Time Intervals and Counting in Point Processes

Bernard Picinbono

Time Intervals and Counting in Point Processes

Bernard Picinbono, *Fellow, IEEE*

Reference of this text : IEEE Trans. Inf. Theory, Vol. 50, N 6 pp. 1336-1340, June 2004.

Abstract

Time point processes can be analyzed in two different ways: by the number of points in arbitrary time intervals or by distance between points. This corresponds to two distinct physical devices: counting or timeinterval measurements. We present an explicit calculation,v alid for arbitrary regular processes,of the statistical properties of time intervals such as residual or life time in terms of counting probabilities. For this calculation, we show that these intervals must be considered as random variables defined by conditional distributions.

Index Terms

Counting,point processes,time measurements.

I. Introduction

Point processes play an important role in many areas of physics and information sciences. They appear on a microscopic scale in the description of particle emission, and, for example, optical communication

Manuscript received

B. P. is with the Laboratoire des Signaux et Systèmes (L2S), a joint laboratory of the C.N.R.S. and the École Supérieure d'Électricité, Plateau de Moulon, 3 rue Joliot–Curie 91192, Gif sur Yvette, France. The L2S is associated with the University of Paris–Sud, France. E-mail: bernard.picinbono@lss.supelec.fr, .

at a very lowlevel of intensity requires the use of statistical properties of photons or photoelectrons [1], [2]. On the other hand, at a macroscopic level many areas such as traffic problems or computer communications require the use of point process statistics [3].

There are two approaches to describe point processes theoretically or to study them experimentally. The first one makes use of counting procedures in one or several nonoverlapping time intervals. The appropriate physical devices for this approach are counters. A limit aspect of counting appears in coincidence experiments in which the time intervals of counting are so small that they can only contain one or zero point [4].

On the otherhand, it is possible to analyze point processes by using time intervals between points measurements. This introduces the concept of residual time, orsur vival time, orw aiting time of order n, which is the time distance between an arbitrary time instant and the nth point of the processes following this instant. It is also possible to study the life time which is the time distance between successive ornonsuccessive points of the processes.

In the stationary case, the calculation of the probability distributions of residual or life times in terms of counting probabilities is known [5], [6]. However, in many practical situations, the stationarity assumption cannot be introduced and it appears that the direct transposition of the results obtained in the stationary case is not possible. The main reason is that time intervals must be considered as random variables (RV) defined by conditonal distributions.We shall see that this remark is of no importance in the stationary case but it must be taken into account for nonstationary processes. The omission of this fact has resulted in many incorrect expressions appearing in classical books on point processes. This is one of the reasons for analyzing the problem again and more carefully.

Before going further, let us introduce some general concepts and notation that will be used throughout the correspondence. As indicated in the title, we are interested in time point processes, which means that the points are time instants.

We assume that the point processes studied are defined only in a time interval $(T_i; T)$, where T_i and T are the beginning and the end of the processes, respectively. For the sake of simplicity we take T_i as the origin of time, or $Ti = 0$.

We denote by $N[t_1, t_2)$ the number of points in the interval $[t_1, t_2)$. It is a discrete-valued RV and the point process is entirely defined if for any set of nonoverlapping intervals$[t_i, t_i + \Delta t_i)$ the joint probability distribution of the RVs $\{N[t_i, T_i + \Delta t_i]\}$ is known. These probabilities are denoted *counting probabilities*, and we shall use the notations

$$p_i(t, \tau) \overset{\triangle}{=} P\{N(t, \tau) = i\}. \tag{1}$$

II. Residual time of Order n

A. General Results

Let t be an arbitrary time instant satisfying $0 = T_i \leq t \leq T$. The residual time of order n is the RV $R_n(t)$ equal to the distance between the origin T_i and the nth point of the process posterior to t. It is fundamental to note that this RV does not exist if there are less than n points posterior to t, or if the event $N[t;T) < n$ is realized. Consequently, the distribution function (DF) of $R_n(t)$ defined by $F_h(t,\tau) = P[R_n(t) \leq \tau]$, $t \leq \tau \leq T$ is the conditional probability

$$F_n(t,\tau) = P(\{N[t,\tau) \geq n\}|\{N[t,T) \geq n\}) = \frac{P(\{N[t,\tau) \geq n\}.\{N[t,T) \geq n\})}{P(N[t,T) \geq n)} \qquad (2)$$

As $\tau < T$, the numerator is equal to

$$P(N[t,\tau) \geq n) = 1 - \sum_{i=0}^{n-1} p_i(t,\tau) \qquad (3)$$

The denominator has the same structure but $p_n(t,\tau)$ is replaced by $\pi_i = p_i(t,T)$. This yields

$$F_n(t,\tau) = \frac{1}{1 - \sum_{i=0}^{n-1} \pi_i} \left[1 - \sum_{i=0}^{n-1} p_i(t,T)\right] . \qquad (4)$$

It results directly from (2) that $F_n(t,\tau)$ is effectively a DF, which means a nondecreasing function varying from 0 to 1 when τ varies from t to T.

The probability density function (pdf) $f_n(t,\tau)$ is the derivative of this expression with respect to τ, when it does exist, or

$$f_n(t,\tau) = \frac{-1}{1 - \sum_{i=0}^{n-1} \pi_i} \sum_{i=0}^{n-1} \frac{\partial p_i(t,\tau)}{\partial \tau} . \qquad (5)$$

It is clear that (4) or (5) establish a relation between counting probabilities and statistics of the residual time, which is the objective of this correspondence.

B. Stationary Case

In this case, $p_i(t,\tau) = p_i(t-\tau))$ and T tends to infinity. As a result, $p_i = 0$ and (4) becomes

$$F_n(t,\tau) = 1 - \sum_{i=0}^{n-1} p_i(\tau - t) ,$$

where $p_i(\tau - t)$ is the probability of counting i points between t and τ. This is the classical expression for stationary point processes.

This shows clearly the difference between the stationary and nonstationary cases. When the point process is stationary, it is not necessary to consider a conditional distribution because the event introducing the condition is realized with probability 1. Indeed, except when the process has a zero density, there is always an infinite number of points posterior to any time instant t.

C. Poisson Processes

Consider a nonstationary Poisson process defined by a density $\lambda(t)$ equal to zero if t is not in the interval $[T_i, T)$. For the following calculations, it is worth introducing the quantity $d_n(m)$ associated to any Poisson distribution of mean value m and defined by

$$d_n(m) \triangleq \sum_{i=0}^{n-1} p_i = \exp(-m)\left[1 + m + ... + \frac{m^{n-1}}{(n-1)!}\right] \tag{6}$$

This is obviously the probability that a Poisson RV of mean m takes a value smaller than n. With this notation (4) can be written as

$$F_n(t,\tau) = \frac{1 - d_n(m)}{1 - d_n(M)}$$

where

$$m = m(t,\tau) = \int_t^\tau \lambda(\theta)d\theta, \quad M = M(t,\tau) = \int_t^T \lambda(\theta)d\theta \tag{7}$$

In order to calculate the pdf, we note that the derivative of $d_n(m)$ defined by (6) with respect to m is $-m^{n-1}\exp(-m)/(n-1)!$. Consequently, the pdf of the residual time is

$$f_n(t,\tau) = \frac{1}{1 - d_n(M)}\ \mathrm{e}^{m(t,\tau)}\frac{m^{n-1}(t,\tau)}{(n-1)!} =\ \lambda(\tau),\ 0 \leq \tau \leq T. \tag{8}$$

This expression can be obtained directly from the properties of a Poisson process. Note that the factor $[1 - d_n(M)]^{-1}$ in (8) ensures that $f_n(t,\tau)$ is indeed a pdf. This factoris usually forgotten, which introduces a function that is not a pdf. This, for example, is the case for [7, eq. (3.3)], [8, eq. (2.20)], [9, eq. (2.3.6)], or [10, eq. (8.8)]. The reason for these expressions is that the calculations do not take into account the fact that the residual time is an RV defined by a conditional DF.

In order to visualize the effect of the nonstationarity, consider the example of a Poisson process with a density equal to λ in the interval $[0, T)$ and zero outside. The value of the pdf of the residual time $R_n(0) = R_n$ is

$$f_n(\tau) = \frac{1}{1 - d_n(M)}\ \frac{(\lambda\tau)^{n-1}}{(n-1)!}\ \lambda\mathrm{e}^{-\lambda\tau}\ ,\quad 0 \leq \tau \leq T \tag{9}$$

and zero outside this interval. It is easy to calculate the mean value of this RV which is

$$\mathrm{E}(R_n) = \frac{n}{\lambda}\ \frac{1 - d_{n+1}(M)}{1 - d_n(M)} = \frac{n}{\lambda}\ c_n(M), \tag{10}$$

where d_n and M are defined just above. When $T \to \infty$, $c_n(M) \to 1$ and $\mathrm{E}(R_n)$ tends to the value n/λ. This is in agreement with the fact that in a stationary Poisson process of density λ the distances between successive points are independent and identically distributed (i.i.d.) random variables of mean value $1/\lambda$. Thus, the term $c_n(M)$ can be considered as a correction factor due to the nonstationary character of the process. This correction factor is a function of M and it is clear that the normalization factor $1 - d_n(M)$ in the pdf is fundamental to obtain the correct mean value.

The difference between stationary and nonstationary Poisson processes with constant density is especially important when the density becomes very small. In the stationary case, the mean value $1/\lambda$ tends to infinity when $\lambda \to 0$. This is of course impossible in our particular case of Poisson process with a density equal to zero outside the interval $[0, T)$. For $\lambda \to 0$, the mean value $\mathrm{E}(R_n)$ defined by (10) tends to $T[n/n+1]$. For example, for $n = 1$ this gives $T = 2$, and this can also be obtained from the pdf $f_1(\tau)$ which tends to $1/T$ and shows that the RV R_1 is uniformly distributed in the interval $|0, T)$.

D. Compound Poisson Processes

Compound Poisson processes, sometimes called doubly stochastic processes [8], are Poisson point processes in which the density $\lambda(t)$ is a random function [10]. They play an important role in many areas of physics orinfor mation sciences and especially in optical communications. Indeed, it can be shown by various arguments that they describe the point process of the detection of photons and that the random density is proportional to the random intensity of the optical field [11].

Forthese processes, the calculation of p_i and π_i require an ensemble average over $\lambda(t)$. Then all the previous calculations can be used again with the only difference that $d_n(m)$ and $d_n(M)$ are replaced by their expectation values with respect to $\lambda(t)$. The pdf of the residual time thus becomes

$$f_n(t,\tau) = \frac{1}{1-\mathrm{E}[d_n(M)]}$$

$$\times \mathrm{E}\left\{ \frac{[\int_t^\tau \lambda(\theta)d\theta]^{n-1}}{(n-1)!}\lambda(\tau)\exp\left[-\int_t^\tau \lambda(\theta)d\theta\right]\right\}, \qquad (11)$$

In the case where $\lambda(t)$ is stationary, $\mathrm{E}[d_n(M)] = 0$, and we find once again a known classical expression (see [10, p. 348]).

III. Multiple time Distances

Instead of studying a single time distance it is possible to jointly analyze two or several such distances. In order to simplify the presentation, we will restrict ourselves to the analysis of the case of two distances of order one, the extensions to other cases introducing only notational complexity but no conceptual difficulties.

Considerthe two RVs $R_1(t)$ and $R_2((t)$ as defined at the beginning of Section II. By construction, they satisfy the condition $R_1(t) < R_2((t)$. Let $f(t, \tau_1, \tau_2)$ be theirjoint pdf. A calculation transposing to this case the method used above yields

$$f(t,\tau_1,\tau_2) = \frac{-1}{1-\pi_0-\pi 1}\,\frac{\partial^2 p_{10}(t,\tau_1,\tau_2)}{\partial\tau_1\partial\tau_2} \qquad (12)$$

for $t \leq \tau_1 \leq \tau_2 \leq T$ and zero otherwise. In this expression, $p_{10}(t, \tau_1, \tau_2)$ is the probability that the intervals $[t, \tau_1)$ and $[\tau_1, \tau_2 =$ contain 1 and 0 points, respectively.

In the case of a nonstationary Poisson process, this pdf is

$$f(t, \tau_1, \tau_2) = \frac{1}{1 - d_2(M)} \lambda(\tau_1)\lambda(\tau_2) \exp[-m(t, \tau_2)], \tag{13}$$

where $d_2(M)$ is defined by (6) and (7). It is easy to verify that this pdf is normalized in its domain of definition $t \leq \tau_1 \leq \tau_2 \leq T$.

IV. Conclusion

The purpose of this correspondence was to calculate the statistical properties of the distances between points of a point process in terms of the statistics of counting in some time intervals. This problem was solved a long time ago for stationary processes, but the extension of the results for nonstationary processes has so far been presented in an incorrect way.

We have shown that this extension requires the consideration of time distances between points as conditional random variables and the errors appearing in literature stemmed from overlooking this point. The consequences of this were analyzed in this correspondence for the residual time and some numerical examples on pure and compound Poisson processes illustrate the calculations.

The analysis was extended forthe life time of any regularpoint process. It was shown that the pdf of the life time can be deduced in terms of counting probabilities in one single interval.

Finally, the procedure was extended for multiple time distances and the expression of the statistics of the distance was given in terms counting probabilities in adjacent intervals.

References

[1] J. R. Pierce, E. C. Posner, and E. R. Rodemich, The capacity of photon counting channel, *IEEE Trans. Inform. Theory,* vol. IT 27, , pp. 61-73, Jan. 1981..

[2] B. Saleh and M. Teich, *Fundamentals of Photonics..* New York:Wiley, 1991.

[3] E. Gelenbe and G. Pujolle, *Introduction to Queuing Networks.* New York: Wiley, 1987.

[4] O. Macchi, The coincidence approach to stochastic point proccesses, *Adv. Appl. Probab.,* vol. 7, pp. 83-122, 1975.

[5] J. A. McFadden, The axis crossing intervals of random functions, *IEEE Trans. Inform. Theory,* vol. IT-4, pp. 14-24, Mar. 1958.

[6] J. A. McFadden,On the lengths of intervals in a stationary point process, *J. Roy. Statist. Soc.,* vol. B 24, pp. 364382, 1962.

[7] D. R. Cox and V. Isham, *Point Processes.* London, U.K.: Chapman and Hall, 1980.

[8] D. L. Snyderand M. I. Miller, *Random Point Processes in Time and Space.* New York: Springer-Verlag, 1991.

[9] E. P. Kao, *An Introduction to Stochastic Processes.* Boston, MA: Duxbury, 1997.

[10] B. Picinbono, *Random Signals and Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1993.

[11] J. Klauderand E. Sudarshan, *Fundamentals od Quantum Optics.* New York: Benjamin, 1968.

[12] H. Cramer and M. R. Leadbetter, *Stationary and Related Stochastic Processes.* New York: Wiley, 1967.