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Bernard Picinbono

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Bernard Picinbono, *Fellow, IEEE*

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Abstract

In many questions of signal processing, it is important to use the concepts of instantaneous amplitude or phase of signals. This is especially the case in communication systems with amplitude or frequency modulation. These concepts are often introduced empirically. However, it is well known that the correct approach for this purpose is to use the concept of analytic signal. Starting from this point, we show some examples of contradictions appearing when using other definitions of instantaneous amplitude or frequency that are commonly admitted. This introduces the problem of characterizing pure amplitude-modulated or pure phase-modulated signals. It is especially shown that whereas amplitude modulated signals can be characterized by spectral considerations, this is no longer the case for phase modulated signals.

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B. P. with the Laboratoire des Signaux et Systèmes (L2S), a joint laboratory of the C.N.R.S. and the École Supérieure d'Électricité, Plateau de Moulon, 3 rue Joliot–Curie 91192, Gif sur Yvette, France. The L2S is associated with the University of Paris–Sud, France. E-mail: bernard.picinbono@lss.supelec.fr, .

I. INTRODUCTION

INSTANTANEOUS amplitude and phase are basic concepts in all the questions dealing with modulation of signals appearing especially in communications or information processing. Let us remember that a purely monochromatic signal such as $a\cos(\omega t+\phi)$ cannot transmit any information. For this purpose, a modulation is required, and one of the simplest possible to introduce is amplitude modulation. Let $m(t)$ be a positive function corresponding to the information o be transmitted. By multiplying the carrier frequency signal $\cos(\omega_0 t)$ by $m(t)$, we obtain the signal

$$x(t) = m(t)\cos(\omega_0 t) \tag{1}$$

and it is commonly admitted that is $m(t)$ the instantaneous amplitude of the signal $x(t)$. This appears in many textbooks, especially in [1, p. 237]. On the other hand, the need for phase or frequency modulation requires the definition of the instantaneous phase. By a reasoning similar to the previous one concerning the amplitude, it is commonly admitted that the signal

$$x(t) = a\cos(\phi t) \tag{2}$$

has a constant amplitude a and an instantaneous phase $\phi(t)$. Furthermore, the instantaneous angular frequency is given by the derivative of $\phi(t)$. This is a generalization of the procedure applied in the case where the phase is linear in time or in the form $\phi(t) = \omega_0 t + \phi$, giving the frequency . These definitions also appear in many textbooks or papers. For example, the following is written on [2, p. 645]: The argument $\theta(t)$ in any signal having the form $\cos[(\theta(t)]$ is called the instantaneous phase, and $(1/2\pi)d[\theta(t]/dt$ is called the instantaneous frequency. The same definition appears, for example, in [3, p. 480], [4, p. 260], and [5, p. 144].

Even if the previous definitions appear quite natural and are widely used in practical applications dealing with signal modulation, we immediately note that they cannot be satisfactory. In order to make this point clear, let us discuss a very simple example. Suppose that the function $m(t)$ appearing in (1) is bounded or satisfies $0 \leq m(t) \leq a$. As a result, we have $|1/a| \leq 1$. It is then possible to introduce a unique function $\phi(t)$ satisfying

$$0 \leq \phi(t) \leq \pi \;;\; \cos[\phi(t)] = (1/a)x(t). \tag{3}$$

By using this well-defined function, we obtain

$$x(t) = m(t)\cos[\omega_0 t] = a\cos[\phi(t)]. \tag{4}$$

This shows that the signal $x((t)$ can be considered frequency as well as amplitude modulated. In other words, its instantaneous amplitude is $m(t)$ as well as a, and moreover, its instantaneous phase is either $\omega_0 t$ or $\phi(t)$.

More generally, starting from a given signal $x(t)$, it is possible to introduce an infinite number of pairs $[a(t), \phi(t)]$ such that

$$x(t) = a(t)\cos[\phi(t)]. \tag{5}$$

This leads to the conclusion that the definitions given previously, even if they are widely used, are incoherent because they do not associate with a given real signal a well-defined pair of functions that are the instantaneous amplitude and phase of $x(t)$. Therefore, these definitions must be reformulated in such a way that any given signal $x(t)$ corresponds to one well-defined pair $[a(t), \phi(t)]$, allowing us to write $x(t)$, as in (5).

In reality, the solution of this problem is well known and explicitly introduced in [6, p. 50]. A recent review paper on this question [7] also gives a good introduction on the discussion leading to the standard definition of the quantities studied hereafter. Moreover, this paper contains a list of references corresponding to the history of this problem.

The classical definition of the instantaneous amplitude and phase of a real signal is recalled in the next section. This definition introduces the concept of a canonical pair $[a(t), \phi(t)]$, and it is therefore interesting to find conditions ensuring that a given pair of functions $a(t)$ and $\phi(t)$ is canonical. When the instantaneous phase is linear, or when the signal looks like (1), it is possible to characterize a canonical pair only from the spectral properties of the signal. This becomes much more difficult when the instantaneous phase is no longer linear, and this especially appears for signals with constant amplitude, which are called phase signals, which are at the foundation of phase- or frequency-modulation systems. The properties of such signals are analyzed in the following section, and from them, we can deduce various properties of canonical pairs. The practical consequences of these results and their extension for random signals are finally investigated.

II. Definitions and Canonical Pairs

A. *Definitions*

Let us recall the classical way to define without ambiguity the instantaneous amplitude and phase of a real signal $x(t)$. The problem is to write $x(t)$ as in (5) but by using a pair of functions $[a(t), \phi(t)]$ that is in a one-to-one correspondence with $x(t)$. For this purpose, we associate with $x(t)$ its analytic signal (AS) $z(t)$ (see [6, p. 48]). It is obtained from $x(t)$ by filtering it using a filter with the frequency response $H(\nu)$ equal to 2 for $\nu > 0$ and to 0 for $\nu < 0$. Conversely, it is obvious that $x(t)$ = Re$[z(t)]$, where Re means the real part. Therefore, if $z(t)$ is real, there is indeed a one-to-one correspondence between $x(t)$ and $z(t)$. On the other hand, a complex function is an AS if its Fourier transform is zero for negative frequencies. It is clear that this function is the AS of its real part.

As $z(t)$ cannot be a real function because its Fourier transform $Z(\nu)$ is zero for $\nu < 0$, it can be written as

$$z(t) = a(t)\exp[\phi(t)], \tag{6}$$

where the phase $\phi(t)$ is defined modulo 2π, and $a(t)$ is nonnegative. As a conclusion, using the AS makes it possible to associate with any real signal a unique pair $[a(t), \phi(t)]$ called in the following the canonical pair associated with $x(t)$.

Definition: Let be a real signal and the canonical pair $[a(t), \phi(t)]$ associated with it. The function $a((i)$ appearing in this pair is the instantaneous amplitude of $x((t)$, and $\phi((t)$ is its instantaneous phase. The instantaneous frequency is the derivative with respect to time of $\phi((t)$.

The introduction of the AS is not at all recent, and among the principal papers in this field, we can note [8][12]. There are some questions concerning the physical meaning of the AS, and some of them are mentioned and discussed in [7]. Furthermore, it is shown in [13] and [14] that starting from some a priori physical assumptions, the only possible definition of the instantaneous amplitude and phase is the one given just above. However, it is worth pointing out that other physical conditions lead to another definitions that are not discussed below [15].

Once the definition is given, the question that immediately follows is to characterize a canonical pair or to show what the conditions are on $a(t)$ and $\phi(t)$ in order to ensure that (6) is an AS, which means that its Fourier transform is zero for negative frequencies. This is essential when verifying whether or not the classical definitions given above are correct. In fact, there is no a priori reason for $[m(t)], \omega_0 t]$ appearing in (1) or $[a(t), \phi(t)]$ appearing in (2) to be canonical.

B. Spectral Characterization of a Canonical Pair

Consider first a signal of the form (1). It corresponds to an amplitude modulation of a pure monochromatic signal with the carrier frequency ω_0. The nonnegative function $m(t)$ is the instantaneous amplitude of $x(t)$ if and only if $m(t)\exp(j\omega_0 t)$ is an AS.

Let $M(\nu)$ be the Fourier transform (FT) of $m(t)$. As $m(t)$ is real, $M(\nu) = M^*(-\nu)$. The FT of $m(t)\exp(j\omega_0 t)$ is, of course, $M(\nu - \nu_0)$, with $\nu_0 = \omega/2\pi$. As a result, we obtain that $M(\nu - \nu_0) = 0$ for ν negative or that $m(t)$ is the instantaneous amplitude of $x(t)$ given by (1) if and only if $M(\nu)$ is zero for $|\nu| > \nu_0$. Physically, this means that $m(t)$ is a low-frequency bandlimited signal. In all that follows, we call such a signal a low-frequency(ν_0) signal. Similarly, a high-frequency (B) signal is characterized by the fact that its FT vanishes for $|\nu| < B$.

This discussion shows that it is possible to characterize a canonical pair such as $[a(t)], \omega_0 t$ uniquely by a spectral condition on $a(t)$. Starting from this example of amplitude modulation, it was tempting to try to use spectral methods for the characterization of more general pairs of functions $[a(t), \phi(t)]$ Unfortunately, we shall see that the task becomes immediately impossible.

Saying that $a(t)\ exp[j\phi(t)]$ is an AS is equivalent to saying that the Hilbert transform of $a(t)\ cos[\phi(t)]$ is equal to $a(t)\ sin[\phi(t)]$ (see [6, p. 49]). It is therefore appropriate to make use of the so-called Bedrosian theorem [16] dealing with the Hilbert transform of a product of two real functions $x_1(t)$ and $x_2(t)$. A very simple derivation of this theorem and some extensions can be found in [17]. The main result is as follows: Let $X_1(\nu)$ and $X_2(\nu)$ be the FTs of $x_1(t)$ and $x_1(t)$, respectively. If $X_1(\nu) = 0$ for $\nu > B$ and $X_2(\nu) = 0$ for $\nu < B$, then

$$H[x_1(t)x_2(t)] = x_1(t)H[x_2(t)] \tag{7}$$

where $H[.]$ means the Hilbert transform. A direct application of this result shows that if $a(t)$ is a low-frequency (B) signal , and $\cos[\phi(t)]$ a high-frequency (B) signal, or if their spectra do not overlap, then

$$H\{a(t)\cos[\phi(t)]\} = a(t)H\{\cos[\phi(t)]\}. \tag{8}$$

However, this does not at all imply that

$$H\{\cos[\phi(t)]\} = \sin[\phi(t)] \tag{9}$$

as stated by many authors and even recently in [7]. If (9) were true, it would also be possible to characterize a canonical pair only by spectral considerations, as for amplitude modulation. Furthermore, as the constant signal has an FT limited to the frequency zero, (9) would be true for any signal $\cos[\phi(t)]$ without a low-frequency component. This result would be especially attractive, suppressing all the questions discussed in the introduction, when presenting some comments on (2). Let us now show that (9) has no reason to be true when only spectral properties of $\cos[\phi(t)]$ are introduced. For this purpose, we shall propose an elementary counterexample of this property. It is obvious that (9) implies that

$$\cos^2[\phi(t)] + \sin^2[\phi(t)] = 1. \tag{10}$$

Let $x(t)$ be defined by

$$x(t) = \mathrm{sinc}(2Bt)\cos(\omega_0 t), \tag{11}$$

where $\mathrm{sinc}(x) = \sin(\pi x)/\pi x$, and $\omega_0 > 2\pi B$. As $|x(t)| < 1$, it is possible to introduce a phase $\phi(t)$ uniquely defined if $0 \leq \phi(t) \leq \pi$ and such that $x(t) = \cos[\phi(t)]$. As the FT of $\mathrm{sinc}2Bt$ is zero for $|\nu| > B$, the FT of $x(t)$ does not contain low-frequency components because of the assumption on ω_0. Applying (9) then gives $H[x(t)]\sin[\phi(t)]$, and as a consequence, $x^2((t) + H^2[x(t)] = 1$. However, it is obvious that this equality is not correct. In fact, by applying the Bedrosian theorem, we obtain

$$H[x(t)] = \mathrm{sinc}(2Bt)\sin(\omega_0 t) \tag{12}$$

and $x^2(t) + H^2[x(t)] == \text{sinc}^2(2Bt)$.

This shows that contrary to a common well-established idea, it is not possible to justify (9) by introducing only spectral considerations. This point will become much clearer in the next section. In fact, (9) implies that

$$z(t) = \exp[j\phi(t)] \tag{13}$$

is an AS or that the pair$[1, \phi(t)]$ is canonical. We shall now see that this requires very specific properties of the structure of the phase $\phi(t)$.

III. Phase Signals

A. General Structure

Phase signals are real signals with constant instantaneous amplitude. They can be expressed as (2) but with the condition that $\exp[j\phi(t)]$ is an AS. As a consequence, (9) is satisfied. For such signals, all the information is contained in the instantaneous phase (or frequency), and phase signals are then the basic elements of phase or frequency modulation.

The condition that $\exp[j\phi(t)]$ is an AS requires very specific properties on the phase $\phi(t)$. These properties have been analyzed in the framework of coherence problems in optics [18] but, more precisely, in the framework of the study of analytic functions, and especially in [19, ch. 7]. We present here the results that are the most important for our arguments without the proofs, which are out of the scope of this discussion and can be found in [19].

The most general structure of the AS of a nonsingular phase signal is

$$z(t) = b(t)\exp[j(\omega_0 t + \theta)], \tag{14}$$

where θ is arbitrary, ω_0 is nonnegative, and $b(t)$ is a Blaschke function defined by

$$b(t) = \prod_{k=1}^{N} \frac{t - z_k}{t - z_k^*}\ , \quad z_k \in P_+, \tag{15}$$

where P_+ is the half plane of the complex plane defined by $\text{Im}(z) > 0$. The quantity ω_0 is the carrier frequency and can be equal to zero. The expression nonsingular means that the instantaneous phase of $z(t)$ remains finite for finite values of t. When the number N of factors in the product is not finite, there are other constraints on the complex numbers z_k due to convergence problems. Here, we avoid these questions by assuming that N is finite. In this case, the interpretation of (15) is very simple. In order to be an AS, the function $b(t)$, for complex values of t, must have all its poles in the half plane Im . In order to have a modulus equal to one, each pole must be associated with a corresponding zero symmetric of this pole with respect to the real axis. This procedure is well known in filter theory: Stable phase filters have the same number of poles and zeros, and these zeros are symmetric to the poles with respect to the imaginary axis. The stability and causality conditions imply that all the poles are in the left half plane of the complex plane.

It is obvious that $|b(t)| = 1$, which implies that $|z(t)| = 1$. Let us now explain why $z(t)$ is an AS. For this, we must analyze the structure of the FT $B(\nu)$ of $b(t)$. As N is finite, $b(t)$ is a rational function in t. If all the z_ks are distinct, we can write

$$b(t) = 1 + \sum_{k=1}^{N} \frac{c_k}{t - z_k^*} \ , \tag{16}$$

where

$$c_k = \lim_{t \to z_k^*} (t - z_k^*) b(t) \tag{17}$$

As a consequence, we have

$$B(\nu) = \delta(\nu) + \sum_{k=1}^{N} C_k(\nu), \tag{18}$$

where $C_k(\nu)$ is the FT of $c_k(t - z_k^*)^{-1}$. Because of the localization of z_k^* in the complex plane, we deduce that $C_k(\nu) = 0$ for $\nu < 0$, which implies that $B(\nu) = 0$ for $\nu < 0$ and ensures that $b(t)$ is an AS. Finally, as $\omega_0 > 0$, $z(t)$ also is an AS. The reasoning can be extended without difficulty when some poles z_k are no longer distinct.

The phase of $b(t)$ is, of course

$$\phi_k(t) = \mathrm{Arg}[b(t)], \mathrm{mod} 2\pi, \tag{19}$$

and, as a result, we can say that any phase signal can be written as (2), where $\phi(t)$ must have the form

$$\phi(t) = \theta + \omega_0 t + \phi_b(t), \mathrm{mod} 2\pi. \tag{20}$$

In practice, the continuity of the phase leads to suppress the term $\mathrm{mod}(2\pi)$, and this convention is adopted in all that follows.

This most general phase is defined by N complex parameters z_k and two real parameters ω_0 and θ. Furthermore, it is obvious that the phases $\phi_k(t)$ are the sum of N phases of the factors appearing in the product (15). Let $b_k((t)$ be equal to $(t - z_k)(t - z_k^*)^{-1}$, and let $\phi k(t)$ be its phase. This gives

$$\phi(t) = \theta + \omega_0 t + \sum_{k=1}^{N} \phi_k(t). \tag{21}$$

By introducing the real and imaginary parts of z_k or $z_k = a_k + j b_k$, one obtains

$$\phi_k(t) = 2\mathrm{Arctg} \frac{b_k}{a_k - t} \ , \quad -\pi/2 \leq \phi_k(t) \leq \pi/2. \tag{22}$$

At this step, we come back to the problem discussed in the introduction. The signal (2) is phase modulated only if its phase takes the form (20), and as this is not in general the case, its amplitude is not constant, and it must be expressed as in (5).

B. Properties of Phase Signals

Having the most general structure of phase signals, we will now present some of their properties, which allows a better understanding of their structure.

Property 3.1:*A phase signal contains only two spectral lines corresponding to its carrier frequency.* This is a direct consequence of the structure of the Blaschke function appearing in (15). Its FT$B(\nu)$ is given by (18), which can be written as

$$B(\nu) = \delta(\nu) + B_c(\nu). \tag{23}$$

The function $B_c(\nu)$ describes the continuous part of the FT of $B(\nu)$. It is a sum of N components $C_k(\nu)$ that are bounded and equal to zero for $\nu < 0$. This implies that $B_c(\nu)$ is also bounded and equal to zero for $\nu < 0$, and thus $B(\nu)$ exhibits only one Dirac component, or a spectral line, at the frequency zero. Because of the exponential term in (14), the FT of $z(t)$ is $Z(\nu)\exp(j\theta)B(\nu - \nu_0)$, and this means that there is only one spectral line at the carrier frequency $\nu_0 = \omega_0/2\pi$. By using the Hermitian symmetry, we deduce that $z(t)$ has only two spectral lines at the frequencies $\pm\nu_0$.

This property can be used in a reciprocal way, indicating that all the signals containing more that one spectral line in the range of positive frequencies cannot be phase signals and, therefore, exhibit a nonconstant instantaneous amplitude.

Property 3.2: *A phase signal with a nonzero carrier frequency) is a high-frequency*$(\nu_0$ *signal*. This is a direct consequence of the form of the FT $Z(\nu)$ analyzed just above. As $B(\nu) = 0$ for $\nu < 0$, $Z(\nu) = 0$ for $\nu < \nu_0$ and $X(\nu) = 0$ for $|\nu| < \nu_0$.

The converse is, of course, not true. There is no reason for a high-frequency signal to be a phase signal because this frequency condition does not imply the structure (14). A simple counterexample appears with a high-frequency signal containing more than two spectral lines and, thus, does not satisfy Property 3.1.

Property 3.3: *The FT of the AS of a phase signal is zero for all the frequencies smaller than the carrier frequency* ν_0 *where the spectral line is located.* This is a direct consequence of (23) and of the fact that the FT $Z(\nu)$ of$z(t)$ is proportional to $B(\nu - \nu_0)$.

Property 3.4: *A phase signal cannot be a low-frequency*(B) *signal except when it is monochromatic.* The monochromatic case appears when $b(1)$, and $z(t)$ is therefore $\cos(\omega_t + \theta)$, that is, of course, a low-frequency signal. Except in this case, the property means that it is impossible to find a frequency B such that $Z(\nu) = 0$ for $\nu > B$. This property can be shown by two procedures. In the first, we simply note that the functions $C_k(\nu)$ of (18) are exponential functions for $\nu > 0$ when the poles are distinct.

However, it is well known that a sum of a finite number of exponential functions cannot be zero for all the frequencies satisfying $\nu > B$.

It is also possible to show this property by contradiction. Suppose then that there exists a frequency B such that $Z(\nu) = 0$ for $\nu > B$. It results from this assumption and from Property 3.3 that the FT of $z(t)$ is zero outside the frequency interval $\nu_0 < \nu < B$ and has a spectral line at the carrier frequency ν_0. Consider now the signal $z^*(t)$. Its FT is equal to $Z^*(-\nu)$, and this FT is zero outside the frequency interval $-B < -\nu < -\nu_0$ and has a spectral line at the frequency $-\nu_0$. Let us now introduce the signal

$$w(t) = z^*(t) \exp(2\pi j f t), \tag{24}$$

with $f > B$. This signal obviously satisfies $|w(t)| = 1$, and the frequency condition ensures that it is an AS. It then has the general form (14) and (15) and must satisfy Property 3.3. However, its spectral line is at the frequency $f - \nu_0$, and $W(\nu)$ is not zero for $\nu < f - \nu_0$. This is in contradiction with Property 3.3, which shows the result.

Property 3.5: *Frequency Shift.* If in (14) we replace ω_0 with ω_1, with $\omega_1 > \omega_0$, we obtain a complex signal that is still an AS. As a consequence, if $x(t)\cos[\phi(t)]$ is a phase signal, which means that its phase has the structure (20), $x'(t = a\cos|\Delta\omega t + \phi(t)])$, where $\Delta\omega = \omega_1 - \omega_0$, is still a phase signal with the carrier frequency ω_1. This especially implies that $x'(t)$ is a high-frequency(ω_1) signal.

Property 3.6: *Instantaneous Frequency of a Phase Signal:* It is obtained by differentiating the instantaneous phase. The most general form of this phase is given by (21) and (22), and differentiating this equation yields

$$\omega(t) = \omega_0 + 2\sum_{k=1}^{N} \frac{b_k}{b_k^2 + (a_k - t)^2} \ . \tag{25}$$

A similar equation has been obtained in [20] and [21] by using a rather different procedure.

As the coefficients b_k are positive, because of the localization of the zeros z_k, we deduce that the instantaneous frequency $\omega(t)$ is always greater than ω_0. This is another illustration of the fact that the FT of $z((t)$ is zero for $\nu < (1/2\pi)\omega_0$. If all the z_ks are zero, then $b(t)$ defined by (15) is equal to one, and the instantaneous frequency is tsimply ω_0.

Some other comments can be presented on the structure of the instantaneous frequency of a phase signal, and this is especially relevant in all those questions dealing with frequency or phase modulation of signals. The information carried by the instantaneous frequency of a phase signal is entirely in the term

$$\omega_m(t) = 2\sum_{k=1}^{N} \frac{b_k}{b_k^2 + (a_k - t)^2} \ , \tag{26}$$

where the index m stands for the modulation term. We note that this function tends to zero when $|t| \to \infty$. This especially means that $\omega_m(t)$ cannot be a periodic function, and this is related to the fact that a phase signal cannot have spectral lines, except those coming from the carrier angular frequency ω_0.

Furthermore, we note that $\omega_m(t)$ is a rational function in t. The polynomials appearing in the numerator and the denominator have the degrees $2N-2$ and $2N$, respectively. As N isarbitrary, we deduce that by using the $2N$ parameters a_k and b_k, it is possible to approximate a large class of functions. The most limiting constraint on these functions comes from the necessary behavior for $|t| \to \infty$. In fact, $\omega_m(t)$ decreases at infinity in $|t|^{-2}$, which is a strong restriction on the instantaneous frequency.

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