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Some Properties of Lattice Autoregressive Filters

Bernard Picinbono, *Fellow, IEEE* and Messaoud Benidir

Laboratoire des Signaux et Systèmes (L2S), Division Signaux

The pdf copy of the final published text can be obtained from the first author at the following address : bernard.picinbono@lss.supelec.fr

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Abstract

An autoregressive filter is defined either by the components of the regression vector or by the reflection coefficients appearing in its lattice representation. The mathematical expression of the regression vector in terms of the reflection coefficients is very complex but many structural properties can be obtained without this exact expression. In this paper, we present some examples of such structural properties, and we apply these results to prove some extrema1 properties of stable filters such as the maximum value of the components of the regression vector or the maximum value of its norm. Moreover, some properties of the boundary of the stability domain are discussed.

I. Introduction

An autoregressive (AR), or recursive, discrete time filter is a filter deduced from the difference equation

$$y_k - \sum_{i=1}^{n} a_i y_{k-i} = u_k, \tag{1}$$

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B. P. and M. B. are with the Laboratoire des Signaux et Systèmes (L2S), a joint laboratory of the C.N.R.S. and the École Supérieure d'Électricité, Plateau de Moulon, 3 rue Joliot–Curie 91192, Gif sur Yvette, France. The L2S is associated with the University of Paris–Sud, France. E-mail: bernard.picinbono@lss.supelec.fr, .

where u_k, y_k, and a_i are the input, output, and the components of a vector $\mathbf{a}$, respectively. This vector is called the regression vector because when u_k is a white noise the linear prediction of y_k in terms of all its past is

$$\hat{y}_k = \sum_{i=1}^{n} a_i y_{k-i}, \tag{2}$$

which is a linear regression. The transfer function of the causal filter defined by (1) is, of course,

$$H(z) = \frac{z^n}{z^n - \sum_{i=1}^{n} a_i z^{n-i}} \ , \tag{3}$$

which clearly introduces the equivalent terminology of an all poles filter.

It is well known that the regression vector $\mathbf{a}$ can be expressed in terms of the so-called reflection coefficients k by using the Levinson recursion [1 , p . 271, [2] , which allows us to calculate a regression vector of order m, a_m in terms of a_{m-1} by

$$\mathbf{a}_{m-1}^{m} = \mathbf{a}_{m-1} - k_m \mathbf{a}_{m-1}^{(-)} \tag{4}$$

$$a_m^m = k_m. \tag{5}$$

In these equations, a_m^m is the last component of $\mathbf{a}_m$ and $\mathbf{a}_{m-1}^m$ is a vector of order $(m-1)$ deduced from $\mathbf{a}_m$ by suppressing its last component. Moreover, $\mathbf{a}_{m-1}^{(-)}$ is deduced from $\mathbf{a}_{m-1}$ by inversion of the order of the components Then using recursively (4) and (5) from 1 to n, we deduce the vector $\mathbf{a}_n$ in terms of the set of reflexion coefficients k_i , $1 \leq i \leq n$.

Unfortunately, the explicit expression of the components of $\mathbf{a}_n$ in terms of k_i is not simple and, as an example, we give the first three vectors

$$\mathbf{a}_1 = k_1 \tag{6}$$

$$\mathbf{a}_2 = [k_1 - k_1 k_2, k_2]^T \tag{7}$$

$$\mathbf{a}_3 = [k_1 - k_1 k_2 - k_2 k_3, k_2 - k_1 k_3 + k_1 k_2 k_3, k_3]^T. \tag{8}$$

Finally, another way to represent the transfer function is to use a sequence of polynomials defined by

$$P_m(z) = 1 - \mathbf{a}_m^T \mathbf{Z}_m, \tag{9}$$

where $\mathbf{Z}_m$ is the vector

$$\mathbf{Z}_m = [z, z^2, z^3, \ldots, z^m]^T. \tag{10}$$

Of course, the transfer function (3) is given by

$$H(z) = [P_n(z^{-1})]^{-1}, \tag{11}$$

and it is equivalent to study the polynomial $P_n(z)$ or the transfer function H(z).

As a consequence, it appears that if the roots of $P_n(z)$ are z_k, $1 \leq k \leq n$, then the poles of $H_n(z)$ are z_k^{-1} and, in particular, the stability condition for the AR filter is that the roots z_k are outside the unit circle.

The purpose of this paper is the following one. In Section I1 we present some general properties of the regression vector $\mathbf{a}$ which are direct consequences of the Levinson recursion and then obtained without calculating the components of this vector. In Section III we introduce the stability condition and we deduce some properties of the regression vector of stable filters. In particular, we give the expression of the vector $\mathbf{a}$ of maximum norm for a stable filter. This question was motivated by an interpolation problem appearing in estimation theory [4]. In Section IV we present some simple sufficient conditions for stability of an AR filter. The conditions are particularly important in an adaptive filtering context where the coefficients a_i's are time varying. In these cases, it is important to secure the stability with conditions involving almost no calculations. In Section V, the stability problem is considered more carefully and we specify the limits of the stability domain in the space of the regression vector $\mathbf{a}$. This problem was partially considered in [5]. More precisely we give some geometrical properties of the boundary of the stability domain in the $\mathbf{a}$ domain. For the points belonging to this boundary, we give the structure of the transfer function which allows us to indicate which poles are on the unit circle. Finally, Section VI is devoted to some general expressions of the regression vector in terms of the reflection coefficients.

II. Some Structural Properties

In this section we present a sequence of properties directly deduced from the Levinson recursion and interpreted in the lattice representation.

Property 2.1 : Each component of $\mathbf{a}_n$ is a polynomial in k_i with a degree smaller or equal to n.

Proof: The property is true until the order 3, from (6) to (8). If it is true at the order m, we deduce immediately from (4) that it is also true at the order $m+1$.

Property 2.2 : The powers of k_i in the polynomials introduced by 2.1 are either 0 or 1.

Proof: The property is true for $\mathbf{a}_i$, $i \leq 3$. If it is true for $\mathbf{a}_n$, it is also true for $\mathbf{a}_{n+1}$ because in (4) $\mathbf{a}_n$ and $\mathbf{a}_n^{(-)}$ do not depend on k_{n+1}.

Property 2.3 : The polynomials introduced in 2.1 are sums of products of k_i with the signs $+$ if the number of k_i is odd and $-$ if it is even.

Proof: It is also by induction, and a direct consequence of the multiplication by $-k_{n+1}$ in (4).

Property 2.4 : If the reflection coefficients are negative, the components of $\mathbf{a}$ are also negative.

Proof: It is a direct consequence of 2.3.

Property 2.5 : If $\mathrm{Sg}(k_i) = (-1)^{i+1}$, $1 \leq i \leq n$, then $\mathrm{Sg}(a_i) = (-1)^{i+1}$, $1 \leq i \leq n$, where $\mathrm{Sg}(f)$ is the sign of f.

Proof: The property is true for $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, from (6) to (8). Let us suppose that the signs of the componentsof $\mathbf{a}_{m-1}$ are

$$\mathbf{a}_{m-1} : [+, -, +, -, \ldots, (-1)^m]. \tag{12}$$

For $a_{m-1}^{(-)}$ we obtain similarly

$$\mathbf{a}_{m-1}^{(-)} : (-1)^m[+, -, +, -, \ldots, (-1)^m]. \tag{13}$$

Then as $k_m = (-1)^{m+1}|k_m|$, we deduce that the signs of $-k_m\mathbf{a}_{m-1}^{(-)}$ appearing in (4) are

$$-k_m\mathbf{a}_{m-1}^{(-)} : [+, -, +, -, \ldots, (-1)^m]. \tag{14}$$

and by using (4), the property is true for $\mathbf{a}_{m-1}^{m}$. As it is true for the last component given by (5), it is true for $\mathbf{a}_i$, $1 \leq i \leq n$.

Property 2.6 : The number of terms appearing in the polynomials introduced in 2.1 is for each component $[C_n^1, C_n^2, \ldots, C_n^n]$.

Proof: These coefficients, often denoted $\binom{n}{r}$, are the binomial coefficients. The property is true for $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. If it is true for n, the number of terms appearing in $\mathbf{a}_{n+1}$ is from (1-4)

$$[C_n^1+, C_n^n, C_n^2+, C_n^{n-1}, \ldots, C_n^i+, C_n^{n-i+1}, \ldots, C_n^n, C_n^n]$$

Indeed $\mathbf{a}_n$ and $k_{n+1}\mathbf{a}_n^{(-)}$ cannot have common terms because k_{n+1} is not present in the expression of the components of $\mathbf{a}_n$. But as $C_n^p = C_n^{n-p}$ and

$$C_n^{p-1} + C_n^p == C_{n+1}^p, \tag{15}$$

we deduce immediately

$$C_n^i + C_n^{n-i+1} = C_{n+1}^i. \tag{16}$$

As the last component of $\mathbf{a}_{n+1}$ is k_{n+1}, if the property is true for $\mathbf{a}_n$, it is also true for $\mathbf{a}_{n+1}$.

Property 2.7 : The total degrees of the polynomials appearing in the components of $\mathbf{a}_n$, are

$$[2, 4, 6, \ldots, 2n, 2n-1, 2n-3, \ldots 3, 1] \tag{17}$$

for $\mathbf{a}_{2n}$, and

$$[2, 4, 6, \ldots, 2n, 2n+1, 2n-1, 2n-3, , \ldots 3, 1] \tag{18}$$

for $\mathbf{a}_{2n+1}$.

Proof: f we consider $\mathbf{a}_3$ given by (8), we see that the three components are polynomials in k_i , respectively, with degrees 2, 3, 1. Let us suppose that (17) is true. The degrees of the components of $k_{2n+1}\mathbf{a}_{2n^{(-)}}$ are, becauseof the multiplication by k_{2n+1}

$$k_{2n+1}\mathbf{a}_{2n}^{(-)} : [2, 4, \dots, 2n-2, 2n, 2n+1, 2n-1, \dots, 5, 3].$$

As the last component of $\mathbf{a}_{2n+1}$ is of degree 1, we deduce that the degree of $\mathbf{a}_{2n+1}$ is given by (18). By the same procedure, if $\mathbf{a}_{2n+1}$ has the property (18), $\mathbf{a}_{2n+2}$ has the property (17).

III. Some Consequences of Stability

All the properties indicated in Section II are direct consequences of the Levinson recursion (4)-(5), without any consideration of stability problems. In this section, we will discuss some consequences on the regression vector deduced from the stability condition $|k_i| < 1$, $1 \leq i \leq n$.

Consequence 3.1: If an AR filter is stable, then the components of the regression vector satisfy $|a_i| < C_n^i$.

Proof: It is a direct consequence of 2.6. Indeed, a_i is a polynomial with C_n^i terms which are product of reflection coefficients k_i. As $|k_j| < 1$, the absolute values of these terms are also smaller than 1 which gives immediately $|a_i| < C_n^i$.

Comment: It is clear that this condition does not ensure that the AR filter is stable. Indeed, it is well known that the stability condition cannot be expressed as a condition on each component, but is a condition on all the a_i together. On the other hand, if one a_i is greater than C_n^i, the filter is certainly unstable.

Consequence 3.2: If an AR filter is stable, then we have $\sum |a_i| < 2^n - 1$.

Proof: It is a direct consequence of 3.1. Indeed, we deduce from it that

$$\sum |a_i| < \sum C_n^i = 2^n - 1. \tag{19}$$

Comments: As for 3.1, this condition is a necessary condition for stability of an AR filter, but does not secure this stability.

Consequence 3.3: If an AR filter is stable, then the magnitude of the regression vector is smaller than $[C_{2n}^n - 1]^{1/2}$.

Proof: We deduce that the square of the square of the magnitude of $\mathbf{a}_n$ satisfies

$$||\mathbf{a}_i||^2 < \sum_{i=1}^{n} (C_n^i)^2 = \sum_{i=0}^{n} (C_n^i)^2 - 1. \tag{20}$$

In order to calculate this term, we write two expansions of $(a+b)^{2n}$

$$(a+b)^{2n} = \sum_{k=0}^{2n} C_{2n}^k a^{2n-k} b^k, \tag{21}$$

$$[(a+b)^n]^2 = \sum_i \sum_j C_n^i C_n^j a^{2n-i-j} b^{i+j}, \tag{22}$$

By identification we deduce immediately

$$C_{2n}^n = \sum_{i=0}^{n} C_n^i C_n^{n-i} = \sum_{i=0}^{n} [C_n^i]^2, \tag{23}$$

and with (20) we deduce

$$||\mathbf{a}_i||^2 < C_{2n}^n - 1. \tag{24}$$

Consequence 3.4: If an AR filter is stable, then the maximum of the magnitude of the regression vector is reached for two vectors corresponding to $k_i = -1$ and to $k_i = (-1)^{i-1}$, $1 \leq i \leq n$, respectively, and its value is $[C_{2n}^n - 1]^{1/2}$.

Proof: For $k_i = -1$ we deduce from 2.3, 2.4, and 2.6 that the components of $\mathbf{a}_n$, are

$$\mathbf{a}_n^T = -[C_n^1, C_n^2, , \ldots, C_n^n], \tag{25}$$

which means that for this vector the upper bound given by (20) is reached.

For $k_i = (-1)^{i-1}$ we will prove that the corresponding vector is

$$\mathbf{a}_n^T = [C_n^1, -C_n^2, , C_n^3 \ldots, (-1)^{n-1} C_n^n]. \tag{26}$$

Let us suppose that this property is true for n, and let us calculate $\mathbf{a}_{n+1}$ with $k_{n+1} = (-1)^n$. We deduce that

$$-k_{n+1}\mathbf{a}_n^{(-)T} = [C_n^n, -C_n^{n-1}, C_n^{n-2}, \ldots, (-1)^{n-1} C_n^1], \tag{27}$$

and applying (4) and (5), we deduce that

$$\mathbf{a}_{n+1}^{nT} = [C_{n+1}^1, -C_{n+1}^2, C_{n+1}^3, \ldots, (-1)^{n-1} C_{n+1}^n], \tag{28}$$

As the last component of $\mathbf{a}_{n+1}$ is $k_{n+1} = (-1)^n = (-1)^n C_{n+1}^{n+1}$, we see that $\mathbf{a}_{n+1}$ has the structure of $\mathbf{a}_n$ given by (26). Of course, the upper bound of (20) is reached for this vector.

Comment: This property is important in some interpolation problems. Indeed it has been shown in another paper [4] that if we consider the signal y_k defined by (1) where u_k is a white noise of variance ϵ^2, the interpolation of y_n in terms of all its past and future can be obtained with an error given by

$$\eta^2 = \epsilon^2 [1 + \sum a_i^2]^{-1} = \epsilon^2 [1 + ||\mathbf{a}_n||^2]^{-1}. \tag{29}$$

For a given value of ϵ^2 this error is minimum $|\mathbf{a}_n|$ is maximum, provided that the AR filter is stable. Using (24) we see that this minimum error is

$$\eta_{min}^2 = \epsilon^2 (n!)^2 / (2n)!. \tag{30}$$

Consequence 3.5: The two regression vectors of 3.4 correspond to two AR filters with poles of order n located in +1 and -1.

Proof: The denominator of $H(z)$ defined by (3) is

$$D(z) = z^n - \sum a_i z^{n-i}. \tag{31}$$

If the regression vectors are $\mathbf{a}_n$ given by (25) or (26), we obtain, respectively,

$$D_+(z) = (z+1)^n \tag{32}$$

$$D_-(z) = (z-1)^n. \tag{33}$$

IV. Simple Sufficient Conditions for Stability

All the properties indicated in Section I11 are consequences of the stability and are then necessary conditions for stability. But conversely they can be satisfied for unstable filters and are then not sufficient conditions for stability.

Of course such necessary and sufficient conditions do exist and are well known [6], [7]. But, in general, they are very complicated to check and require a lot of calculation. In practice, they are equivalent to the calculation of all the reflections coefficients k_i from a given regression vector in order to check that $|k_i| < 1$.

In many problems, and particularly in an adaptive context, it is necessary to have simple conditions securing the stability of the filter. By simple we mean conditions which can be verified almost without calculation.

Starting from the idea that if $|\mathbf{a}|$ is small, the k_is also are small and the filter is stable, we will introduce some very simple sufficient conditions for stability.

For this purpose, let us associate to any vector $\mathbf{a}$ the functions

$$S_n(\mathbf{a}) \triangleq \sum_{i=1}^{n} |a_i| \tag{34}$$

$$T_n(\mathbf{a}) \triangleq \sum_{i=1}^{n} a_i^2 = ||\mathbf{a}||^2 \tag{35}$$

$$A_n(\mathbf{a}) \triangleq \max_{1 \leq i \leq n}(|a_i|). \tag{36}$$

These functions have been considered in Section III, and, in particular, their upper bounds for stable filters have been obtained.

In this section we are interested in some sufficient conditions on these functions which secure the stability of the corresponding AR filter. Of course it is obvious that these functions are very simple, .in terms of calculation from the components of the regression vector.

At first let us recall a known result.

Result 4. I: If $S_n < 1$, then the corresponding AR filter is stable.

Proof: We have to prove that if $S_n < 1$, the polynomial $P_n(z)$ defined by (9) has all its roots outside the unit circle. Let us suppose the contrary and consider a root z_0 such that $|z_0| < 1$. As z_0 is a root we have, from (9),

$$\sum_{i=1}^{n} a_i z_0^i = 1, \tag{37}$$

which gives

$$1 = \left| \sum_{i=1}^{n} a_i z_0^i \right| \leq \sum_{i=1}^{n} |a_i||z_0^i| \leq S_n, \tag{38}$$

because $|z_0| < 1$. Thus, $S_n \geq 1$ which is in contradiction with the initial condition. Then all the roots are outside the unit circle.

Comment: We deduce from 3.2. and 4.1. that if $S_n < 1$, the AR filter is stable, and if $S_n > 2^n - 1$, it is unstable. It is clear that if $1 < S_n < 2^n - 1$, it can be either stable or unstable, as we will see now.

Result 4. 2: For every $s > 1$ it is possible to find an unstable AR filter with regression vector $\mathbf{a}$ such that $S_n(\mathbf{a}) = s.$.

Proof: Let us suppose that $a_1 = a_2 = \ldots = a_{n-1} = 0$. Then $s = |a_n| = |k_{n-1}|$, and as $s > 1$ this particular AR filter is unstable.

Result 4. 3: If $T_n < l/n$, then the corresponding AR filter is stable.

Proof: From.the Schwarz inequality we deduce that

$$S_n^2(\mathbf{a}) = \left[\sum_{i=1}^{n} |a_i| \right]^2 \leq n \sum_{i=1}^{n} a_i^2 = n T_n(\mathbf{a}), \tag{39}$$

and if $T_n < 1/n$, then $S_n < 1$, which secures the stability.

Result 4. 4: For every $t > 1/n$ it is possible to find an unstable AR filter with regression vector $\mathbf{a}$ such that $T_n(\mathbf{a}) > t$.

Proof: Let us consider the filter such that $a_i = a > 0, \forall i$. For this filter $T_n = na^2$, and as $t > 1/n$, then $a > 1/n$, or can be written $1/n + \epsilon$, $\epsilon > 0$. It is possible to calculate the reflection coefficients of this filter, using (4) and (5). For example, we obtain easily

$$k_1 = \frac{1 + n\epsilon}{1 - n\epsilon(n-1)}, \tag{40}$$

if $\epsilon < [n(n-1)]^{-1}$, which proves that this filter is unstable.

Result 4. 5: If $A_n < l/n$, then the corresponding AR filter is stable.

Proof: This condition means that $|a_i| < 1/n$, $1 \leq i \leq n$, and it results that $S_n < 1$, which gives the stability.

Result 4. 6: If $A_n > l/n$, then the AR filter may be unstable.

Proof: It is the same as for 4.4. By taking all the a_i's equal to a, the condition means that a may be greater than l/n and, in this case, the filter is unstable.

Finally, it is worth noticing that .the condition $A_n < 1/n$ implies $T_n < 1/n$ which also implies $S_n < 1$. In other words, the simplest condition ($A_n < 1/n$), which needs absolutely no calculation, is also the most restrictive.

These sufficient stability conditions are generalizations to all orders of a well-known conditions for second-order filters [8, p. 211].

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