Eigenlogic

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This work presents an operational and geometric approach to logic. It starts from the multilinear elective decomposition of binary logical functions in the original form introduced by George Boole [1]. A justification on historical grounds is presented bridging Boole's theory and the use of his arithmetical logical functions with the axioms of Boolean algebra using sets and quantum logic.

It is shown that the algebraic polynomial formulation can be naturally extended to operators in vector spaces. In this way propositional logic can be formalized in linear algebra by using combinations of tensored elementary operators. The original and principal motivation of this work is for applications in the new field of quantum information, differences are outlined with more traditional quantum logic approaches. This formulation is named Eigenlogic [2].

The interesting feature is that the eigenvalues of these operators are the truth values of the corresponding logical connective and the associated eigenvectors correspond to one of the fixed combinations of the inputs (interpretations). The outcome of a "measurement" or "observation" on a logical observable will give the truth value of the associated logical proposition, and becomes "interpretable" when applied to its eigenspace leading to a natural analogy with the measurement postulate in quantum mechanics. The following diagram summarizes this point of view:

$$\text{eigenvalues} \longrightarrow \text{truth values} \quad ; \quad \text{eigenvectors} \longrightarrow \text{interpretations} \quad ;$$
$$\text{logical operators} \longrightarrow \text{logical connectives}$$

One can generalize to eigenvalues different from the Boolean binary values $\{0,1\}$ for example with $\{+1,-1\}$ associated to self-inverse unitary operators [3]. In general one can associate a binary logical operator with whatever couple of distinct eigenvalues $\{\lambda_1,\lambda_2\}$ the corresponding family of logical operators can be found by Lagrange-Cayley-Hamilton matrix interpolation methods. The extension from binary to many-valued logic is then considered by defining specific operators using multivariate interpolation. The interesting property is that a unique seed operator generates the complete logical family of operators for a given m-valued n-arity system.

This method can be applied to the synthesis of binary and multivalued quantum logical gates. Eigenlogic brings a correspondence between control logic (David Deutsch's quantum logical gate paradigm) and ordinary propositional logic. Several of the logical observables turn out to be well-known quantum gates. It is well known that the 2-quibit entangling *Control-Z* $\boldsymbol{Cz}$ gate in association with 1-quibit gates is a universal quantum gate set. In Eigenlogic the $\boldsymbol{Cz}$ gate is the conjunction (AND) self-inverse Eigenlogic operator. Following this approach a new design method of the universal Toffoli gate, using $\boldsymbol{T}$ gates is proposed.

Ternary-logic quantum gates using qutrits lead to less complex circuits, the design of a balanced qutrit arithmetic full-calculator circuit is realized using an Eigenlogic approach [4].

In Eigenlogic all propositional binary and multivalued logic can be built on the basis of a complete family of commuting logical observables. With non-eigenvectors the logical operators are no more diagonal and correspond to propositions with a fuzzy logic interpretation [3]: the degree of truth corresponding to the fuzzy membership function defined by the mean value (Born rule) applied on the logical observables. Also when using two maximally incompatible logical families such as those generated by the $\boldsymbol{X}$ and $\boldsymbol{Z}$ gates one gets an interesting outlook: the usual Grover gate turns out to be the self-inverse Eigenlogic inclusive disjunction operator (OR) in the X system and can be interpreted in the Z system as a predicative logical existential connective. This could permit to extend the Eigenlogic approach including first-order logic.

References

1. G. Boole, *The Mathematical Analysis of Logic. Being an Essay To a Calculus of Deductive Reasoning*, 1847.
2. Z. Toffano, "Eigenlogic in the spirit of George Boole", ArXiv:1512.06632, 2015.
3. F. Dubois, Z. Toffano,"Eigenlogic: a Quantum View for Multiple-Valued and Fuzzy Systems", Quantum Interaction Conference QI 2016. Lecture Notes in Computer Science, vol 10106. Springer, pp. 239-251, 2017.
4. Z. Toffano, F. Dubois, "Interpolating Binary and Multivalued Logical Quantum Gates", Proceedings of the 4th International Electronic Conference on Entropy and Its Applications. MDPI Proceedings, 2, 4, 152, 2018.