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**Abstract**

Relationships between ergodicity and structures of fourthorder spectral moments are investigated. In particular it is shown that second-order ergodicity of a random process is directly related to the distribution of these moments on the normal manifolds of the frequency domain. This result is illustrated by various examples.

**Index Terms**

Ergodicity, normal processes, shot noise, stationary and normal manifolds, trispectrum.

## I. Introduction and Notations

Higher order statistics, or, more precisely, statistics of order higher than two, are used in various areas of signal processing and information sciences. The purpose of this correspondence is to show that they can also be useful for testing ergodic properties of some stochastic processes (SP). From a strictly

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mathematical point of view this has been known for a long time [1], [2]. However, these papers are written in a very abstract form, and the engineering community has considered the same problem again with a quite different approach, and especially in the case of harmonic processes [3], [4]. This correspondence has an intermediary perspective: it is not especially mathematically oriented and not limited to harmonic processes.

Let $x(t)$ be a continuous-time real and strictly stationary SP. In order to estimate its mean value m it is appropriate to use its causal time average $y_T(t)$ given by

$$y_T(t) = (1/T)\int_{t-T}^{t} x(\theta)d\theta. \tag{1}$$

It is said that $x(t)$ is weakly first-order (WFO) ergodic if $y_T(t)$ tends to $m$ in the quadratic mean sense when $T$ tends to infinity. The characterization of this kind of ergodicity requires only the knowledge of second-order properties of $x(t)$: This means that conditions for ergodicity can be deduced from the behavior of the correlation function

$$\gamma(\tau) = E[x(t)x(t-\tau)] - m^2,$$

or from the structure of its Fourier transform $\Gamma(f)$, the power spectrum of $x(t)$: In order to simplify the presentation we assume that this spectrum can be expressed as

$$\Gamma(f) = \Gamma_b(f) + \sum_{i=1}^{N} \gamma_i \delta(f - f_i), \tag{2}$$

where $\Gamma_b(f)$ is nonnegative, bounded, and integrable. Physically, this means that the spectrum of $x(t)$ contains a continuous part, with a density $\Gamma_b(f)$, and a finite number of spectral lines at frequencies $f_i$. In this case it is known that $x(t)$ is WFO ergodic if and only if there is no spectral line at the frequency zero [5].

It is obvious that WFO ergodicity does not imply that the time average of $x(t)x(t-s)$ converges to the ensemble average $E[x(t)x(t-s)]$ for any $s$, which characterizes weak second-order (WSO) ergodicity. For this, it is necessary to use the correlation function of $x(t)x(t-s)$, which is a fourth-order moment of $x(t)$: The most general form of such a moment is

$$m_4(\mathbf{t}) = m_4(t_1, t_2, t_3, t_4) = E[x(t_1)x(t_2)x(t_3)x(t_4)] \tag{3}$$

and we shall also extensively use its Fourier transform

$$M_4(\mathbf{f}) = M_4(f_1, f_2, f_3, f_4)$$

called fourth-order spectral moment. These functions are discussed in detail in [5, p. 236]. In particular, it is shown that if $x(t)$ is stationary, the spectral moment $M_4(\mathbf{f})$ is equal to zero except when $fT^{\mathbf{r}} = 0$, where $\mathbf{r}$ is the vector $[1,1,1,...,1]^T$ : This allows us to write $M_4(\mathbf{f}) == \Gamma_3(\mathbf{f})\delta(\mathbf{f}^T\mathbf{r})$, where $\Gamma_3(\mathbf{f})$ is called the moment trispectrum of $x(t)$. Therefore, $M_4(\mathbf{f})$ and $\Gamma_3(\mathbf{f})$ contain the same information and in the following discussion it will be more convenient to use $M_4(\mathbf{f})$ instead of $\Gamma_3(\mathbf{f})$. The equation

$$\mathbf{f}^T\mathbf{r} = f_1 + f_2 + f_3 + f_4 = 0$$

defines the stationary manifold in the frequency domain. For our discussion it is necessary to recall the definition of the normal manifolds (NM).

Suppose that $x(t)$ is a normal SP defined by its covariance function $\gamma(\tau)$ or its spectrum $\Gamma(f)$: The fourth-order moments of $x(t)$ can be expressed in terms of $\gamma(\tau)$ by a well-known formula (see [5, p. 278]). By Fourier transformation of this expression we obtain

$$M_4(\mathbf{f}) = \sum_N \Gamma(f_i)\Gamma(f_k)\delta(f_i + f_j)\delta(f_k + f_l). \tag{4}$$

The letterN (Normal) means that this sum contains three terms defining the three normal manifolds (NM), subsets of the stationary manifold, and defined by

$$f_i + f_j = 0 \;\; ; \;\; f_k + f_l = 0, \tag{5}$$

where the sequence $[i,j,k,l]$ takes the three isctincts values $[1,2,3,4]$, $[1,3,2,4]$, and $[1,4,2,3]$.

It is clear that (4) is unique up to an exchange of either $f_i$ and $f_j$ or $f_k$ and $f_l$, due to the delta term and the symmetry of $\Gamma(f_i)$. Furthermore, (4) shows that on the NMs the density is not arbitrar but appears as a product of spectral densities $\Gamma(f_i)\Gamma(f_j)$.This form of density is called a normal density. This can be summarized by saying that the fourth-order spectral moment of a normal SP is uniquely distributed in the NMs with a normal density. For the following discussion it is worth pointing out that there are submanifolds of the NMs that can play a certain role.

## II. Conditions for Second-Order Ergodicity

In order to apply the result indicated above, it is necessary to calculate the power spectrum of $y(t;s) = x(t)x(t-s)$. For this purpose, and to simplify the discussion, let us introduce some general assumptions. Suppose that $x(t)$ is an SP with zero mean value and a power spectrum denoted by $\Gamma(f)$. We assume that its fourth-order spectral moment can be written as

$$M_4(\mathbf{f}) = \sum_N A(f_i, f_k)\delta(f_i + f_j)\delta(f_k + f_l) + B(f_1, f_2, f_3)\delta(f_1 + f_2 + f_3 + f_4). \tag{6}$$

The first three terms of this expression correspond to a distribution on the NMs with density $A(f_i, f_k)$ and the last term is a distribution on the stationary manifold with density $B(\mathbf{f})$. In the normal case we have $B(\mathbf{f})$ and $A(f_i, f_k) = \Gamma(f_i)\Gamma(f_k)$. It is clear that $B(\mathbf{f})$ is the contribution to the fourth-order spectral moment that does not possesses the normal manifolds as support. The relation (6) is verified by all the models of SPs analyzed below. Furthermore, there is no example of SP discussed in the literature for which the terms $A(f_i, f_k) = 0$, and, therefore, the structure (6) seems quite general.

Weak second-order ergodicity is related to the structure of the power spectrum of $y(t; s) = x(t)x(t-s)$ Its covariance function is

$$\gamma_y(\tau; s) = m_4(t, t-s, t-\tau, t-s-\tau) - \gamma^2(s), \tag{7}$$

where $\gamma(.)$ is the covariance function of $x(t)$, and $m_4$ its fourth-order moment. The power spectrum of $y(t; s)$ is the Fourier transform of $\gamma_y(\tau; s)$, or

$$\Gamma_y(f; s) = \int \gamma_y(\tau; s) \exp(-2\pi j f\tau) d\tau, \tag{8}$$

Expressing $m_4$ of (7) in terms of its Fourier transform $M_4(\mathbf{f})$ yields

$$m_4(t, \tau, s) \int\int\int\int M_4(\mathbf{f}) g(\mathbf{f}) d\mathbf{f}, \tag{9}$$

with $g(\mathbf{f}) = \exp[2\pi j\{\mathbf{f}^T\mathbf{r}t - (f_3 + f_4)\tau - (f_2 + f_4)s\}]$. The stationary manifold $\mathbf{f}^T\mathbf{r} = 0$ appears immediately in this expression. Indeed, because of the assumption of stationarity, the function $m_4$ in (9) does not dependon $t$ This is ensured by the relation $\mathbf{f}^T\mathbf{r} = 0$, and therefore the first exponential term of the integral is equal to one. By inserting thielation into (8), performing the integration with respect to $\tau$, which introduces the term $\delta(f + f_3 + f_4)$, and afterwards the integration in $f_4$, we obtain

$$\Gamma_y(f; s) = \int\int\int M_4(f_1, f_2, f_3, -f - f_3) \exp[2\pi j(f_2 - f - f_3)s] df_1 df_2 df_3 - c\delta(f), \tag{10}$$

wihj $c = \gamma^(s)2$.

Let us now calculate the contribution to $\Gamma_y(f; s)$ of the various terms appearing in $M4(\mathbf{f})$ The last term of (6) introduces the term $\delta(\mathbf{f}^T\mathbf{r})$ Used in (10) this gives $\delta(f_1 + f_2 - f)$. As a consequence, the contribution of the density $B(\mathbf{f})$ on the stationary manifold introduces the term

$$T_B = \int\int B(f_1, f - f_1, f_3) \exp[2\pi j(f_1 + f_3)s] df_1 df_3. \tag{11}$$

Let us now calculate the contribution $T_1$ of the first NM appearing in (6). This introduces the term $A(f_1, f_2)\delta(f_1 + f2)\delta(f_3 - f - f_3$ in the integral (10). As a result, this term is

$$T_1 = \delta(f) \int\int A(f_1, f_3) \exp[2\pi j(f_1 + f_3)s] df_1 df_3. \tag{12}$$

By following the same procedure we find that the two terms due to the last two manifolds of (5) are equal to

$$T_2 = \int A(f_1, f - f_1) df_1,$$

$$T_3 = \int\int A(f_1, f - f_1)\exp[2\pi j(2f_1 - f)s]df_1 df_3. \tag{13}$$

Finally, te term continig the factor $\delta(\nu)$ in $\Gamma_y(f; s)$ is

$$T_\delta = \int\int A(f_1, f_3)\exp[2\pi j(f_1 + f_3)s]df_1 df_3 - \gamma^2(s). \tag{14}$$

In order to test WSO ergodicity by applying the result indicated in the Introduction, we have to verify whether or not there is a spectral line at the frequency zero. If the coefficient $T_\delta$ in (14) is equal to zero and if the other terms are bounded, there is no spectral line, and $x(t)$ is WSO ergodic. On the other hand, lack of ergodicity can come from several origins. This can appear if $T_\delta$ in (14) is not equal to zero and if the other terms are bounded. In these cases, it appears that the spectral line, and therefore the ergodicity property, is entirely due to the structure of the terms $A(f_1, f_3)$ on the NMs. However, there are more complicated situations, the best example being the case where $x(t)$ is normal, as analyzed in the next section.

The previous analysis was only devoted to second-order ergodicity and the corresponding calculations require the use of spectral moments up to the fourth order. It is clear that similar calculations can be extended to ergodicity of an order higher than two. The principles are the same but the detailed expressions are more complicated. It is obvious that testing weak ergodicity of order $n$ requires the use of spectral moments of order $2n$ However, it appears that the role of the NMs remains the same.

## III. EXAMPLES

### A. Normal Case

Suppose that $x(t)$ is a normal SP. This implies that the last term of (14) is zero. Suppose first that there is no spectral line in the spectrum of $x(t)$. This means that the second term of (2) is zero. In this case, the coefficient $T$of (14) is zero and the second term is bounded. As a consequence, $x(t)$ is WFO and WSO ergodic. Suppose now that there is one spectral line in the spectrum of $x(t)$ at a nonzero frequency. As $x(t)$ is real, its power spectrum is symmetric, and our assumption allows us to write (2) in the form

$$\Gamma(f) = \Gamma_b(\nu) + \gamma[\delta(f - f_i) + \delta(f + f_i)]$$

It is clear that these spectral lines do not change the fact that $x(t)$ is WFO ergodic. Furthermore, the coefficient $T$ of (14) is still equal to zero. This does not mean that there is no spectral line at the frequency zero. Indeed, by coupling the terms $\delta(f - f_i)$ and $\delta(f + f_i)$ the second integral of (14) yields the term which introduces a spectral line at the frequency zero. Therefore, even if $x(t)$ is WFO ergodic, it is not WSO ergodic. So we see that the spectral line at the frequency zero comes not from (14) but from its second term, because $A(f_1, f_3)$ is not bounded. In conclusion, ergodicity of normal SPs requires that there is no spectral line at all in the spectrum of $x(t)$ This is in accordance with a known result, shown by a completely different method (see [6, p. 157]): a normal process is ergodic if and only if its spectral distribution is continuous everywhere. As the power spectrum is the derivative of the spectral distribution, this is equivalent to saying that there is no spectral line in the spectrum.

### B. *Spherical Invariant Processes*

A process $x(t)$ is said to be spherically invariant if it can be written as $x(t) = Au(t)$, where $u(t)$ is a normal SP and $A$ a random variable independent of $u(t)$. For simplicity we assume that the mean values of $A$ and $u(t)$ are zero. Furthermore, we assume that the variance of $A^2$ is not zero, which means that $A$ is not a random variable taking only two values $+a$ and $a$. The results from these assumptions are that $x(t)$ cannot be normal and, therefore, its fourth-order spectral moment has no reason to take the form (4). We shall, however, see that this spectral moment satisfies (6), with specific values of $A$ and $B$. For this, note that the power spectrum of $x(t)$ is $m_2\Gamma_u(f)$, where $m_2$ is the variance of $A$ and $\Gamma_u(f)$ the power spectrum of $u(t)$. We assume that this spectrum does not contain a spectral line. As$u(t)$ is normal. As a result, the fourth-order spectral moment of $x(t)$ is given by (6) where the term $B(f_1, f_2, f_3)$ is zero and

$$A(f_i, f_k) = m_4\Gamma(f_i)\Gamma(f_k).$$

This implies that the last two terms of (14) are bounded. The first term can be expressed as $\delta(f)m_4m_2^2\gamma_u^2(s)$ where $m_4 = E(A^4)$ and $\gamma_u(.)$ is the correlation function of $u(t)$. The conclusion is straightforward: the SP $x(t)$ is WSO ergodic if and only $m_4 - m_2^2 = 0$. As this expression is the variance of $A^2$, this is equivalent to saying that $A2^2$ is not random. This implies either that $A$ is not random or that $A$ is a random variable taking only two values $\pm a$, so that $A^2 = a^2$ If $A$ is not random, $Au(t)$. As a conclusion, the only spherically invariant SP that are WSO are the normal processes. This property has also been obtained by completely different methods in [7]. Note finally that x(t) is an example of SP for which M4(f) is uniquely distributed on the normal manifolds, which means that the term B(f) is zero, and the nonnormal character is due to the fact that the density in the NMs is not normal.

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