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***Abstract*— This paper presents a design method for robust output feedback control of linear, discrete-time, invariant systems affected by both state and output additive disturbances. The method relies on the so-called convex lifting, which is defined on the $N-$step controllable set. It is proven that the proposed method guarantees the recursive feasibility and robust stability in the sense that the closed loop converges to a given robust positively invariant set as time tends to infinity. Moreover, the method only requires the resolution of a linear programming problem at each sampling instant. Finally, a numerical example is considered to illustrate the methodology.**

I. INTRODUCTION

Robust control is still an active topic in control theory, although extensive studies have been dedicated to over several decades, see among the others [16], [27]. In these robust control problems, full knowledge of the state is required, however, in practice measurement is usually affected by external noises. Therefore, suitable observer is required to estimate the state and to integrate in control design. As a consequence, control strategy becomes output feedback, since it is a function of outputs. Among the available methods, a basic idea is to transform a robust output feedback control problem into a robust state feedback counterpart, see [6]. Accordingly, robust control methodologies can be applied. Note that if the underlying system is linear and subject to linear constraints, a linear Luenberger observer can be deployed. On this direction, model predictive control (MPC) strategies were proposed in [14], [26]. A common point of MPC strategies is that suitable terminal constraints, describing a robust positively invariant set, are imposed at the end of the prediction horizon, leading to robust stability [15]. However, this also leads to an exponential increase of the number of constraints along the prediction horizon and subsequently to a demanding online evaluation.

To reduce the computational complexity, set-theoretic methods can also be of use by determining a priori a suitable control Lyapunov function, as employed in [4]. This method only requires to solve a linear program at each sampling instant. However, it also requires the contractivity of the feasible region over which this control Lyapunov function is defined. The computation of such a contractive set is usually based on an iterative procedure, known to be demanding. Recall that the vertex control presented in [10] also requires the contractivity of the feasible region. As an extension, the interpolation based approach was put forward in [18], in which the interpolation gain serves as a control Lyapunov function. Accordingly, robust stability is ensured by the non-increase of this gain along the dynamics. Moreover, the so-called convex lifting approach has recently been presented in [22], [24]. Unlike the above set-theoretic methods, this concept is not a control Lyapunov function, however the contractivity of the feasible region is still required.

This paper presents an alternative method, relying on a suitable convex lifting as an extension of the method in [22], [24]. More precisely, this convex lifting is defined over an $N-$step controllable set, known not to be contractive. Similar to these set-theoretic methods, terminal constraints are not imposed. However, unlike a control Lyapunov function, this function is equal to 0 over a given robust positively invariant set and strictly positive outside this region. In addition, this function is shown to be strictly decreasing along the closed-loop dynamics outside this invariant set, leading to its convergence to 0 as time tends to infinity. Accordingly, robust stability is guaranteed by means of the convergence of the closed-loop dynamics to a given robust positively invariant set. Meanwhile, this method also requires to solve a linear program at each sampling instant, it thus could be useful for systems with fast dynamics, e.g. cantilever beam system [9].

Nomenclature

Throughout the paper, $\mathbb{R}, \mathbb{N}, \mathbb{N}_{>0}$ denote the field of real numbers, the set of nonnegative integers and the positive integer set, respectively. The index set $\mathcal{I}_N$ is also defined as $\mathcal{I}_N = \{1, 2, \ldots, N\}$ with respect to a given $N \in \mathbb{N}_{>0}$. A polyhedron is defined as the intersection of finitely many closed halfspaces. A polytope is defined as a bounded polyhedron. Also, $\mathcal{V}(P)$ denotes the set of vertices of polytope P. We use $\operatorname{int}(S)$ to denote the interior of a full-dimensional set S and $\operatorname{conv}(S)$ denotes the convex hull of S. Also, given a set S and a matrix A of suitable dimension, we use AS to denote the following set: $AS = \{Ax : x \in S\}$. Given two sets $S_1, S_2 \subset \mathbb{R}^d$, by $S_1 \backslash S_2$, we denote the following set: $S_1 \backslash S_2 := \left\{x \in \mathbb{R}^d : x \in S_1,\ x \notin S_2\right\}$. Also, the Minkowski sum of these two sets, denoted by $S_1 \oplus S_2$, is defined as follows: $S_1 \oplus S_2 := \{x_1 + x_2 : x_1 \in S_1,\ x_2 \in S_2\}$. Finally, the Potryagin difference $S_1 \ominus S_2$ is defined by: $S_1 \ominus S_2 := \{x : x \oplus S_2 \subseteq S_1\}$.

II. PROBLEM SETTINGS

In this paper, we consider a linear time-invariant system, affected by both additive state and output disturbances:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + w_k \\ y_k &= Cx_k + v_k \end{aligned} \tag{1}$$

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where x_k, u_k denote the state and control variable at time k, while w_k, v_k represent the additive state and output disturbances at time k, respectively. These variables and disturbances satisfy:

$$x_k \in \mathbb{X},\ u_k \in \mathbb{U},\ w_k \in \mathbb{W},\ v_k \in \mathbb{V}, \quad (2)$$

where the constraint sets $\mathbb{X} \subset \mathbb{R}^{d_x}, \mathbb{U} \subset \mathbb{R}^{d_u}, \mathbb{W} \subset \mathbb{R}^{d_x}, \mathbb{V} \subset \mathbb{R}^{d_y}$ are assumed to be polytopes, containing the origin in their interior, with given $d_x, d_u, d_y \in \mathbb{N}_{>0}$. This system is assumed to satisfy the following assumption.

Assumption 1: The pair $(A,\ B)$ is controllable and the pair $(A,\ C)$ is observable.

This paper aims to design robust output feedback controller which is able to cope with the state and output disturbances, and to stabilize system (1), while guaranteeing the constraints in (2). To tackle this problem, a common approach is to make use of a Luenberger observer. More precisely, this observer is written in the following form:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L(y_k - \hat{y}_k),\ \hat{y}_k = C\hat{x}_k. \quad (3)$$

If one defines $e_k = x_k - \hat{x}_k$, then we obtain the following autonomous system, according to (1) and (3),

$$e_{k+1} = (A - LC)e_k + w_k - Lv_k. \quad (4)$$

Note that since $w_k \in \mathbb{W}$ and $v_k \in \mathbb{V}$, then $w_k - Lv_k$, considered as additive disturbances of system (4), is also bounded in $\mathbb{W} \oplus (-L\mathbb{V})$, i.e., $w_k - Lv_k \in \mathbb{W} \oplus (-L\mathbb{V})$. Therefore, if one chooses a gain L such that $A - LC$ is strictly stable, then it is proven in [8] that a robust positively invariant set exists (this concept will be recalled in Sect. III). Such a gain L can be determined by means of the Riccati equation for the system $(A^T,\ C^T)$.

For ease of presentation, let Ω_e denote a robust positively invariant set for the autonomous system (4). Accordingly, for any initial error state $e_0 \in \Omega_e$, then $e_k \in \Omega_e$ for all $k \in \mathbb{N}_{>0}$. The observer system (3) can be written in the following form:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + Lv_k + LCe_k,\ \hat{y}_k = C\hat{x}_k. \quad (5)$$

We remark that if the initial error state $e_0 \in \Omega_e$, then $Lv_k + LCe_k$ is also considered as additive disturbance for the observer system (5). Furthermore, such an additive disturbance is also bounded in $L\mathbb{V} \oplus LC\Omega_e$. Note that $\mathbb{X}, \mathbb{V}, \mathbb{W}$ are assumed to be polytopes, therefore Ω_e is bounded, leading to the boundedness of $L\mathbb{V} \oplus LC\Omega_e$. If we compute a polytopic robust positively invariant set Ω_e for the autonomous system (4), then we can ensure that the additive disturbance $Lv_k + LCe_k$ is bounded in polytope $L\mathbb{V} \oplus LC\Omega_e$. Also, it can easily be observed that $0 \in \mathrm{int}(L\mathbb{V} \oplus LC\Omega_e)$, since $0 \in \mathrm{int}(\mathbb{V})$ and $0 \in \mathrm{int}(\Omega_e)$. As a consequence, we now return to a robust control design problem for the linear system (5) affected by bounded additive state disturbances, whose variables and parameters are bounded as below:

$$\hat{x}_k \in \mathbb{X} \ominus \Omega_e,\ u_k \in \mathbb{U},\ Lv_k + LCe_k \in L\mathbb{V} \oplus LC\Omega_e, \quad (6)$$

under the assumption that $\mathbb{X} \ominus \Omega_e$ is non-empty. In the sequel, we will present a robust control design method for system (5), which relies on a suitable convex lifting. Unlike the counterpart presented in [22], [24], a convex lifting employed in this paper is defined over the $N-$step controllable set, usually known not to be contractive. Note that the contractivity property is always required in most of existing methods related to piecewise linear control Lyapunov functions [4], [10], [17]. It is worth emphasizing that robust stability in this paper is not in the sense of Lyapunov. Instead, we will prove that such a constructive convex lifting is strictly decreasing along the closed-loop dynamics outside a given robust positively invariant set and consequently the closed loop of the observer system (5) is convergent to this set, leading to robust stability of system (1).

III. Construction of suitable convex lifting

Before going to the main result, we recall some important concepts which are of use later in the proposed control design method. Positive invariance concept has been investigated in many studies [1]–[3], [5] and deployed in different control design strategies. In case the underlying system is affected by disturbances, the robust positive invariance concept is of use instead.

Definition 3.1: Given an admissible control law $u = K\hat{x} \in \mathbb{U}$, a set $\Omega_{\hat{x}} \subseteq \mathbb{X} \ominus \Omega_e$ is called *robust positively invariant* with respect to system (5) subject to constraints (6) iff $(A + BK)\Omega_{\hat{x}} \oplus L\mathbb{V} \oplus LC\Omega_e \subseteq \Omega_{\hat{x}}$.

Similar to determining an observer gain, such a local controller $u = K\hat{x} \in \mathbb{U}$ in Definition 3.1 can also be obtained via the Riccati equation for the pair $(A,\ B)$. According to this local controller, one can use existing algorithms to compute a robust positively invariant set $\Omega_{\hat{x}}$, see for instance [8], [13], [17], [25]. Hereafter, $\Omega_{\hat{x}}$ is assumed to be a full-dimensional polytope.

Another important concept is the feasible region. Unlike the one deployed in [24], in this paper we make use of the $N-$step controllable set as the feasible region with respect to a given $N \in \mathbb{N}_{>0}$. Its definition is recalled below.

Definition 3.2: Consider system (5) subject to constraints (6). Let a robust positively invariant set $\Omega_{\hat{x}}$ and $N \in \mathbb{N}_{>0}$ be given. A set denoted by $\mathcal{K}_N(\Omega_{\hat{x}}) \subseteq \mathbb{X} \ominus \Omega_e$ is called the *$N-$step controllable set* if any point, belonging to this set, can reach $\Omega_{\hat{x}}$ in N steps, while staying inside $\mathbb{X} \ominus \Omega_e$ despite any disturbances in $L\mathbb{V} \oplus LC\Omega_e$, i.e.,

$$\begin{aligned}
&\mathcal{K}_0(\Omega_{\hat{x}}) = \Omega_{\hat{x}},\\
&\mathcal{K}_N(\Omega_{\hat{x}}) = \Big\{\hat{x}_0 \in \mathbb{X} \ominus \Omega_e : \exists u_0, \ldots, u_{N-1} \in \mathbb{U} \ \text{ s.t.}\\
&\qquad \hat{x}_N \in \Omega_{\hat{x}},\ \forall v_i \in \mathbb{V},\ e_i \in \Omega_e,\ i \in \{0\} \cup \mathcal{I}_{N-1}\Big\}.
\end{aligned}$$

For brevity, the computation of $\mathcal{K}_N(\Omega_{\hat{x}})$ is referred to [12] for further details. We now present the construction of a suitable function according to the above ingredients, which will be deployed in the proposed control design. For ease of presentation, let $\ell_N(\hat{x})$ denote such a function defined over $\mathcal{K}_N(\Omega_{\hat{x}})$. According to its construction presented in the sequel, $\ell_N(\hat{x})$ was proven in [19], [24] to be a convex lifting, therefore, we will call this function a convex lifting throughout the rest of the paper. For simplicity, the formal definition of this concept and related ones are referred to [20], [21], [23]. This function should be convex, nonnegative, equal to 0 over $\Omega_{\hat{x}}$, strictly positive over $\mathcal{K}_N(\Omega_{\hat{x}})\backslash\Omega_{\hat{x}}$. Furthermore, it should satisfy $\ell_N(v) > \max_{\hat{x}\in\mathcal{K}_{i-1}(\Omega_{\hat{x}})} \ell_N(\hat{x})$ for all $v \in \mathcal{V}(\mathcal{K}_i(\Omega_{\hat{x}}))\backslash\mathcal{K}_{i-1}(\Omega_{\hat{x}})$. All these requirements are of use later to prove the convergence of $\ell_N(\hat{x})$ to 0 along the trajectories, as it will be clear later. To construct $\ell_N(\hat{x})$, we start with the following initial point, for $h_0 = 0$ and a given constant $h_1 > 0$:

$$\begin{aligned} V_0 &:= \left\{ \begin{bmatrix} v^T & h_0 \end{bmatrix}^T : v \in \mathcal{V}(\Omega_{\hat{x}}) \right\} \subset \mathbb{R}^{d_x+1} \\ V_1 &:= \left\{ \begin{bmatrix} v^T & h_1 \end{bmatrix}^T : v \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}})) \right\} \cup V_0 \\ \Pi_1 &:= \operatorname{conv}(V_1) \\ \ell_1(\hat{x}) &:= \min_z z \quad \text{s.t.} \quad \begin{bmatrix} \hat{x}^T & z \end{bmatrix}^T \in \Pi_1. \end{aligned} \tag{7}$$

The above function $\ell_1(\hat{x})$ represents the following properties.

Lemma 3.1: Function $\ell_1(\hat{x})$ is continuous, convex and piecewise affine.

The proof follows as a direct consequence of Theorems IV-3 and IV-4 in [7]. The following important properties of $\ell_1(\hat{x})$ are of use later.

Lemma 3.2: Function $\ell_1(\hat{x})$ satisfies:

1) $0 \leq \ell_1(\hat{x}) \leq h_1$ for all $\hat{x} \in \mathcal{K}_1(\Omega_{\hat{x}})$;
2) $\ell_1(\hat{x}) = 0$ for $\hat{x} \in \Omega_{\hat{x}}$;
3) $\ell_1(\hat{x}) > 0$ for $\hat{x} \in \mathcal{K}_1(\Omega_{\hat{x}})\backslash\Omega_{\hat{x}}$;
4) $\ell_1(\hat{x}) = h_1$ for $\hat{x} \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))\backslash\Omega_{\hat{x}}$.

Proof: Any point $\begin{bmatrix} \hat{x}^T & z \end{bmatrix}^T \in \Pi_1$ can be expressed as:

$$\begin{aligned} &\alpha_0(v), \alpha_1(v) \geq 0, \sum_{v\in\mathcal{V}(\Omega_{\hat{x}})} \alpha_0(v) + \sum_{v\in\mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))} \alpha_1(v) = 1, \\ &\begin{bmatrix} \hat{x} \\ z \end{bmatrix} = \sum_{v\in\mathcal{V}(\Omega_{\hat{x}})} \alpha_0(v) \begin{bmatrix} v \\ 0 \end{bmatrix} + \sum_{v\in\mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))} \alpha_1(v) \begin{bmatrix} v \\ h_1 \end{bmatrix}. \end{aligned} \tag{8}$$

Accordingly, one can easily observe that

$$0 \leq z = \sum_{v\in\mathcal{V}(\Omega_{\hat{x}})} \alpha_0(v)0 + \sum_{v\in\mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))} \alpha_1(v)h_1 \leq h_1,$$

leading to claim 1). Note that the left-hand inequality becomes equality only if $\alpha_1(v) = 0$ for all $v \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))$, leading to $\hat{x} \in \Omega_{\hat{x}}$, as stated in claim 2). Moreover, for any point $\hat{x} \in \mathcal{K}_1(\Omega_{\hat{x}})\backslash\Omega_{\hat{x}}$ expressed above, there exists at least one vertex $v \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))$ such that $\alpha_1(v) > 0$, leading to $z > 0$ and subsequently to claim 3). Finally, since $\ell_1(\hat{x})$ is convex over $\mathcal{K}_1(\Omega_{\hat{x}})$, this function attains its maximal value h_1 at vertices of $\mathcal{K}_1(\Omega_{\hat{x}})$. Note however that any $\hat{x} \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}})) \cap \Omega_{\hat{x}}$ can be expressed as in (8) with $\alpha_1(v) = 0$ for all $v \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))$, yielding $\ell_1(\hat{x}) = 0 < h_1$. In other words, $\ell_1(\hat{x}) = h_1$ for $\hat{x} \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))\backslash\Omega_{\hat{x}}$. ■

In order to construct a convex lifting $\ell_N(\hat{x})$ as required above, we construct intermediate functions $\ell_i(\hat{x})$ for $i \in \mathbb{N}_{>0}$ such that $\ell_N(\hat{x}) = \ell_i(\hat{x})$ for all $\hat{x} \in \mathcal{K}_i(\Omega_{\hat{x}})$ and for all $i \in \mathcal{I}_N$. Starting from (7), the construction of $\ell_N(\hat{x})$ is presented step-by-step in the sequel. Suppose function $\ell_i(\hat{x})$, associated with the polytopic partition $\left\{\mathcal{X}_j^{(i)}\right\}_{j\in\mathcal{I}_{M(i)}}$ of $\mathcal{K}_i(\Omega_{\hat{x}})$, is written in the form:

$$\ell_i(\hat{x}) = (a_j^{(i)})^T\hat{x} + b_j^{(i)} \quad \text{for} \quad \hat{x} \in \mathcal{X}_j^{(i)}. \tag{9}$$

We also define the following function:

$$\sigma_i(\hat{x}) = \max_{j\in\mathcal{I}_{M(i)}} (a_j^{(i)})^T\hat{x} + b_j^{(i)} \quad \text{for} \quad \hat{x} \in \mathbb{R}^{d_x}. \tag{10}$$

Accordingly, $\ell_{i+1}(\hat{x})$ is constructed as follows, with respect to a given constant $\epsilon > 0$:

$$\begin{aligned} &h_{i+1} := \min_h h \text{ s.t. } \sigma_i(v) + \epsilon \leq h, \forall v \in \mathcal{V}(\mathcal{K}_{i+1}(\Omega_{\hat{x}}))\backslash\mathcal{K}_i(\Omega_{\hat{x}}) \\ &V_{i+1} := \left\{ \begin{bmatrix} v^T & h_{i+1} \end{bmatrix}^T : v \in \mathcal{V}(\mathcal{K}_{i+1}(\Omega_{\hat{x}})) \right\} \cup V_i \\ &\Pi_{i+1} := \operatorname{conv}(V_{i+1}) \\ &\ell_{i+1}(\hat{x}) := \min_z z \quad \text{s.t.} \quad \begin{bmatrix} \hat{x}^T & z \end{bmatrix}^T \in \Pi_{i+1}. \end{aligned} \tag{11}$$

The constructions (9)–(11) are repeated until $i = N - 1$. Similar to $\ell_1(\hat{x})$ shown in Lemma 3.1, $\ell_{i+1}(\hat{x})$ for all $i \in \mathcal{I}_{N-1}$ possess the following properties:

Lemma 3.3: Functions $\ell_{i+1}(\hat{x})$ are convex, continuous, piecewise affine, for all $i \in \mathcal{I}_{N-1}$.

Lemma 3.4: $\sigma_i(\hat{x}) = \ell_i(\hat{x})$ for all $\hat{x} \in \mathcal{K}_i(\Omega_{\hat{x}})$, $\forall i \in \mathcal{I}_N$.

Proof: Since $\ell_i(\hat{x})$ is a convex, piecewise affine function, proven in Lemma 3.3, it can be written as follows:

$$\ell_i(\hat{x}) = \max_{j\in\mathcal{I}_{M(i)}} (a_j^{(i)})^T\hat{x} + b_j^{(i)} \quad \text{for} \quad \hat{x} \in \mathcal{K}_i(\Omega_{\hat{x}}).$$

Accordingly, the representation of $\sigma_i(\hat{x})$ in (10) leads to $\sigma_i(\hat{x}) = \ell_i(\hat{x})$ for all $\hat{x} \in \mathcal{K}_i(\Omega_{\hat{x}})$. ■

We now prove that function $\ell_N(\hat{x})$ satisfies the aforementioned requirements.

Lemma 3.5: For $i \in \mathcal{I}_N$, we obtain

1) $0 < h_i < h_{i+1}$
2) $\ell_i(\hat{x}) = h_i$ for $\hat{x} \in \mathcal{V}(\mathcal{K}_i(\Omega_{\hat{x}}))\backslash\mathcal{K}_{i-1}(\Omega_{\hat{x}})$
3) $0 \leq \ell_i(\hat{x}) \leq h_i$
4) $\ell_{i+1}(\hat{x}) = \ell_i(\hat{x})$ for $\hat{x} \in \mathcal{K}_i(\Omega_{\hat{x}})$.

Proof: We will prove claims 1), 2) and 3) in the same time. As proven in Lemma 3.2, for $i = 1$, claims 2) and 3) hold true and $0 < h_1$, we now prove that $h_1 < h_2$. In fact, as $\sigma_1(\hat{x})$ is convex over $\mathcal{K}_2(\Omega_{\hat{x}})$, it thus attains its maximal value at vertices of $\mathcal{K}_2(\Omega_{\hat{x}})$. Accordingly, there exists a vertex $v \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}}))\backslash\mathcal{K}_1(\Omega_{\hat{x}})$ such that $\sigma_1(v) \geq \sigma_1(\hat{x})$ for all $\hat{x} \in \mathcal{K}_2(\Omega_{\hat{x}})$. Therefore, if one chooses $\hat{x} \in \mathcal{V}(\mathcal{K}_1(\Omega_{\hat{x}}))$, then (11) yields $h_2 \geq \epsilon + \sigma_1(\hat{x}) = \epsilon + \ell_1(\hat{x}) = \epsilon + h_1 > h_1$.

To prove claim 3) for $i = 2$, we consider $\begin{bmatrix} \hat{x}^T & z \end{bmatrix}^T \in \Pi_2$, this point can be described as below

$$\begin{aligned} &\alpha_j(v) \geq 0, \sum_{j=0}^{2} \sum_{v\in\mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v) = 1 \\ &\begin{bmatrix} \hat{x}^T & z \end{bmatrix}^T = \sum_{j=0}^{2} \sum_{v\in\mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v) \begin{bmatrix} v^T & h_j \end{bmatrix}^T. \end{aligned} \tag{12}$$

As a consequence of claim 1) for $i = 1$, we obtain

$$0 \leq z = \sum_{j=0}^{2} \sum_{v\in\mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v)h_j \leq h_2,$$

leading to claim 3) for $i=2$.

To prove claim 2) holds true for $i=2$, consider any point $\left[\hat{x}^T\ z\right]^T \in \Pi_2$, this can be written as in (12). Note that $\ell_2(\hat{x})$ is convex over $\mathcal{K}_2(\Omega_{\hat{x}})$, therefore $\ell_2(\hat{x})$ attains its maximal value h_2 at vertices of $\mathcal{K}_2(\Omega_{\hat{x}})$. However, any point $\hat{x} \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}})) \cap \mathcal{K}_1(\Omega_{\hat{x}})$ can be expressed as in form (12) with $\alpha_2(v)=0$, $\forall v \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}}))$, leading to $\ell_2(\hat{x}) \leq h_1 < h_2$. Roughly speaking, $\ell_2(\hat{x}) = h_2$ for $\hat{x} \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}})) \backslash \mathcal{K}_1(\Omega_{\hat{x}})$. The proof of claims 1), 2) and 3) for $N \geq i \geq 3$ follows the same arguments as deployed above.

For claim 4), we first prove that it holds for $i=1$. In fact, as proven in claim 1) that $h_2 \geq h_1 + \epsilon$, then the construction in (11) leads to $h_2 \geq \sigma_1(v) + \epsilon$ for all $v \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}}))$. Accordingly, any point $\left[\hat{x}^T\ z\right]^T \in \Pi_2$ described as in (12) yields:

$$
\begin{aligned}
z &= \sum_{j=0}^{2} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v) h_j && \text{(13a)}\\
&\geq \sum_{j=0}^{1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v) \sigma_1(v) \\
&\qquad + \sum_{v \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}}))} \alpha_2(v)(\sigma_1(v)+\epsilon) && \text{(13b)}\\
&\geq \sigma_1(\hat{x}) + \sum_{v \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}}))} \alpha_2(v)\epsilon \geq \sigma_1(\hat{x}). && \text{(13c)}
\end{aligned}
$$

Note that inclusion (13c) is obtained due to the convexity of $\sigma_1(\hat{x})$. Also, the right-hand inequality of (13c) becomes equality only if $\alpha_2(v)=0$ for all $v \in \mathcal{V}(\mathcal{K}_2(\Omega_{\hat{x}}))$, leading to $\hat{x} \in \mathcal{K}_1(\Omega_{\hat{x}})$. Therefore, $\ell_2(\hat{x}) = \sigma_1(\hat{x}) = \ell_1(\hat{x})$ for all $\hat{x} \in \mathcal{K}_1(\Omega_{\hat{x}})$. Suppose claim 4) holds until i, leading to $h_j \geq \sigma_i(v)$ for all $v \in \mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))$ and all $j \in \mathcal{I}_i$. We prove that it holds for $i+1$. In fact, following the same argument of use to prove $h_2 \geq h_1 + \epsilon$, we can obtain $h_{i+1} \geq \epsilon + h_i$. Accordingly, any point $\left[\hat{x}^T\ z\right]^T \in \Pi_{i+1}$ can be written in the following form:

$$
\begin{aligned}
&\alpha_j(v) \geq 0,\ \sum_{j=0}^{i+1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v) = 1 \\
&\left[\hat{x}^T\ z\right]^T = \sum_{j=0}^{i+1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v) \left[v^T\ h_j\right]^T .
\end{aligned} \tag{14}
$$

As a consequence, the definition of h_{i+1} in (11) yields:

$$
\begin{aligned}
z &= \sum_{j=0}^{i+1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v) h_j && \text{(15a)}\\
&\geq \sum_{j=0}^{i+1} \sum_{v \in \mathcal{V}(\mathcal{K}_j(\Omega_{\hat{x}}))} \alpha_j(v)\sigma_i(v) + \sum_{v \in \mathcal{V}(\mathcal{K}_{i+1}(\Omega_{\hat{x}}))} \alpha_{i+1}(v)\epsilon \\
&\geq \sigma_i(\hat{x}) + \sum_{v \in \mathcal{V}(\mathcal{K}_{i+1}(\Omega_{\hat{x}}))} \alpha_{i+1}(v)\epsilon \geq \sigma_i(\hat{x}). && \text{(15b)}
\end{aligned}
$$

Note that the left-hand inequality of (15b) is due to the convexity of $\sigma_i(\hat{x})$. The equality holds true only if $\alpha_{i+1}(v) = 0$ for all $v \in \mathcal{V}(\mathcal{K}_{i+1}(\Omega_{\hat{x}}))$, leading to $\hat{x} \in \mathcal{K}_i(\Omega_{\hat{x}})$. In other words, $\ell_{i+1}(\hat{x}) = \sigma_i(\hat{x}) = \ell_i(\hat{x})$ for $\hat{x} \in \mathcal{K}_i(\Omega_{\hat{x}})$. ∎

IV. Robust output feedback control design

This section presents the main result of this paper, i.e., a control design procedure based on the convex lifting $\ell_N(\hat{x})$, constructed in the preceding section. This procedure is summarized in Algorithm 1.

Algorithm 1 Control design procedure

Input: A local controller $u = K\hat{x}$ associated with $\Omega_{\hat{x}}$, $\ell_N(\hat{x})$ and $\mathcal{K}_N(\Omega_{\hat{x}})$.

Output: optimal controller $u^*(\hat{x}_k)$ at each instant.

1: Compute $\ell_N(\hat{x}_k)$.

2: **If** $\hat{x}_k \in \Omega_{\hat{x}}$, then $u^*(\hat{x}_k) = K\hat{x}_k$.

3: **Else** Solve the following problem:

$$
\begin{aligned}
\left[\gamma^*\ (u_k^*)^T\right]^T &= \arg\min_{\gamma,\, u_k} \gamma \\
\text{s.t. } & \ell_N(A\hat{x}_k + Bu_k + w) \leq \gamma \ell_N(\hat{x}_k), \\
& (A\hat{x}_k + Bu_k) \oplus (L\mathbb{V} \oplus LC\Omega_e) \subseteq \mathcal{K}_N(\Omega_{\hat{x}}), \\
& u_k \in \mathbb{U},\ \gamma \geq 0,\ \forall w \in \mathcal{V}(L\mathbb{V} \oplus LC\Omega_e).
\end{aligned} \tag{16}
$$

4: $u^*(\hat{x}_k) = u_k^*$.

5: **End**

6: $k \leftarrow k+1$. Return to step 1.

We now need to prove that the controller designed in Algorithm 1 can both guarantee the recursive feasibility and closed-loop stability of system (5), while still satisfying constraints (6). For ease of presentation, we first prove that the above controller makes function $\ell_N(\hat{x}_k)$ strictly decreasing along the closed-loop dynamics outside $\Omega_{\hat{x}}$. This observation is formally stated in the following result.

Proposition 4.1: Consider controller designed in Algorithm 1 and function $\ell_N(\hat{x})$, then for any $\hat{x} \in \mathcal{K}_N(\Omega_{\hat{x}}) \backslash \Omega_{\hat{x}}$, it follows that

$$\ell_N(A\hat{x} + Bu^*(\hat{x}) + w) < \ell_N(\hat{x}) \quad \text{for all} \quad w \in L\mathbb{V} \oplus LC\Omega_e.$$

Proof: First, we will prove that any $v \in \bigcup_{i \in \mathcal{I}_N} \mathcal{V}(\mathcal{K}_i(\Omega_{\hat{x}})) \backslash \Omega_{\hat{x}}$ satisfies $\ell_N(Av + Bu^*(v) + w) < \ell_N(v)$ for all $w \in L\mathbb{V} \oplus LC\Omega_e$. In fact, for ease of presentation, define the following index, with respect to $\hat{x} \in \mathcal{K}_N(\Omega_{\hat{x}})$, $i^*(\hat{x}) = \min\limits_{j \in \mathcal{I}_N \cup \{0\}} j$ s.t. $\hat{x} \in \mathcal{K}_j(\Omega_{\hat{x}})$. Obviously, for an $\hat{x} \in \mathcal{K}_N(\Omega_{\hat{x}})$, the definition of $i^*(\hat{x})$ is unique. According to claim 2) of Lemma 3.5, for any $v \in \bigcup_{i \in \mathcal{I}_N} \mathcal{V}(\mathcal{K}_i(\Omega_{\hat{x}})) \backslash \Omega_{\hat{x}}$, it yields

$$\ell_N(v) = h_{i^*(v)} > 0,\ i^*(v) \geq 1. \tag{17}$$

Also, there exists a control law $u(v) \in \mathbb{U}$ such that $Av + Bu(v) + w \in \mathcal{K}_{i^*(v)-1}(\Omega_{\hat{x}})$, leading to

$$\ell_N(Av + Bu(v) + w) \leq h_{i^*(v)-1},\ \forall w \in L\mathbb{V} \oplus LC\Omega_e, \tag{18}$$

as shown in claim 3) of Lemma 3.5. As a consequence of claim 1) in Lemma 3.5, inclusions (17) and (18) lead to

$$\ell_N(Av + Bu(v) + w) \leq h_{i^*(v)-1} < h_{i^*(v)} = \ell_N(v). \tag{19}$$

Otherwise, if $v \in \mathcal{V}(\Omega_{\hat{x}})$, then there also exists a controller $u(v) = Kv$ such that $Av + Bu(v) + w \in \Omega_{\hat{x}}$ for all $w \in$

$L\mathbb{V} \oplus LC\Omega_e$, because $\Omega_{\hat{x}}$ is a robust positively invariant set. Therefore, claim 4) of Lemma 3.5 and claim 2) of Lemma 3.2 lead to

$$\ell_N(Av + Bu(v) + w) = \ell_N(v) = 0. \quad (20)$$

Consider any point $\hat{x} \in \mathcal{K}_N(\Omega_{\hat{x}}) \backslash \Omega_{\hat{x}}$, there exists a region $\mathcal{X}_j^{(N)}$ in the polytopic partition $\left\{\mathcal{X}_j^{(N)}\right\}_{j \in \mathcal{I}_{M^{(N)}}}$ of $\mathcal{K}_N(\Omega_{\hat{x}})$ such that $\hat{x} \in \mathcal{X}_j^{(N)}$. Accordingly, $\hat{x}$ can be expressed as:

$$\hat{x} = \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) v,\ \alpha(v) \geq 0,\ \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) = 1. \quad (21)$$

This yields

$$\begin{aligned} \ell_N(\hat{x}) &= \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) \ell_N(v) && \text{(22a)} \\ &> \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) \ell_N(Av + Bu(v) + w) && \text{(22b)} \\ &\geq \ell_N(A\hat{x} + B \sum_{v \in \mathcal{V}(\mathcal{X}_j^{(N)})} \alpha(v) u(v) + w) && \text{(22c)} \\ &\geq \ell_N(A\hat{x} + Bu^*(\hat{x}) + w). && \text{(22d)} \end{aligned}$$

Note that equality (22a) holds because $\ell_N(\hat{x})$ is affine over $\mathcal{X}_j^{(N)}$. Since $\hat{x} \in \mathcal{K}_N(\Omega_{\hat{x}}) \backslash \Omega_{\hat{x}}$, there exists at least one vertex $v \in \mathcal{V}(\mathcal{X}_j^{(N)}) \backslash \Omega_{\hat{x}}$ such that $\alpha(v) > 0$. Accordingly, inclusions (19) and (20) lead to the strict inequality (22b). Moreover, inclusion (22c) holds due to the convexity of $\ell_N(\hat{x})$. Finally, inequality (22d) is obtained because of $u(v) \in \mathbb{U}$ for all $v \in \mathcal{V}(\mathcal{X}_j^{(N)})$ and the convexity of $\mathbb{U}$. The proof is complete. ■

Proposition 4.2: Consider controller designed in Algorithm 1 and function $\ell_N(\hat{x})$, then for any $\hat{x} \in \Omega_{\hat{x}}$, it yields

$$\ell_N(A\hat{x} + Bu^*(\hat{x}) + w) = \ell_N(\hat{x}) = 0,\ \forall w \in L\mathbb{V} \oplus LC\Omega_e.$$

Proof: Due to the robust positive invariance of $\Omega_{\hat{x}}$, for any $\hat{x} \in \Omega_{\hat{x}}$, it yields for all $w \in L\mathbb{V} \oplus LC\Omega_e$,

$$A\hat{x} + Bu^*(\hat{x}) + w = (A + BK)\hat{x} + w \in \Omega_{\hat{x}}. \quad (23)$$

Accordingly, claim 2) of Lemma 3.2 leads to

$$\ell_N(A\hat{x} + Bu^*(\hat{x}) + w) = 0,\ \forall w \in L\mathbb{V} \oplus LC\Omega_e. \quad (24)$$

The proof is complete. ■

The main results of the paper are presented in the sequel.

Theorem 4.3: Consider system (5) subject to constraints (6). Then, for any initial point $\hat{x}_0 \in \mathcal{K}_N(\Omega_{\hat{x}})$, controller designed in Algorithm 1 guarantees the recursive feasibility and robust stability.

Proof: The recursive feasibility is guaranteed by suitable $\gamma < 1$ according to Propositions 4.1 and 4.2. Moreover, Proposition 4.1 shows that $\ell_N(\hat{x}_k)$ is strictly decreasing along the closed-loop dynamics outside $\Omega_{\hat{x}}$, it is also bounded in $[0, h_N]$, therefore $\{\ell_N(\hat{x}_k)\}_{k=0}^{\infty}$ is convergent to 0. As a consequence, $\hat{x}_k$ converges to $\Omega_{\hat{x}}$ as time tends to infinity, leading to robust stability of system (5). ■

Theorem 4.4: Consider system (1) subject to constraints (2). If the initial points $e_0 \in \Omega_e$ and $\hat{x}_0 \in \mathcal{K}_N(\Omega_{\hat{x}})$, then controller designed in Algorithm 1 also guarantees the recursive feasibility and robust stability.

Proof: As Ω_e is a robust positively invariant set of system (4), then $e_k \in \Omega_e$, when $e_0 \in \Omega_e$. Also, as shown in Theorem 4.3, controller designed in Algorithm 1 also guarantees the recursive feasibility for system (5), therefore $\hat{x}_k \in \mathbb{X} \ominus \Omega_e$. As a consequence, $x_k = e_k + \hat{x}_k \in \mathbb{X}$. Moreover, this controller satisfies the constraints in (2), leading to the recursive feasibility for system (1).

As for robust stability of system (1) and controller designed in Algorithm 1, it can be observed that $\hat{x}_k$ converges to $\Omega_{\hat{x}}$, while e_k is bounded in Ω_e, therefore $x_k = e_k + \hat{x}_k$ is convergent to $\Omega_{\hat{x}} \oplus \Omega_e$ as time tends to infinity, leading to its robust stability. ■

Recall that the vertex control in [10] and the one in [4] cannot guarantee that the closed-loop dynamics converge to a given robust positively invariant set, i.e., the result in Proposition 4.1. Note also that the construction of $\ell_N(\hat{x})$ becomes more demanding when the dimension of the state space increases, since it requires vertex enumeration. However, as this construction is performed offline, it is reasonable to assume that powerful computational resources are available.

V. NUMERICAL EXAMPLE

In this section, a numerical example is considered to illustrate the proposed robust output feedback control design procedure. To this end, a DC-DC converter model is employed and given below:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 1 & 0.0075 \\ -0.143 & 0.996 \end{bmatrix} x_k + \begin{bmatrix} 4.798 \\ 0.115 \end{bmatrix} u_k + w_k \\ y_k &= \begin{bmatrix} -0.5 & 1 \end{bmatrix} x_k + v_k. \end{aligned} \quad (25)$$

Also, the state, control variable and disturbances satisfy:

$$\begin{gathered} -[10\ \ 10]^T \leq x_k \leq [3\ \ 3]^T,\ \|u_k\|_\infty \leq 0.5, \\ \|w_k\|_\infty \leq 0.01,\ \|v_k\|_\infty \leq 0.005. \end{gathered}$$

We choose the observer gain $L = [-0.5640\ 0.4838]$ and a local controller $u = [-0.2313\ 0.1781]\,\hat{x}$. Accordingly, the minimal robust positively invariant set Ω_e is computed, using the algorithm presented in [25], to enlarge the feasible region for the observer system (3). Also, $\Omega_{\hat{x}}$ is chosen as the maximal output admissible set associated with the local controller $u = [-0.2313\ 0.1781]\,\hat{x}$. The set $\Omega_{\hat{x}} \oplus \Omega_e$ and the $20-$step controllable set $\mathcal{K}_{20}(\Omega_{\hat{x}}) \oplus \Omega_e$ are both represented in Fig. 1. Subsequently, a convex lifting $\ell_{20}(\hat{x})$ is constructed following the construction in (7) and (11) with $h_1 = 10^{-2}$ and $\epsilon = 10^{-3}$. This convex lifting is presented in Fig. 2. Also, it is shown in this figure the evolution of $\ell_{20}(\hat{x}_k)$ along the closed-loop dynamics of system (5). It verifies that the convex lifting $\ell_{20}(\hat{x}_k)$ is strictly decreasing along the dynamics outside $\Omega_{\hat{x}}$, leading to the convergence of $\ell_{20}(\hat{x}_k)$ to 0. Roughly speaking, x_k converges to $\Omega_{\hat{x}} \oplus \Omega_e$ as shown more clearly in Fig. 1. Finally, the numerical example of this paper is simulated in the environment of MPT 3.0 [11].

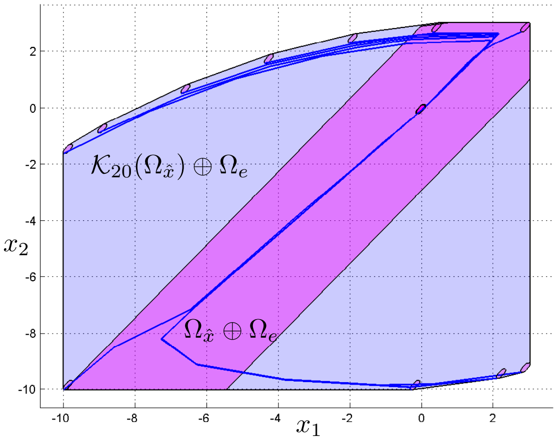

Fig. 1. Maximal output admissible set $\Omega_{\hat{x}} \oplus \Omega_e$, the 20−step controllable set $\mathcal{K}_{20}(\Omega_{\hat{x}}) \oplus \Omega_e$ and the closed-loop dynamics of system (1).

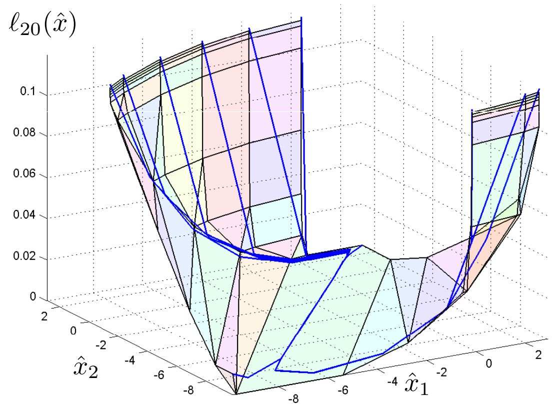

Fig. 2. The constructed convex lifting $\ell_{20}(\hat{x})$ over $\mathcal{K}_{20}(\Omega_{\hat{x}})$ with $h_1 = 10^{-2}$ and $\epsilon = 10^{-3}$ and its strict decrease along the closed-loop dynamics of the observer system (3) outside $\Omega_{\hat{x}}$.

VI. Conclusions

This paper presented a method for robust output feedback control of linear systems affected by both state and output additive disturbances. The method made use of a suitable convex lifting, which was shown to be strictly decreasing along the closed-loop dynamics outside a given robust positively invariant set. Accordingly, the closed-loop dynamics were shown to be convergent to this invariant set, leading to robust stability. The method was shown to only require solving a linear program at each sampling instant. Finally, a numerical example was considered to illustrate the method.

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