Model reduction for linear delay systems using a delay-independent balanced truncation approach

B. Besselink, Antoine Chaillet, N. van de Wouw

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***Abstract*— A model reduction approach for asymptotically stable linear delay-differential equations is presented in this paper. Specifically, a balancing approach is developed on the basis of energy functionals that provide (bounds on) a measure of energy related to observability and controllability, respectively. The reduced-order model derived in this way is again a delay-differential equation, such that the method is structure preserving. In addition, asymptotic stability is preserved and an a priori bound on the reduction error is derived, providing a measure of accuracy of the reduction. The results are illustrated by means of application on an example.**

I. Introduction

Models of engineering systems or physical phenomena can often be represented in terms of dynamical systems with time delays. Examples include models of machine tool vibrations, control over communication networks, or population dynamics, see the books [11], [15], [5] for an overview. In addition, accurate models of such systems are typically of high order, motivating the need for developing model reduction techniques for delay differential equations. This paper addresses this problem by developing a model reduction technique for linear delay systems.

Methods for model reduction of finite-dimensional linear systems are well developed (see [1], [2] for overviews) and popular approaches are given by balanced truncation [16], [7] and moment matching techniques via Krylov subspaces [6].

For systems with time delays (or, more generally, infinite-dimensional systems), finite-dimensional approximations have been considered on the basis of Fourier series [10], Padé approximations [9], or using the Hankel operator [8]. An overview of such methods is given in [17].

Next, methods for model reduction of delay differential equations have been developed by extending methods for finite-dimensional systems. A moment matching approach using Krylov methods was presented in [14]. Here, as before, a finite-dimensional reduced-order model is obtained; as a consequence, the delay structure is not preserved in the reduction. Another perspective on moment matching for systems with time delays is given in [19], where both finite-dimensional and infinite-dimensional approximations are considered. As a class of reduced-order models is characterized in [19], this has the potential to select a reduced-order model that preserves asymptotic stability properties of the original high-order delay differential equation.

An extension of balanced truncation towards systems with delays is presented in [13], based on characterizing measures of controllability and observability similar to those used in balanced truncation for finite-dimensional systems. This method preserves the delay-structure in the reduced-order model, but asymptotic stability is not necessarily preserved. The method in [20] does provide such guarantee and in addition directly exploits reduction techniques for finite-dimensional linear systems by decomposing the delay system in a finite-dimensional part and an infinite-dimensional delay operator. This method also guarantees a bound on the reduction error. Finally, an alternative perspective is given in [21], where the model reduction problem is formulated as a rank-constrained optimization problem.

In this paper, a balancing approach for model reduction of asymptotically stable delay systems is presented. This approach is based on computing bounds on energy functionals that provide a measure of observability and controllability and the use of these bounds in a balancing procedure. These bounds take the form of Lyapunov-Krasovskii functionals and hold regardless of the size of the delay, leading to a delay-independent reduction procedure. In particular, this reduction procedure features the following properties, which form the main contributions of this paper. First, the reduction is structure-preserving, i.e., the reduced-order model is again in the form of a delay-differential equation, albeit with a reduced set of equations. This allows for accurately capturing the infinite-dimensional nature of the original high-order delay system. Second, the reduced-order model is guaranteed to be asymptotically stability and, third, an a priori error bound is available that provides a measure of the accuracy of the reduction.

The remainder of this paper is outlined as follows. The problem setting is detailed in Section II, before the energy functionals and their bounds are presented in Section III. Section IV discusses the model reduction procedure and the properties of the reduced-order delay system. An illustrative example is given in Section V and conclusions are stated in Section VI.

Notation: The field of real (complex) numbers is denoted by $\mathbb{R}$ ($\mathbb{C}$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. Given a symmetric matrix $X \in \mathbb{R}^{n\times n}$, $X \succ 0$ ($X \succcurlyeq 0$) indicates that it is positive (semi-)definite. The Banach space of continuous functions from an interval

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$\mathcal{T} \subset \mathbb{R}$ into $\mathbb{R}^n$ is represented as $\mathcal{C}(\mathcal{T}, \mathbb{R}^n)$. Similarly, $\mathcal{L}_2(\mathcal{T}, \mathbb{R}^n)$ denotes the class of square integrable functions from $\mathcal{T}$ into $\mathbb{R}^n$.

II. Problem setting

Consider the linear delay-differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_{\mathrm{d}}x(t-\tau) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, and output $y(t) \in \mathbb{R}^p$ for all $t \geq t_0$. The initial condition for (1) is given by the function segment $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, such that

$$x(t) = \varphi(t), \qquad \forall t \in [t_0 - \tau, t_0]. \tag{2}$$

Next, the function segment $x_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ defined as $x_t(s) = x(t+s)$, $s \in [-\tau, 0]$ characterizes the state of (1) at time $t \geq 0$, such that the initial condition (2) can also be written as $x_{t_0} = \varphi$.

In this paper, model reduction of systems of the form (1) is pursued under the assumption that (1) is asymptotically stable. Specifically, a model of the same form is sought that approximates the input-output behavior of (1), but whose state ξ_t is in $\mathcal{C}([-\tau, 0], \mathbb{R}^k)$ with $k < n$. Note that, even though this "reduced-order" state remains infinite-dimensional, this is regarded as model reduction as the number of equations in the first equation in (1) is reduced. In this setting, the problem of finding a reduced-order delay-differential equation is considered that, first, preserves asymptotic stability of the original high-order system and, second, satisfies an a priori error bound in order to characterize the accuracy of the reduction.

III. Observability and controllability functionals

The model reduction approach developed in this paper will be based on energy functions that respectively provide a measure for observability and controllability of the delay system. First, the observability functional is defined as a measure of energy associated with observing the output of (1).

Definition 1: The observability functional of (1) is the functional $L_{\mathrm{o}} : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ defined as

$$L_{\mathrm{o}}(\varphi) = \int_0^\infty |y(t)|^2 \,\mathrm{d}t, \tag{3}$$

where $y(t) = Cx(t) = Cx_t(0)$ is the output of (1) for initial condition $x_0 = \varphi$ and zero input ($u = 0$).

It is clear that the observability functional exists (i.e., the integral (3) is bounded) if the system (1) is asymptotically stable. Next, a measure for the energy associated with controlling (1) is given by the controllability functional.

Definition 2: The controllability functional of (1) is the functional $L_{\mathrm{c}} : \mathcal{D}_{\mathrm{c}} \to \mathbb{R}$ defined as

$$L_{\mathrm{c}}(\varphi) = \inf \left\{ \int_{-\infty}^0 |u(t)|^2 \,\mathrm{d}t \;\middle|\; u \in \mathcal{L}_2((-\infty, 0], \mathbb{R}^m), \lim_{T\to\infty} x_{-T} = 0,\, x_0 = \varphi \right\}, \tag{4}$$

where x_t is the solution of (1) for input u and $\mathcal{D}_{\mathrm{c}} \subset \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ the domain of L_{c}, i.e., the collection of function segments φ for which $L_{\mathrm{c}}(\varphi)$ is well-defined.

Remark 1: The definition of the energy functionals in Definitions 1 and 2 is motivated by the energy functions that form the basis of balanced truncation for finite-dimensional linear systems, see, e.g., [16], [7], [1]. In this case, these energy functions are characterized by the observability and controllability Gramian, respectively. $\lhd$

A characterization of the observability functional in Definition 1 is provided as follows.

Lemma 1: Consider the asymptotically stable delay-differential equation (1). If there exist matrices $Q \succ 0$ and $Q_{\mathrm{d}} \succcurlyeq 0$ such that

$$\begin{bmatrix} A^{\mathrm{T}}Q + QA + Q_{\mathrm{d}} + C^{\mathrm{T}}C & QA_{\mathrm{d}} \\ A_{\mathrm{d}}^{\mathrm{T}}Q & -Q_{\mathrm{d}} \end{bmatrix} \preccurlyeq 0, \tag{5}$$

then the functional $E_{\mathrm{o}} : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ defined as

$$E_{\mathrm{o}}(\varphi) = \varphi^{\mathrm{T}}(0)Q\varphi(0) + \int_{-\tau}^0 \varphi^{\mathrm{T}}(s)Q_{\mathrm{d}}\varphi(s)\,\mathrm{d}s, \tag{6}$$

satisfies

$$E_{\mathrm{o}}(\varphi) \geq L_{\mathrm{o}}(\varphi) \tag{7}$$

for all $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and L_{o} as in Definition 1.

Proof: In order to prove the lemma, let x_t be the solution of (1) for initial condition $x_0 = \varphi$ and zero input and consider $E_{\mathrm{o}}(x_t)$. Note that, by (6), $E_{\mathrm{o}}(x_t)$ can be written as

$$E_{\mathrm{o}}(x_t) = x^{\mathrm{T}}(t)Qx(t) + \int_{t-\tau}^t x^{\mathrm{T}}(s)Q_{\mathrm{d}}x(s)\,\mathrm{d}s, \tag{8}$$

with $x(t+s) = x_t(s)$, $s \in [-\tau, 0]$. Then, time-differentiation of E_{o} along trajectories of (1) yields

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\{E_{\mathrm{o}}(x_t)\} = \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^{\mathrm{T}} M_{\mathrm{o}} \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}, \tag{9}$$

with

$$M_{\mathrm{o}} = \begin{bmatrix} A^{\mathrm{T}}Q + QA + Q_{\mathrm{d}} & QA_{\mathrm{d}} \\ A_{\mathrm{d}}^{\mathrm{T}}Q & -Q_{\mathrm{d}} \end{bmatrix}, \tag{10}$$

and where the dynamics (1) is used to obtain (9) (recall that $u = 0$). Employing the condition (5) in (9)–(10) leads to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\{E_{\mathrm{o}}(x_t)\} \leq -x^{\mathrm{T}}(t)C^{\mathrm{T}}Cx(t) = -|y(t)|^2, \tag{11}$$

where $y(t) = Cx_t(0)$ is the output corresponding to the trajectory x_t. Integration of the result (11) over the interval $[0, T]$ gives

$$E_{\mathrm{o}}(x_T) - E_{\mathrm{o}}(x_0) \leq -\int_0^T |y(t)|^2 \,\mathrm{d}t, \tag{12}$$

where it is recalled that $x_0 = \varphi$. Moreover, due to asymptotic stability, it holds that

$$\lim_{T\to\infty} E_{\mathrm{o}}(x_T) = E_{\mathrm{o}}(0) = 0, \tag{13}$$

such that (12) leads, for $T \to \infty$, to

$$E_{\mathrm{o}}(\varphi) \geq \int_0^\infty |y(t)|^2 \, \mathrm{d}t. \tag{14}$$

This proves the desired result (7) by recalling the definition of L_{o} in (3). ■

The controllability functional admits a similar characterization, as shown in the following lemma.

Lemma 2: Consider the delay-differential equation (1). If there exist matrices $P \succ 0$ and $P_{\mathrm{d}} \succcurlyeq 0$ such that

$$\begin{bmatrix} AP + PA^{\mathrm{T}} + P_{\mathrm{d}} + BB^{\mathrm{T}} & A_{\mathrm{d}}P \\ PA_{\mathrm{d}}^{\mathrm{T}} & -P_{\mathrm{d}} \end{bmatrix} \preccurlyeq 0, \tag{15}$$

then the functional $E_{\mathrm{c}} : \mathcal{C}([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}$ defined as

$$E_{\mathrm{c}}(\varphi) = \varphi^{\mathrm{T}}(0) R \varphi(0) + \int_{-\tau}^{0} \varphi^{\mathrm{T}}(s) R_{\mathrm{d}} \varphi(s) \, \mathrm{d}s, \tag{16}$$

with $R = P^{-1}$ and $R_{\mathrm{d}} = RP_{\mathrm{d}}R$, satisfies

$$E_{\mathrm{c}}(\varphi) \leq L_{\mathrm{c}}(\varphi) \tag{17}$$

for all $\varphi \in \mathcal{D}_{\mathrm{c}} \subset \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and L_{c} as in Definition 2.

Proof: In order to prove the lemma, the matrix $R = P^{-1}$ is defined, such that a congruence transformation of (15) with a block-diagonal matrix $\mathrm{blkdiag}\{R, R\}$ leads to the equivalent condition

$$\begin{bmatrix} A^{\mathrm{T}}R + RA + R_{\mathrm{d}} + RBB^{\mathrm{T}}R & RA_{\mathrm{d}} \\ A_{\mathrm{d}}^{\mathrm{T}}R & -R_{\mathrm{d}} \end{bmatrix} \preccurlyeq 0, \tag{18}$$

with $R_{\mathrm{d}} = RP_{\mathrm{d}}R$. Next, application of the Schur complement shows that (18) (and, hence, (15)) is equivalent to

$$\begin{bmatrix} A^{\mathrm{T}}R + RA + R_{\mathrm{d}} & RA_{\mathrm{d}} & RB \\ A_{\mathrm{d}}^{\mathrm{T}}R & -R_{\mathrm{d}} & 0 \\ B^{\mathrm{T}}R & 0 & -I \end{bmatrix} \preccurlyeq 0, \tag{19}$$

which will form the basis for the remainder of the proof.

Consider a solution x_t to (1) corresponding to an input $u \in \mathcal{L}_2((-\infty, 0], \mathbb{R}^n)$ and satisfying the conditions in (4), i.e., $\lim_{T\to\infty} x_{-T} = 0$ and $x_0 = \varphi$. Since $E_{\mathrm{c}}(x_t)$ can be written as

$$E_{\mathrm{c}}(x_t) = x^{\mathrm{T}}(t) R x(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(s) R_{\mathrm{d}} x(s) \, \mathrm{d}s, \tag{20}$$

with $x(t+s) = x_t(s)$, $s \in [-\tau, 0]$, it follows that time-differentiation of E_{c} along the trajectories of (1) leads to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\{E_{\mathrm{c}}(x_t)\} = \begin{bmatrix} x(t) \\ x(t-\tau) \\ u(t) \end{bmatrix}^{\mathrm{T}} M_{\mathrm{c}} \begin{bmatrix} x(t) \\ x(t-\tau) \\ u(t) \end{bmatrix}. \tag{21}$$

with

$$M_{\mathrm{c}} = \begin{bmatrix} A^{\mathrm{T}}R + RA + R_{\mathrm{d}} & RA_{\mathrm{d}} & RB \\ A_{\mathrm{d}}^{\mathrm{T}}R & -R_{\mathrm{d}} & 0 \\ B^{\mathrm{T}}R & 0 & 0 \end{bmatrix}. \tag{22}$$

The use of (19) in (21)–(22) leads to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\{E_{\mathrm{c}}(x_t)\} \leq |u(t)|^2, \tag{23}$$

after which integration over the interval $[-T, 0]$ yields

$$E_{\mathrm{c}}(x_0) - E_{\mathrm{c}}(x_{-T}) \leq \int_{-T}^{0} |u(t)|^2 \, \mathrm{d}t. \tag{24}$$

Letting $T \to \infty$ and noting that $\lim_{T\to\infty} E_{\mathrm{c}}(x_{-T}) = 0$, it follows that

$$E_{\mathrm{c}}(\varphi) \leq \int_{-\infty}^{0} |u(t)| \, \mathrm{d}t, \tag{25}$$

where the condition $x_0 = \varphi$ is used. Since the input function u is chosen arbitrarily, the inequality (25) also holds for the input u that achieves the minimization in (4). Consequently, (25) implies the desired result (17), finalizing the proof. ■

The functional E_{o} in (6) provides an upper bound on the observability functional L_o in Definition 1, whereas E_{c} in (16) is a lower bound to the controllability functional L_{c} in Definition 2. These bounds, rather than the observability and controllability functionals themselves, will be used as a basis for model reduction. Namely, it will be shown that the structure of the bounds (6) and (16) is beneficial for the development of a model reduction procedure that preserves asymptotic stability and provides an a priori error bound.

Remark 2: Even though the controllability functional can in general only be defined on a restricted domain $\mathcal{D}_{\mathrm{c}}$ (see Definition 2), its bound E_{c} in (16) can be defined for all function segments in $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ (provided that (15) holds). As the latter will be used as a basis for model reduction, the reduced-order model will be well-defined. ◁

Remark 3: The functionals E_{o} in (6) and E_{c} in (16) are similar to Lyapunov-Krasovskii functionals as often exploited in stability analysis of time-delay systems, see [11]. In fact, if the matrices in (5) or (15) are negative definite instead of merely negative semi-definite, they imply delay-independent asymptotic stability of (1). ◁

IV. Model reduction by truncation

Before discussing the use of (bounds on) the observability and controllability functionals in Definitions 1 and 2 in the scope of model reduction, the general reduction procedure of truncation is presented. To this end, a partitioned form of the dynamics (1) is considered in which $x(t)$ and the function segments x_t are partitioned as

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_t = \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}, \tag{26}$$

with $x_1(t) \in \mathbb{R}^k$, $x_{1,t} \in \mathcal{C}([-\tau, 0], \mathbb{R}^k)$ and $k < n$. The corresponding partitioning of the system matrices yields

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \, A_{\mathrm{d}} = \begin{bmatrix} A_{\mathrm{d},11} & A_{\mathrm{d},12} \\ A_{\mathrm{d},21} & A_{\mathrm{d},22} \end{bmatrix}, \, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{27}$$

and $C = [\, C_1 \; C_2 \,]$. Using the partitioning (26), (27), a reduced-order approximation of (1) can be obtained by truncation as

$$\begin{aligned} \dot{\xi}(t) &= A_{11}\xi(t) + A_{\mathrm{d},11}\xi(t-\tau) + B_1 u(t), \\ \hat{y}(t) &= C_1 \xi(t), \end{aligned} \tag{28}$$

with $\xi(t) \in \mathbb{R}^k$ for each t and function segments $\xi_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^k)$. Here, $\xi(t)$ provides an approximation of $x_1(t)$ in the partitioned coordinates (26).

The following property holds for the observability functional of the reduced-order system (28).

Lemma 3: Let the condition (5) be satisfied for symmetric matrices $Q \succ 0$ and $Q_{\rm d} \succcurlyeq 0$ of the form

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad Q_{\rm d} = \begin{bmatrix} Q_{\rm d,11} & Q_{\rm d,12} \\ Q_{\rm d,21} & Q_{\rm d,22} \end{bmatrix}. \tag{29}$$

Then, the observability functional $\hat{L}_{\rm o}$ of the reduced-order system (28) exists and the functional $\hat{E}_{\rm o} : \mathcal{C}([-\tau, 0], \mathbb{R}^k) \to \mathbb{R}$ given as

$$\hat{E}_{\rm o}(\hat{\varphi}) = \hat{\varphi}^{\rm T}(0) Q_1 \hat{\varphi}(0) + \int_{-\tau}^{0} \hat{\varphi}^{\rm T}(s) Q_{\rm d,11} \hat{\varphi}(s)\, {\rm d}s, \tag{30}$$

satisfies $\hat{E}_{\rm o}(\hat{\varphi}) \geq \hat{L}_{\rm o}(\hat{\varphi})$ for all $\hat{\varphi} \in \mathcal{C}([-\tau, 0], \mathbb{R}^k)$.

Proof: Since the matrices Q and $Q_{\rm d}$ in (29) are such that (5) holds, it follows that

$$\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}^{\rm T} \begin{bmatrix} A^{\rm T}Q + QA + Q_{\rm d} + C^{\rm T}C & QA_{\rm d} \\ A_{\rm d}^{\rm T} Q & -Q_{\rm d} \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \preccurlyeq 0 \tag{31}$$

for any matrix T. After choosing $T = [\, I_k \;\, 0\,]^{\rm T}$, it can be checked by using the partitioning (27) that the left-hand side of (31) provides an inequality of the form (5) for the reduced-order system (28). Then, it follows from (12) in the proof of Lemma 1 that, for all $T \geq 0$,

$$\hat{E}_{\rm o}(\hat{\varphi}) \geq \int_0^{\rm T} \left|\hat{y}(t)\right|^2 {\rm d}t + \hat{E}_{\rm o}(\xi_T), \tag{32}$$

where ξ_t is the solution of (28) for $\xi_0 = \hat{\varphi}$ and for $u = 0$. As $\hat{E}_{\rm o}(\hat{\varphi})$ is well-defined (i.e., bounded), it follows after taking the limit for $T \to \infty$ that $\hat{L}_{\rm o}$ exists and that the bound $\hat{E}_{\rm o}(\hat{\varphi}) \geq \hat{L}_{\rm o}(\hat{\varphi})$ for all $\hat{\varphi} \in \mathcal{C}([-\tau, 0], \mathbb{R}^k)$. ∎

The counterpart of Lemma 3 for the controllability functional is given as follows.

Lemma 4: Let the condition (15) be satisfied for symmetric matrices $P \succ 0$ and $P_{\rm d} \succcurlyeq 0$ of the form

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, \quad P_{\rm d} = \begin{bmatrix} P_{\rm d,11} & P_{\rm d,12} \\ P_{\rm d,21} & P_{\rm d,22} \end{bmatrix}. \tag{33}$$

Then, the functional $\hat{E}_{\rm c} : \mathcal{C}([-\tau, 0], \mathbb{R}^k) \to \mathbb{R}$ given as

$$\hat{E}_{\rm c}(\hat{\varphi}) = \hat{\varphi}^{\rm T}(0) R_1 \hat{\varphi}(0) + \int_{-\tau}^{0} \hat{\varphi}^{\rm T}(s) R_{\rm d,11} \hat{\varphi}(s)\, {\rm d}s, \tag{34}$$

with $R_1 = P_1^{-1}$ and $R_{\rm d,11} = R_1 P_{\rm d,11} R_1$ satisfies $\hat{E}_{\rm c}(\hat{\varphi}) \leq \hat{L}_{\rm c}(\hat{\varphi})$ for all $\hat{\varphi} \in \hat{\mathcal{D}}_{\rm c} \subset \mathcal{C}([-\tau, 0], \mathbb{R}^k)$. Here, $\hat{L}_{\rm c}$ is the controllability functional for the reduced-order system (28) and $\hat{\mathcal{D}}_{\rm c}$ the domain on which it is well-defined.

Proof: The proof is similar to the first part of the proof of Lemma 3 and is omitted for the sake of brevity. ∎

The results of Lemmas 3 and 4 thus state that the observability and controllability functionals of a reduced-order system obtained by truncation can be bounded by relevant parts of the energy functionals of the original system (1) when the matrices Q and P have a suitable block-diagonal form, see (29) and (33). Even though these results hold for any matrices Q and P satisfying this block-diagonal form, a more specific diagonal form for these matrices is assumed in the remainder of this section. This leads to the following definition.

Definition 3: A realization (1) is said to be balanced if there exist matrices $Q \succ 0$, $Q_{\rm d} \succcurlyeq 0$ satisfying (5) and matrices $P \succ 0$, $P_{\rm d} \succcurlyeq 0$ satisfying (15) such that

$$Q = P = \Sigma := \begin{bmatrix} \sigma_1 I_{m_1} & 0 & \cdots & 0 \\ 0 & \sigma_2 I_{m_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_q I_{m_q} \end{bmatrix} \tag{35}$$

with $\sigma_i > \sigma_{i+1} > 0$, $i \in \{1, \ldots, q-1\}$ and $\sum_{i=1}^{q} m_i = n$.

The following standard result guarantees the existence of such balanced realization.

Lemma 5: Let there exist matrices $Q \succ 0$ and $Q_{\rm d} \succcurlyeq 0$ such that (5) holds and matrices $P \succ 0$ and $P_{\rm d} \succcurlyeq 0$ such that (15) holds. Then, there exists a change of coordinates $x(t) = Tz(t)$ such that the realization given by the new coordinates is balanced, i.e., the nonsingular matrix T can be chosen such that $T^{\rm T} Q T = T^{-1} P T^{-{\rm T}} = \Sigma$, with Σ as in (35).

Proof: The existence of such matrix T follows from standard results in linear algebra on simultaneous diagonalization (e.g., [12]). This result also forms the foundation of balancing for finite-dimensional linear systems, see [1]. ∎

When truncation is applied for asymptotically stable delay systems in a balanced realization as in Definition 3, this stability property is preserved. This result is stated next.

Theorem 6: Let the asymptotically stable system (1) be in a balanced realization and consider the reduced-order delay-differential equation (28) obtained by truncation for k such that $k = \sum_{i=1}^{r} m_i$ for some $r > 0$. Then, the reduced-order system is asymptotically stable.

Proof: This result can be proven by exploiting ideas in the proof of stability preservation for balanced truncation of finite-dimensional linear systems originally shown in [18], see also [4]. Details are omitted. ∎

Moreover, for truncation of a balanced realization, the following error bound holds.

Theorem 7: Let the asymptotically stable system (1) be in a balanced realization and consider the reduced-order delay-differential equation (28) obtained by truncation for k such that $k = \sum_{i=1}^{r} m_i$ for some $r > 0$. Then, for any common input function $u \in \mathcal{L}_2([0, \infty), \mathbb{R}^m)$ and initial conditions $\varphi = 0$ and $\hat{\varphi} = 0$ for (1) and (28), respectively, their output trajectories satisfy the error bound

$$\int_0^T |y(t) - \hat{y}(t)|^2\, {\rm d}t \leq \varepsilon^2 \int_0^T |u(t)|^2\, {\rm d}t \tag{36}$$

for all $T \geq 0$ and where the error bound (gain) ε is given as

$$\varepsilon = 2 \sum_{i=r+1}^{q} \sigma_i, \tag{37}$$

with σ_i as in (35).

Proof: The proof can be found in Appendix A. ■

The results in Theorems 6 and 7 thus suggest a model reduction procedure in which, first, solutions to (5) and (15) are sought. Second, the balancing transformation of Lemma 5 is employed to obtain a balanced realization (see, e.g., [1] for an explicit procedure to compute T) and finally, truncation is applied to obtain a reduced-order model of the form (28). This reduced-order model is then of the same form as (1), is asymptotically stable and satisfies the a priori error bound (36) which has a similar structure as the error bound for balanced truncation of finite-dimensional linear systems.

Remark 4: An alternative approach towards balanced truncation of delay systems is given in [13], where infinite-dimensional generalizations of the observability and controllability Gramian for finite-dimensional systems are exploited. Then, relevant finite-dimensional parts of these Gramians are selected to compute a coordinate transformation $x(t) = Tz(t)$ that diagonalizes these parts. However, this approach does not lead to reduced-order delay systems that preserve asymptotic stability and satisfy an a priori bound on the reduction error as in the current paper. ◁

V. Illustrative example

To illustrate the reduction procedure developed in Section IV, the model of a heated rod discussed in [14] is considered. The model is a partial differential equation of the form

$$\begin{aligned}\frac{\partial v}{\partial t}(x,t) &= \frac{\partial^2 v}{\partial x^2}(x,t)\\ &\quad + a_0(x)v(x,t) + a_1(x)v(\pi - x, t-1),\end{aligned} \tag{38}$$

with $v(x,t)$ the temperature of the rod at location $x \in [0,\pi]$ at time t, satisfying the boundary conditions $v(0,t) = v(\pi,t) = 0$. The functions a_0 and a_1 are given as $a_0(x) = -2\sin(x)$ and $a_1(x) = 2\sin(x)$, respectively. Discretization in space leads to an asymptotically stable delay-differential equation of the form (1) with $x(t) \in \mathbb{R}^n$, $n = 35$ after choosing the input and output matrices as $B = C^{\mathrm{T}} = \frac{1}{\sqrt{n}}\mathbf{1}_n$, where $\mathbf{1}_n \in \mathbb{R}^n$ is a vector of all ones. Thus, the input and output can respectively be interpreted as a uniform heating of the road and its average (in space) temperature.

To derive the reduced-order models, solutions to (5) and (15) are sought that minimize the trace of Q and P, respectively. Using these matrices, the coordinate transformation of Lemma 5 is computed, after which truncation to order $k = 1$ leads to the reduced-order model

$$\begin{aligned}\dot{\xi}(t) &= -2.69\,\xi(t) + 1.65\,\xi(t-1) - 0.93\,u(t),\\ \hat{y}(t) &= -0.93\,\xi(t).\end{aligned} \tag{39}$$

This model can be checked to be asymptotically stable, as guaranteed by Theorem 6. Moreover, according to Theorem 7, the error bound (36) holds with $\varepsilon = 0.012$.

A comparison of the frequency response functions of the high-order and reduced-order delay-differential equations is depicted in Figure 1, indicating a good approximation. This is confirmed by the magnitude of the error in Figure 2, which

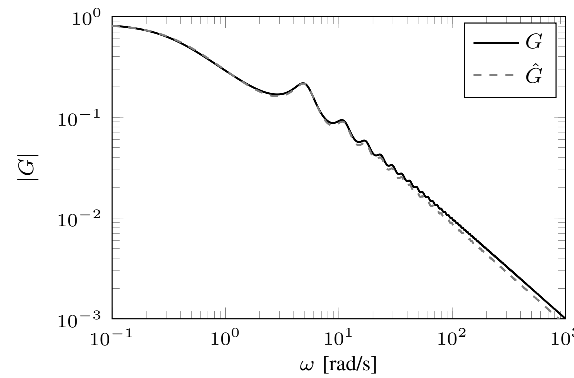

Fig. 1. Magnitude of the frequency response functions G and $\hat{G}$ of the discretized model of the heated rod (38) and the reduced-order approximation (39), respectively. Here, $G(s) = C(sI - A - A_{\mathrm{d}}e^{-s\tau})^{-1}B$ and $\hat{G}$ is defined similarly.

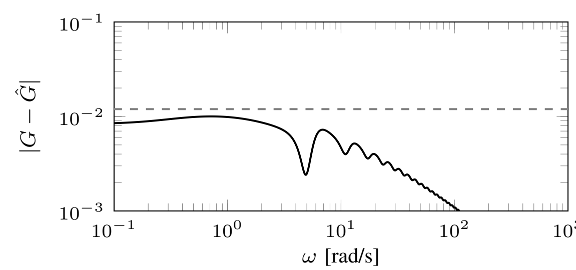

Fig. 2. Magnitude of the frequency response function of the error $G - \hat{G}$, with G and $\hat{G}$ the transfer functions as in Figure 1. The error bound is depicted as the dashed gray line.

also shows that the error bound is not conservative for this example. Finally, it is noted that the scalar reduced-order model (39) accurately captures the repeated resonances in the frequency response function. This behavior cannot be obtained by a finite-dimensional approximation of low-order, indicating the importance of preserving the delay-structure in reduction.

VI. Conclusions

A structure-preserving model reduction procedure for delay-differential equations is presented in this paper, based on the definition of energy functionals that characterize the energy associated with the output and input of the model. A balancing procedure on the basis of these energy functionals is shown to lead to a reduced-order delay-differential equation for which asymptotic stability is preserved. In addition, an a priori error bound is available.

Future work will focus on a delay-dependent approach, which could potentially lead to better reduced-order models if the value of the delay is known.

Appendix A. Proof of Theorem 7

The proof is inspired by a construction in [3]. Following this approach, a one-step reduction is considered first. Here, the state components corresponding to σ_q are discarded by truncation, such that $k = n - m_q$, with m_q the multiplicity of σ_q. In this case, the observability and controllability functionals can be written (using a slight abuse of notation) in partitioned form as

$$E_{\mathrm{o}}(\varphi_1,\varphi_2)=\varphi_1^{\mathrm{T}}Q_1\varphi_1+\varphi_2^{\mathrm{T}}Q_2\varphi_2 \\ +\int_{-\tau}^{0}\begin{bmatrix}\varphi_1(s)\\ \varphi_2(s)\end{bmatrix}^{\mathrm{T}}Q_{\mathrm{d}}\begin{bmatrix}\varphi_1(s)\\ \varphi_2(s)\end{bmatrix}\,\mathrm{d}s \quad (40)$$

and

$$E_{\mathrm{c}}(\varphi_1,\varphi_2)=\varphi_1^{\mathrm{T}}R_1\varphi_1+\varphi_2^{\mathrm{T}}R_2\varphi_2 \\ +\int_{-\tau}^{0}\begin{bmatrix}\varphi_1(s)\\ \varphi_2(s)\end{bmatrix}^{\mathrm{T}}R_{\mathrm{d}}\begin{bmatrix}\varphi_1(s)\\ \varphi_2(s)\end{bmatrix}\,\mathrm{d}s, \quad (41)$$

respectively. Here,

$$Q_1=R_1^{-1}=\mathrm{blkdiag}\{\sigma_1 I_{m_1},\ldots,\sigma_{q-1}I_{m_{q-1}}\}, \quad (42)$$
$$Q_2=R_2^{-1}=\sigma_q I_{m_q}, \quad (43)$$

where the relation $P=R^{-1}$ is used (see Lemma 2) is used. On the basis of the partitioned energy functionals, the functional V is introduced as

$$V(\varphi_1,\varphi_2,\hat{\varphi})=E_{\mathrm{o}}(\varphi_1-\hat{\varphi},\varphi_2)+\sigma_q^2E_{\mathrm{c}}(\varphi_1+\hat{\varphi},\varphi_2), \quad (44)$$

where $\varphi_1\in\mathcal{C}([-\tau,0],\mathbb{R}^k)$, $\varphi_2\in\mathcal{C}([-\tau,0],\mathbb{R}^{n-k})$, and $\hat{\varphi}\in\mathcal{C}([-\tau,0],\mathbb{R}^k)$. In the remainder of this proof, the value of $V(x_{1,t},x_{2,t},\xi_t)$ will be evaluated, i.e., the evolution of V along trajectories $(x_{1,t},x_{2,t})$ of the delay-differential equation (1) and ξ_t of reduced-order delay-differential equation (28). Specifically, by exploiting the characterizations of the observability and controllability functionals in Lemmas 1 and 2 for the partitioned form (27), it can be shown that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\big\{V(x_{1,t},x_{2,t},\xi_t)\big\}\leq-\big|y(t)-\hat{y}(t)\big|^2+(2\sigma_q)^2|u(t)|^2, \quad (45)$$

along trajectories of (1) and (28). Here, the fact that a one-step reduction is considered is crucial to obtain this result, in particular the relation (43). Integration of (45) over the interval $[0,T]$ yields

$$V(x_{1,T},x_{2,T},\xi_T)-V(x_{1,0},x_{2,0},\xi_0) \\ \leq(2\sigma_q)^2\int_0^T|u(t)|^2\,\mathrm{d}t-\int_0^T\big|y(t)-\hat{y}(t)\big|^2\,\mathrm{d}t, \quad (46)$$

where it is noted that $V(x_{1,0},x_{2,0},\xi_0)=0$ due to the choice of zero initial conditions and the structure of V in (44). Moreover, $V(x_{1,t},x_{2,t},\xi_t)\geq 0$ for all $(x_{1,t},x_{2,t},\xi_t)$, which gives the desired result (36) and (37) for $k=n-m_q$.

To prove the result (36) and (37) for arbitrary order $k<n$ (according to the multiplicities m_i of the parameters σ_i in (35)), it is recalled that Lemmas 3 and 4 show that the energy functionals for a reduced-order delay-differential equation of arbitrary order are bounded by the relevant parts of the partitioned energy functionals (40) and (41) of the high-order system. As such, the procedure discussed above can be repeated to obtain a reduced-order model for arbitrary k, where the result (36) and (37) follows from application of the triangle inequality.

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