Singularity in signal theory

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Abstract

A discrete-time random signal is singular if its values are singular random variables defined by a distribution function continuous but with a derivative equal to zero almost everywhere. Singular random signals can be obtained at the output of some linear filters when the input is a discrete-valued white noise. Sufficient conditions for singularity are established. In particular it is shown that if the poles of the filter are inside a circle called the circle of singularity and if the input is white and discrete-valued the output is singular. Computer experiments using histograms at different scales exhibit the structure of singular signals. The influence of input correlation is also analysed. It is shown that when the input is not white, but has a specific Markovian structure, the output can be singular. This is also verified by computer experiments. Finally, mixtures of singular and discrete-valued random signals are analysed.

Index Terms

Statistical signal analysis, Signal and noise modeling, Non-Gaussian signals and noise, Markov processes., Cantor sets, spectral measure.

I. INTRODUCTION

Singularity is a concept introduced in probability theory and related to properties of distribution functions of random variables. Usually only two kinds of random variables are considered: those that are continuous and having a distribution function (DF) with a derivative (probability density function) and those that are discrete with a DF varying by steps defining possible values and their corresponding probabilities. However there is a third kind of random variables: those that are singular.

A random variable (RV) is said to be singular if its DF is continuous but with a derivative equal to zero almost everywhere. Then such a RV is neither continuous (no probability density function) nor discrete because its DF is continuous and not a stepwise function.

Singular RVs are often considered as mathematical curiosities without interest in signal theory and engineering sciences. They are rarely introduced in standard textbooks (Papoulis 1984, Ochi 1990, Pfeifer 1990, Helstrom 1991, Picinbono 1993) and appear only in mathematically oriented books (Loève 1977, Wong and Hajek 1985). Very simple RVs however can be singular. The best example comes from an old result known for more than sixty years. It says that if w_k is a set of independent and identically distributed (IID) RVs taking the values ± 1 with the same probability, then the sum of the series $\sum_{k=0}^{\infty} a^k w_k$ is singular as soon as $|a| < 1/2$. So one of the simplest RV that can be considered is singular. This result is often omitted because its usual proof requires very abstract reasoning. One of the first tasks of this paper is to show its origin and to introduce an elementary proof that can afterwards be used in signal theory. Indeed the series considered above is similar to the output of an exponential discrete-time filter whose input is a discrete valued white noise. This is the case of autoregressive signals widely used in signal processing. Then the question of knowing whether or not this can be extended to other class of signals and systems appears immediately.

The main result of this paper is that singularity of the output of a linear filter depends on two facts: the discrete character of the input, which is a common situation in communication theory, and some specific properties of the filter such as the location of its poles in a circle called the circle of singularity. In order to provide a better understanding of a problem usually widely ignored, and to visualize how singularity can appear for rather simple signals, number of computers experiments are presented.

In the last part of the paper it is also shown that the assumption of whiteness, fundamental in the proof of the basic result, can be partially deleted and we present some examples of coloured input signals leading also to singular outputs. This is especially the case of some Markov processes. Finally mixtures of discrete and and singular random signals can be obtained depending on the properties of the correlation function of the input, and theoretical and experimental examples of such signals are presented.

II. Statement of the problem and review of known results

Let $X(\omega)$ be a RV defined on some probability space. Let $F(x)$ be its distribution function (DF). In all of what follows we do not make any difference between two RVs distinct but equal with probability 1. This means that $X(\omega)$ is entirely defined by its DF $F(x)$. The RV $X(\omega)$ is said to be singular if its DF $F(x)$ is continuous but with a derivative equal to zero almost everywhere.

At the beginning let us remind some fundamental results of probability theory used in the discussion that follows and especially the Lebesgue decomposition theorem (Lukacs 1970). For this let us recall that a RV is said to be continuous if its DF is continuous and has a derivative which is its probability density function (PDF). On the other hand a RV is said to be discrete if its DF varies only by steps of amplitudes p_i at points x_i. This means that the RV takes only the values xi with the probabilities p_i. The Lebesgue theorem says that any DF $F(x)$ can be decomposed in an unique way as a sum of three terms, or that

$$F(x) = a_1 F_c(x) + a_2 F_d(x) + a_3 F_s(x), \;\; a_i \geq 0, \;\; a_1 + a_2 + a_3 = 1. \tag{1}$$

In this equation the three functions $F_i(x)$ are DFs and $Fc(x)$, $Fd(x)$, and $Fs(x)$ are the continuous, discrete and singular components of $F(x)$ respectively. If $a_1 = 1$, the RV is continuous and its PDF is the derivative of $F_c(x)$. If $a_2 = 1$, X is a discrete RV. If, finally, if $a_3 = 1$, the RV X is singular. If one of the coefficients a_i is equal to 1, the DF is said to be pure. The spectrum S_F of a RV is the set of the points of increase of its DF $F(x)$ and the spectral measure (SM) is the Lebesgue measure $L(S_F)$ of this set. It is clear that if the SM is zero the continuous part in (1) is zero, or $a_1 = 0$. Then the RV can be discrete $(a_3 = 0)$, singular $(a_2 = 0)$, or a mixture of a discrete and a singular parts. Then to show that a RV is singular it suffices to show that its SM is zero and that there is no discrete part in the decomposition of its DF.

The fundamental theorem opening all this discussion is the following. Consider a set of RVs w_k independent and identically distributed (IID) with a symmetric Bernoulli distribution. This means that the $w_k's$ take only two values with the same probabilities 1/2. When this is not otherwise explicitly indicated the two possible values of each RV w_k are ± 1. Consider the RV X defined by the series

$$X = \sum_{k=0}^{\infty} a^k w_k, \tag{2}$$

which is convergent provided that $|a| < 1$. Note that the possible values of w_k imply that X is symmetric, which means that X and $-X$ have the same DF. Furthermore, for the same reasons, changing a in $-a$ does not change the DF of X. Then we can assume that $a \geq 0$. The following results holds:

(1) if $a < 1/2$, X is singular;

(2) if $a = 1/2$, X is uniformly distributed in $[-1/2, +1/2]$;

(3) if $1/2 < a = 1$, X is in general continuous, but can also be singular for values of a belonging to a set of zero measure.

Point 2 can be shown directly by a simple calculation. Point 3 is without interest for this paper and the complete set of points of singularity is still a subject of research. Point 1 was shown long time ago in the framework of infinite products of convolutions (Kershner and Wintner 1935) and discussed more recently (Peres and Solomyak 1998, Solomyak et al. 2000). Because of its importance for all what follows, we shall now present a direct proof.

For this purpose consider the partial sum X_N and the rest R_N defined by

$$X_N = \sum_{k=0}^{N} a^k w_k \; ; \; R_N = \sum_{k=N+1}^{\infty} a^k w_k = a^{N+1} \sum_{k=0}^{\infty} a^k w_{N+k+1} \tag{3}$$

The RV X_N takes 2^{N+1} distinct values v_i^N with the same probability $1/2^{N+1}$. The rest R_N satisfies $|R_N| \leq a^{N+1}/(1-a)$, this limit being obtained if all the w_{N+k+1} are equal to $+1$ or -1. As a consequence the SM of RN is smaller than $a^{N+1}[2/(1-a)]$. Since there are 2^{N+1} distinct values v_i^N of XN, the SM of X is smaller than $(2a)^{N+1}[2/'1-a)])$. As this is valid for all N, this SM is zero whenever $a < 1/2$.

Note that this property of the SM is due to the value of a and to the fact that the input w_k has only two possible values. On the other hand the probabilities of these outcomes do not play any role. We shall see later that this is general.

This result means that there is no continuous part in the decomposition of the DF of X, or that $a_1 = 0$ in (1). Let us see now that there is also no discrete component, or $a_2 = 0$. Indeed suppose that this is not the case. This would mean that there is a value x_0 such that the probability that $X = x_0$ is positive. But as x_0 is a value of X, there exists a set of numbers η_k taking only the values ± 1 such that $x_0 = \sum_{k=0}^{\infty} a^k \eta_k$. Furthermore, as the values v_i^N are distinct, this set is unique. Since the RVs w_k are IID and $\mathrm{P}[w_k = \eta_k] = 1/2$, we deduce

$$\mathrm{P}[X = x_0] = \prod_{k=0}^{\infty} \mathrm{P}[w_k = \eta_k] = 0. \tag{4}$$

As a consequence the RV X is singular, or $a_3 = 1$. On the contrary to the property of the SM, it is clear that this proof uses the fact that the w_ks are independent and that their two values have the same probability $1/2$. It is simple to see however that the result remains valid if one of the values has the probability p, except when $p = 0$ or $p = 1$, which corresponds to a situation where the input signal w_k is no longer random.

For the discussion that follows it is important to understand that the singularity comes from two completely distinct properties. The first one depends only on a and on the fact that w_k takes only two values ± 1. But it is insufficient to imply singularity and we shall see later that for some particular probability distributions of the w_k the sum X can be discrete or a mixture of discrete and singular parts. The second properties arises from the whiteness of w_k. It is the combination of these two properties which ensures that X is singular.

Let now present an interpretation of X_N by its tree of construction presented in figure 1. For each value v_i^N of X_N we can associate the value $-v_i^N$. It is obtained simply by changing the signs of the w_ks appearing in (3). This means, as noted above, that the the RV X is symmetric. As a consequence we can consider only positive v_i^Ns For the same reason it is always possible to assume that $a > 0$. With these assumptions we have $v_0^0 = 1$. The two positive values of v_1^1 are $1 - a$ and $1 + a$. The construction of the 8 positive values of X_3 appears in the tree of figure 1. Let us now see that the assumption $a < 1/2$ means that there is no crossing of the branches of the tree. Indeed consider the two branches of the tree starting from a point v_i^N. There is no crossing between all the branches starting from this point if $v_i^N - a^{N+1} + a^{N+2}/(1-a) < v_i^N + a^{N+1} a^{N+2}/(1-a)$. This yields $a < 1/2$. It is clear that the construction of this tree is similar to the one of Cantor sets. This is why it is sometimes said that the RV X has a Cantor-type distribution (Wittke et al. 1988).

Singularity is not limited to random geometric series like (2) but can appear with RVs such as

$$X = \sum_{k=0}^{\infty} h_k w_k, \tag{5}$$

where $h_k > 0$ and the w_ks have the same properties as in (2). In this case X_N and R_N of (3) are written simply by replacing a_k by k_k. The last equality of (3) does not hold.

It is shown without complete proof in p. 66 of (Lukacs 1970) that if

$$\rho = \sum_{k=n+1}^{\infty} h_k < h_n, \ \forall n, \tag{6}$$

then X is singular. In the case where $k_k = a^k$, this yields $a < 1/2$. It is possible to construct a tree as in figure 1 with the h_ks instead of the a^ks. One can then see that the condition (6) implies that there is no crossing between the branches of the tree, which introduces again a Cantor structure (Picinbono and Tourneret 2005).

Before leaving this section let us present a short review of some papers from the engineering literature where the problem of singularity is discussed. The treatment of sequences of Bernoulli RVs appear frequently in the context of digital communications. The first discussion concerning consequences of singularity was presented in (Hill and Blanco 1973). The discussion was limited to geometric series like

(2) and the purpose was to obtain an approximation of the DF for the calculation of the error probability or the performance of communication systems. Extensions of the same problem to Cantor-type distribution was presented in (Wittke et al. 1988). In this paper the condition (6) is explicitly used and various examples of filters satisfying this condition are introduced. However, as indicated by the authors, the condition $h_k > 0$ used in (5) and introducing the Cantor structure is very restrictive and it is not satisfied by a large class of filters containing for example terms like $a^k \cos(\omega k)$. The principle of calculation of the distribution function and of some expectations is then presented and used for the evaluation of the error probability. A rather more theoretical approach of the same problem is presented in (Smith et al. 1993). Finally other calculations of expectations of singular RVs are discussed in (Campbell et al. 1995). In this paper the singularity is introduced from some properties of the entropy of the RVs by using an approach introduced in (Garsia 1962). Similar discussions appear in (Naraghi-Pour et al. 1990, Tourneret et al. 1994).

The first purpose of the present paper is to show that singularity can be introduced from considerations of properties of the poles of a linear filter. Furthermore in all these papers it is assumed that the RVs wk are independent, and this assumption is a corner stone for the introduction of singularity. Then it arises immediately the question of knowing whether or not it can be relaxed without suppressing singularity. This question is discussed in the second part of the paper.

III. Singularity and linear filtering

The previous discussion has an immediate application in the case of some signals obtained by linear filtering and especially auto-regressive signals. A signal x_k is said to be auto-regressive of order 1 [AR(1)] if it is deduced- from a white noise w_k by the linear filter defined in the time domain by the first order recursion $x_k = ax_{k-1} + w_k$. It is then defined by the parameter a called the regression coefficient and by the DF of w_k. This correspondoing input-output relationship is

$$x_n = \sum_{k=0}^{\infty} a^k w_{n-k} \tag{7}$$

It is then a discrete time causal filter with the impulse response a^k. If the input signal is a white symmetric Bernoulli signal, it results from the previous discussion that the RVs x_k are singular as soon that $|a| < 1/2$, and we say that the signal x_k is singular. Indeed in order to come at (2) it suffices to introduce $\hat{w}_k = w_{n-k}$ and it is clear that the RVs $\hat{w}_k$ are still IID Bernoulli. The simplicity of this signal explains why it was said previously that singularity is a common phenomenon.

But these results can be extended to a large of other signals. This can be done either by changing the filter that yields x_k from w_k or by changing the statistical properties of the input.

Let us first present some preliminary considerations. We restrict our discussion to dynamical discrete time filters. These filters are characterized by the facts that they are causal and that their transfer function $H(z)$ is a rational function of z. Such filters are defined by their poles and their zeros and almost all the filters used in signal processing are dynamical. We consider further only IIF filters, because RIF filters cannot introduce singularity. Indeed this property, as seen previously, is due to a series and the input-output relationships in RIF filters is a simple sum. This means that we exclude from our analysis transfer functions with only one pole at the origin.

We consider also white discrete-valued inputs signals w_k, or sequences of IID random variables taking only a finite number q of possible values. Let finally introduce a circle called circle of singularity with center O and with the radius equal to $1/q$. This allows to introduce the following theorem.

Theorem 1: *Let x_k be the output of a dynamical non-RIF filter generated by the input w_k. If w_k is a white signal taking only q values and if the poles of the filter lie inside the circle of singularity, then the output x_k is singular.*

Proof: As previously it is made in two steps: 1. Proof that the SM is zero, 2. Proof that there is no discrete component in the DF.

Let $\mathcal{F}$ be a dynamical filter defined by its transfer function $H(z)$ or its impulse response h_k. Let $\mathcal{F}'$ be the filter with the impulse response $g_k = q^k h_k$. It is obvious that its transfer function is $G(z) = h(z/q)$. This implies that if the poles of $\mathcal{F}$ are z_i, those of $\mathcal{F}'$ are qz_i. The assumption that the poles of $\mathcal{F}$ are inside the circle of singularity implies that the poles of $\mathcal{F}'$ are inside the unit circle, or that $\mathcal{F}'$ is a dynamical filter. As such a filter is stable we deduce that $\sum_{k=0}^{\infty} |g_k| < \infty$. The signal x_p is defined by $x_p = \sum_{k=0}^{\infty} h_k w_{p-k}$ and by introducing $x_p = X$ and $\hat{w}_k = w_{p-k}$ we have $X = \sum_{k=0}^{\infty} h_k \hat{w}_k$, where the $\hat{w}_k$s have the same properties as the w_ks or are IID and take only q values. As indicated above the finite sum X_N and the rest R_N are defined in (3) where a^k is replaced by h_k and w_k by $\hat{w}_k$. Let A be the greatest possible value of w_k. We have then $|R_N| < A\rho_N$ with $\rho_N = \sum_{k=N+1}^{\infty} |h_k|$. As X_N can take only q^N values, the SM S of X satisfies $S \leq 2Aq^N, \rho_N$. This is valid for all N. Then $S \leq 2A \lim_{N\to\infty}(q^N, \rho_N)$. But we have

$$q^N \rho_N = q^N \sum_{k=N+1}^{\infty} |h_k| < \sum_{k=N+1}^{\infty} q^k |h_k| = \sum_{k=N+1}^{\infty} |g_k|, \tag{8}$$

and the limit is 0 because the filter $\mathcal{F}$' is stable or $\sum_{k=0}^{\infty} |g_k| <^{\infty}$. This implies that $S = 0$, or $a_1 = 0$. The proof that $a_2 = 0$ is exactly the same as previously and comes only from the whiteness of the w_ks or from their independence. Another proof of the fact that there is non discrete component is given by Lukacs (1970, p. 65).

It is important to note that this theorem yields only a sufficient condition of singularity. However the condition that the poles lie in the singularity circle is not at all necessary. Examples of this situation can easily be found.

Note is that the possible values of w_k do not play any role in the result. However for a given set of values it is sometimes possible to extend the domain of singularity by more abstract methods (Peres and Solomyak 1998). But if we impose the singularity for all the possible sets of values, we come back to the conditions of Theorem 1.

The final point is that it is in general no longer possible to interpret the result by a tree of construction. This is especially the case when the dynamical filter has complex poles as for example when the impulse response is of the kind $a^k \cos(k\phi)$.

IV. The influence of the correlation

The assumption of independence, or of whiteness of the input, plays a fundamental role in the previous results. It is introduced in all the papers indicated in the list of references. This assumption allows us to show that, even if the SM is zero, there is no discrete component in the DF, of that $a_2 = 0$. Thus appears immediately the question of knowing whether it is still possible to meet singularity in the case of colored inputs.

Consider a filter $\mathcal{F}$ defined by its impulse response hk and satisfying the conditions of Theorem 1. The problem of singularity of the output depends on the properties of the RV X given by (5) where the condition $h_k > 0$ is relaxed. We assume that the w_ks are symmetric Bernoulli, but not necessarily independent. The maximum value of X is $X_m = \sum_{k=0}^{\infty} |h_k|$ which is finite because $\mathcal{F}$ is stable. Let X_N be the partial sum analogue to (3) and defined by

$$X_N = \sum_{k=0}^{N} h_k w_k. \tag{9}$$

It takes at the maximum 2^{N+1} distinct values v_i^N and we assume that this maximum is reached. This assumption of distinct values is obviously satisfied when there is no crossing of the branches of the tree constructed with the h_k. This appears with filters satisfying (6), or for the Cantor-type structure. It is clear that this assumption depends only of the impulse response h_k of the filter. It is however satisfied by a large class of filters which is not discussed here. This means that for any N there is no pair (i, j), $i \neq j$ such that $v_i^N = v_j^N$. Because of the symmetry of the w_ks, the RVs X and X_N are also symmetric and this implies that $v_i^N \neq 0$. Indeed the symmetry and the existence of a zero value would imply that the number of distinct values is odd, which is not the case.

To each value v_i^N we associate a node V_i^N in the tree of construction similar to the one appearing in figure 1. The assumption of distinct values v_i^N means that the nodes of the tree are single, which means that each node V_i^N is reached by only one path coming from only one node V_j^{N-1} at the step $N-1$. This is obviously satisfied when there is no crossing of the branches of the tree constructed with the h_ns. This especially appears if condition (6) is satisfied. However, since h_k is not necessarily positive as in figure 1, it is not possible to restrict this tree to the nodes V_i^N corresponding to positive values v_i^N. Then we assume that there are 2^{N+1} distinct nodes satisfying $v_0^N < v_1^N < v_2^N .... < v_{2^{N+1}-1}^N$. Finally we assume that the nodes V_0^0 and V_1^0 defined by $-v_0^0 = v_1^0 = |h_0|$ are generated from an origin node V which does not correspond to a value of X_N.

The fundamental consequence of the assumption of distinct values v_i^N is that for any N and i there is a unique path going from V to v_i^N. Let $i_k^N(i)$, $0 \leq k \leq N-1$, be the indices j characterizing the nodes V_j^k of this path. These nodes can then be written $V_{i_k^N(i)}^k$.

The problem is to calculate the probabilities

$$p_N(i) = P[X_N = v_i^N], \;\; 0 \leq i \leq 2^{N+1} - 1. \tag{10}$$

When the w_ks are IID this probability is $1/2^{N+1}$. When they are no longer independent, its calculation is much more complicated.

For this we introduce the conditional probability

$$p_N(i,j) = p_N[X_N = v_i^N | X_{N-1} = v_j^{N-1}]. \tag{11}$$

called also transition probability. It has two fundamental properties for the discussion that follows.

The first comes from the fact that, as any probability, it is normalized or satisfies for all j the relation $\sum_{i=0}^{2^{N+1}-1} p(i,j) = 1$. However a node V_j^{N-1} of the tree of construction generates only two nodes V_i^N characterized by the indices $i_+(j)$ and $i_-(j)$ and, according to (9), corresponding to the values $v_{i_\pm}^N v_j^{N-1} \pm h_N w_N$. As a consequence for a given j there is only two terms in the previous sum and we have

$$p_N[i_+(j), j] + p_N[i_-(j), j] = 1 \tag{12}$$

The second starts from the fact that there is only one V_j^{N-1} at the step $N-1$ of the tree generating V_j^N and called $j(i)$ Thus $p_N(i,j)$ is zero except when $j = j(i)$, and the only non-zero values of $p_N(i,j)$ are

$$q_N(i) = p_N[i, j(i)] \tag{13}$$

for $N > 0$ and $q_0(i) = 1/2$.

It results from (11) and from the unicity of the path between V and V_i^N that

$$p_N[(X_N = v_i^N).(X_{N-1} = v_j^{N-1})] = p_{N-1}(j).p_N(i,j)\delta[j - j(i)]. \tag{14}$$

where $\delta[.]$ is the Kronecker delta symbol. By a summation on j, which contains only one term, we obtain

$$p_N(i) = p_{N-1}[j(i)]q_N(i). \tag{15}$$

By repeating this at all the nodes of the unique path between V and V_i^N characterized by the indices $v_k^N(i)$ we obtain

$$p_N(i) = \prod_{k=0}^{N} q_k[i_k^N(i)]. \tag{16}$$

When the RVs w_ks are IID we have of course $q_k[i_k^N(i) = 1/2$, and we find again that the values v_i^N have equal probabilities $1/2^{N+1}$.

The probabilities $p_N(i)$ of (16) are normalized, or $\sum_i p_N(i) = 1$, where the sum is extended to all the indices i from 0 to 2^{N-1}. This property is valid for $N = 0$ because $q_N(i) = 1/2$. Suppose that it is valid at the step $N - 1$. Since each node V_l^{N-1} generates only two nodes the result comes from (12).

The relation (16) is the basis for the discussion of the singularity. Indeed if all the $p_N(i)$ tend to 0 when $N \to \infty$ there is no value v_i^∞ with a finite probability, and this means that the RV cannot have a discrete component and then is singular. This can be specified by the following theorem.

Theorem 2: *Let X be the RV $\sum_{k=0}^\infty h_k w_k$, where h_k is the impulse response of a dynamical non-RIF filter $\mathcal{F}$ and w_k a sequence of Bernoulli RVs. If the poles of F are inside the circle of singularity, if the possible values v_i^N of the partial sums X_N are distinct and if the transition probabilities $q_N(i)$ defined by (13) satisfy*

$$0 < q_N(i) < B < 1, \tag{17}$$

then the RV X is singular, or $a_1 = a_2 = 0$.

Proof: If the poles are inside the circle of singularity, Theorem 1 shows that the SM is zero or $a_1 = 0$. It remains to show that $a_2 = 0$. This is a direct consequence of (16) and (17) because $p_N(i) < (1/2)B^N$, which tends to zero when $N \to 0$.

Comments: It is clear that this situation appears for white input because in this case $q_N(i) = 1/2$. The question that remains concerns the conditions of the theorem on the filter. It is clear from the previous discussion and that if $h_k = a^k$ with $a < 1/2$, these conditions are satisfied. There is a large class of filters satisfying also these conditions. However the question of characterizing all the dynamical filters with poles inside the circle of singularity and introducing distinct values v_i^N remains open. As a matter of fact it is possible to extend this theorem to the case where these values are not distinct, but this introduces other conditions that cannot be presented in this paper.

V. Conclusion

When the input of a causal discrete-time exponential linear filter with impulse response a^k is a Bernoulli white noise, the output is singular for $a < 1/2$. A simple proof of this result was presented and it exhibits two steps. The condition on the parameter a and the fact that the input is a discrete-valued signal implies that the SM of the output is zero, which means that there is no continuous component in its DF. The fact that this DF does not contain a discrete component arises from the whiteness of the input. This result is not specific to exponential filters. We have shown that singularity can appear in many other situations and we have established a sufficient condition for singularity by using the positions of the poles of the filter with respect to the circle of singularity and the whiteness of the input.

This assumption of whiteness can however be partially relaxed without changing the singularity of the output. Some sufficient conditions ensuring the singularity with colored inputs have been established. These conditions are obviously satisfied not only by white noise but also by a large class of correlated signals. It is especially the case of some Markovian signals of finite order. The theoretical analysis also shows that the output generated by colored inputs can be a mixture of a discrete and a singular distribution. Computer experiments in order to verify the theoretical results will be discussed in a forthcoming paper.

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