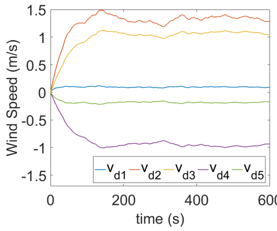
(a) Deloaded electric power. (b) WT inputs

Fig. 4: Local FL controller simulation.

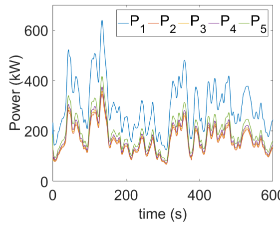
the P_i variables. However, the *absolute* values of all the P_i are influenced by P_i^{fw}, and $v_{d,i}$. This motivates the internal PI control pushing P_i towards P_i^{fw}. Note that by construction $P^{fw} \triangleq [P_1^{fw}, \cdots, P_N^{fw}]^\top$ verifies $\mathscr{L}DP^{fw} = \mathbf{0}$.

VI. Simulations

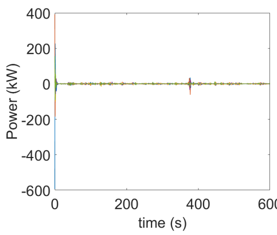
For the following simulations we considered the real CART turbine C_p. This represents the only source of model-plant mismatch. Test under more realistic uncertainties goes beyond the scope of this paper, and it will not be addressed. First of all, let us show the FL local controller behavior. During 600 s, the WT is excited by the effective wind speed signal of Fig. 3. Inputs signal shown in Fig. 4, generated by the FL controller, allow the WT to track a deloaded power reference of 50% with respect to P^o. $|P - P^{ref}|$ goes to zero with a time constant depending on the chosen b_0 in (5), (see Fig. 3). Small persistent error oscillations are due to the C_p mismatch. Concerning the wind farm power sharing simulations, we consider 6 aligned WTs that communicate with their direct neighbor WT. Wind speed signal v_∞, blowing in front of turbine 1 is chosen as the previous simulation one. Turbine 6, being the last one, is required to operate in classic MPPT mode, thus it does not intervene in the consensus control, and its P signal will not be reported. Wind disturbances $v_{d,i}$, and the controlled WT powers are shown in Fig. 5. For this problem, weighted consensus is achieved with performance index $\tau = 24.5$. In order to show consensus achievement we provide two additional figures. Naming $P \triangleq [P_1, \cdots, P_N]^\top$, the first one shows signals $\mathscr{L}DP$, which in ideal conditions should have all zeros entries in steady state. The second one shows DP, where its entries should ideally reach a common value. These simulations are shown in the bottom of Fig. 5, and in a zoomed window in Fig. 6. Note that C_p mismatch, as well as temporary dissatisfaction of Assumption 3 cause persistent small oscillations on the reached weighted consensus.



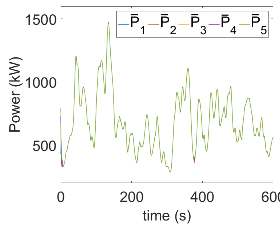
(a) Wind disturbances.



(b) WTs electric power.

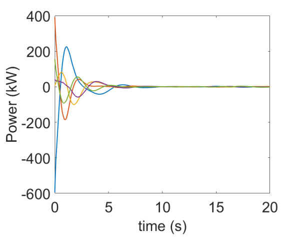


(c) Consensus signals: $\mathscr{L}DP$.

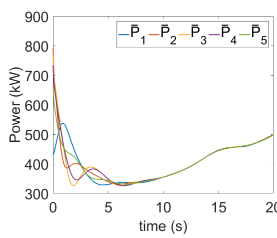


(d) New coordinate power signals: $\bar{P} = DP$.

Fig. 5: Distibuted control simulation.



(a) Zoom on consensus signals: $\mathscr{L}DP$.



(b) Zoom on new coordinate power signals: $\bar{P} = DP$.

Fig. 6: Zoom on system variables during distributed control simulation.

VII. Conclusions

We presented a novel distributed approach to control a wind farm for power maximization under wake effect, based on two control layers. First, the proposed FL local controller allows a WT to track a deloaded power reference by acting on both the rotor speed, and the pitch angle. Then, a PID-like discrete-time weighted consensus control is developed to let the wind farm turbines keep the optimal power sharing under the presence of wind disturbances. Indeed, because of the latter, without this second control layer, WTs would track an absolute power reference which may not respect the optimization step gains. In the near future work we are interested in extending the proposed framework to relax the assumptions on the local controller performance, and to treat other sources of system disturbances.

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