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# Incremental stability of spatiotemporal delayed dynamics and application to neural fields

Georgios Is. Detorakis and Antoine Chaillet

**Abstract**—We propose a Lyapounov-Krasovskii approach to study incremental stability in spatiotemporal dynamics with delays and their capability to be entrained by a periodic input. We then focus on delayed neural fields, which describe the spatiotemporal evolution of neuronal activity. We provide an explicit condition, involving the slope of the activation functions and the strength of coupling, under which delayed neural fields are incrementally stable regardless of communication delays. We finally show how this approach can be used to draw frequency profiles of neuronal populations.

## I. INTRODUCTION

Incremental stability characterizes systems whose solutions all converge to one another, with a transient overshoot “proportional” to the distance between initial states. It can be established through the study of an incremental Lyapunov function provided that its derivative along any two solutions of the system is negative as long as they do not coincide [2].

Incremental stability shares strong similarities with convergent [29], [10] and contractive [25], [17] systems. These similarities have been studied in details in [34]. A key feature that can be deduced from these properties is the ability of the system to be entrained by its input. More precisely, if any of these properties holds uniformly in the applied input then, under mild conditions, the response of the system to any  $T$ -periodic input is asymptotically  $T$ -periodic [25], [29], [35], [8], [26]. While this property is natural for asymptotically stable linear systems, it is far from granted when dynamics are nonlinear. More precisely, examples of systems that have a globally asymptotically stable equilibrium for each constant value of the input, but for which this entrainment property does not hold, can be found in [32], [35].

As observed in [30], the capacity of a system to be entrained by its input is the key ingredient from which one can derive frequency profiles. These take the form of magnitude Bode plots, reflecting the asymptotic amplification of the system in response to harmonic excitations at various frequencies. Unlike in the linear case, their shape may depend on the input amplitude.

These frequency profiles are especially attractive for neuroscience applications. Indeed, oscillations play a fundamental role in brain coordination [6] and altered oscillations

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in specific structures correlate with pathological symptoms, for instance in Parkinson’s disease [21]. The possibility to predict which frequency band is preferably amplified by a given brain structure might be a valuable tool for computational neuroscientists, and could lead to the development of stimulation strategies to restore healthy frequency profiles.

The spatiotemporal dynamics of brain oscillations can be captured by neural fields models [9]. These models take the form of integro-differential equations that abstract the brain structure to a continuous medium. Communication delays can be taken into account in this model, giving rise to delayed neural fields [16], [3], [37]. This feature is particularly relevant as the non-instantaneous communication between neurons, due to axonal propagation and synaptic delays, is believed to play a significant role in the onset and features of brain oscillations [27], [28], [23].

To the best of our knowledge, no systematic tool to guarantee incremental stability of neural fields (with or without delays) has yet been proposed. Incremental stability of delayed systems can be established using Krasovskii [33] or Razumikhin [8], [13] approaches, but these results have not yet been extended to spatiotemporal dynamics.

Here, we propose a Lyapunov-Krasovskii approach to establish incremental stability of spatiotemporal delayed dynamics and provide conditions under which it guarantees the entrainment property. We then focus on delayed neural fields to show that, under the stability condition derived in [16], the system happens to be also incrementally stable. An example motivated by the study of the brain circuitry involved in the motor symptoms of Parkinson’s disease demonstrates the possibility to draw frequency profiles of the dynamics involved, which happen to exhibit a resonance at frequencies associated to motor symptoms.

## II. SPATIOTEMPORAL DELAYED DYNAMICS

Throughout this paper, we make use of the following notation.  $\Omega$  denotes a compact subset of  $\mathbb{R}^q$ ,  $q \in \mathbb{N}_{\geq 1}$ . Given  $n \in \mathbb{N}_{\geq 1}$  and  $\bar{d} \in \mathbb{R}_{\geq 0}$ ,  $\mathcal{U}^n := C^0(\mathbb{R}_{\geq 0}, \mathcal{F}^n)$ ,  $\mathcal{F}^n := L_2(\Omega, \mathbb{R}^n)$ ,  $\mathcal{C}^n := C^0([- \bar{d}; 0], \mathcal{F}^n)$  if  $\bar{d} > 0$ , and  $\mathcal{C}^n := \mathcal{F}^n$  if  $\bar{d} = 0$ . Accordingly, we defined the norms  $\|\cdot\|_{\mathcal{F}}$  and  $\|\cdot\|_{\mathcal{C}}$  as  $\|x\|_{\mathcal{F}} := \sqrt{\int_{\Omega} x(r)^2 dr}$  for all  $x \in \mathcal{F}^n$  and  $\|x\|_{\mathcal{C}} := \sup_{t \in [- \bar{d}; 0]} \|x(t)\|_{\mathcal{F}}$  for all  $x \in \mathcal{C}^n$ . A continuous function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  belongs to class  $\mathcal{K}$  if it is increasing and satisfies  $\alpha(0) = 0$ .  $\alpha \in \mathcal{K}_{\infty}$  if  $\alpha \in \mathcal{K}$  and  $\lim_{s \rightarrow \infty} \alpha(s) = \infty$ . A continuous function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  belongs to class  $\mathcal{KL}$  if  $\beta(\cdot, t) \in \mathcal{K}$  for any  $t \in \mathbb{R}_{\geq 0}$  and, given any  $s \in \mathbb{R}_{\geq 0}$ ,  $\beta(s, \cdot)$  is non-increasing and tends to zero as its argument tends to infinity.

### A. Definition

We consider dynamical systems defined as

$$\dot{x}(t) = f(x_t, u(t)), \quad (1)$$

where  $f : \mathcal{C}^n \times \mathcal{F}^m \rightarrow \mathcal{F}^n$  and  $u \in \mathcal{U}^m$ .  $x(t) \in \mathcal{F}^n$  represents the instantaneous value of the state: at each time instant  $t$ , it is a *function* of the space variable rather than a single point of  $\mathbb{R}^n$ .  $x_t \in \mathcal{C}^n$  represents the history of this function over the latest time interval of length  $\bar{d}$ ; in other words, for each fixed  $\theta \in [-\bar{d}; 0]$ ,  $x_t(\theta) := x(t + \theta)$ . Given any  $t \in \mathbb{R}_{\geq 0}$ ,  $\dot{x}(t) \in \mathcal{F}^n$  is the function defined as  $r \mapsto \frac{\partial}{\partial t}([x(t)](r))$ . System (1) is thus ruled by a differential equation over the Banach space  $\mathcal{C}^n$ . From now on, unless explicitly stated,  $\mathcal{C}^n$ ,  $\mathcal{F}^n$ , and  $\mathcal{U}^m$  will be simply written as  $\mathcal{C}$ ,  $\mathcal{F}$ , and  $\mathcal{U}$ . Throughout the paper, we assume the following.

**Assumption 1 (Uniform regularity)** *f is locally Lipschitz and there exists  $\rho \in \mathcal{K}_\infty$  such that, for all  $\phi, \varphi \in \mathcal{C}$  and all  $v \in \mathcal{F}$ ,  $\|f(\phi, v) - f(\varphi, v)\|_{\mathcal{F}} \leq \rho(\|\phi - \varphi\|_{\mathcal{C}})$ .*

We stress that Assumption 1 ensures that  $f$  maps bounded sets of  $\mathcal{C} \times \mathcal{F}$  to bounded sets of  $\mathcal{F}$ . In other words,  $f$  is completely continuous.

Uniform incremental stability was originally studied in a finite-dimensional context [2]. A natural extension for the class of systems (1) is as follows.

**Definition 1 ( $\delta$ GAS)** *The system (1) is uniformly incrementally stable ( $\delta$ GAS) if it is forward complete and there exists  $\beta \in \mathcal{KL}$  such that, for any pair of initial conditions  $x_0, y_0 \in \mathcal{C}$  and any input  $u \in \mathcal{U}$ , the corresponding solutions of (1) satisfy, for all  $t \in \mathbb{R}_{\geq 0}$ ,*

$$\|x(t; x_0, u) - x(t; y_0, u)\|_{\mathcal{F}} \leq \beta(\|x_0 - y_0\|_{\mathcal{C}}, t). \quad (2)$$

Similarly to incremental stability for finite-dimensional systems [2] and time-delay systems [8], [13],  $\delta$ GAS imposes that any two solutions with the same applied input eventually converge to one another at a rate uniform in the applied input, and that the maximum distance between them in the transients is small if the distance between initial states is small. Similar ingredients are present in the input-output formulation of incremental stability [19], [18], [36].

### B. Krasovskii-Lyapunov approach for incremental stability

Given  $V : \mathcal{C} \rightarrow \mathbb{R}$  locally Lipschitz, we indicate its upper-right Dini derivative along solutions of (1) by

$$\dot{V}^{(1)}(t, \phi, u) := \limsup_{h \rightarrow 0^+} \frac{V(x_{t+h}) - V(x_t)}{h},$$

on the interval of existence of the solution  $x(\cdot)$  of (1) with initial state  $\phi \in \mathcal{C}$  and input  $u \in \mathcal{U}$ . As discussed in [22], Assumption 1 and the fact that all considered inputs are continuous in time ensure absolute continuity of the map  $t \mapsto x_t$ . Consequently, the map  $t \mapsto V(x_t)$  is also absolutely continuous, ensuring that the negativity of  $\dot{V}$  for almost all  $t$  in an interval  $[t_1; t_2]$  implies  $V(x_{t_2}) < V(x_{t_1})$ . Reasoning as

in [31], it might be possible to extend the results presented here to merely measurable locally essentially bounded inputs: see [7] for a spatiotemporal version of this observation.

The following result provides a way to establish  $\delta$ GAS of spatiotemporal delayed dynamics.

**Theorem 1 (Lyapunov-Krasovskii for  $\delta$ GAS)** *Assume that there exist  $\underline{\alpha}, \bar{\alpha} \in \mathcal{K}_\infty$ ,  $\alpha \in \mathcal{K}$ , and a locally Lipschitz functional  $V : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  such that, for any  $x_0, y_0 \in \mathcal{C}$  and any  $u \in \mathcal{U}$ , the solutions of (1)  $x(\cdot) := x(\cdot; x_0, u)$  and  $y(\cdot) := x(\cdot; y_0, u)$  satisfy, for almost all  $t \in \mathbb{R}_{\geq 0}$ ,*

$$\underline{\alpha}(\|x(t) - y(t)\|_{\mathcal{F}}) \leq V(x_t, y_t) \leq \bar{\alpha}(\|x_t - y_t\|_{\mathcal{C}}) \quad (3)$$

$$\dot{V}^{(1)}(t, x_0, y_0, u) \leq -\alpha(\|x(t) - y(t)\|_{\mathcal{F}}). \quad (4)$$

*Then, under Assumption 1, the system (1) is  $\delta$ GAS.*

A specificity of the above result lies in the fact that the dissipation rate in (4) involves the  $\mathcal{F}$ -norm of the incremental state error, rather than its  $\mathcal{C}$ -norm. In other words, it depends on the current value of the incremental state error rather than its prehistory. If the right-hand side of (4) was  $-\alpha(\|x_t - y_t\|_{\mathcal{C}})$ , then it would hold that  $\dot{V}^{(1)} \leq -\alpha \circ \bar{\alpha}^{-1}(V)$  and uniform incremental stability would readily follow from the comparison lemma. Here, the proof is slightly more involved, but follows along the lines of the original Lyapunov-Krasovskii approach [20], [24]. Due to page limits, this proof is moved to the on-line extended version of this work [11].

### C. Entrainment

One interesting feature of  $\delta$ GAS systems is their capability to be entrained by their input.

**Definition 2 (Entrainment)** *The system (1) is said to be entrained by its input if, given any  $T \geq 0$  and any  $T$ -periodic<sup>1</sup>  $u \in \mathcal{U}$ , there exists a  $T$ -periodic solution  $\bar{x}^u : \mathbb{R}_{\geq 0} \rightarrow \mathcal{F}$  of (1) such that, for any  $x_0 \in \mathcal{C}$ ,*

$$\lim_{t \rightarrow \infty} \|x(t; x_0, u) - \bar{x}^u(t)\|_{\mathcal{F}} = 0.$$

The entrainment property thus ensures that, after transients, the system evolves in a periodic fashion at the rhythm of the applied input. We stress that entrainment ensures in particular that, in response to any constant input, there exists a unique equilibrium to which all solutions converge.

In line with [8] with an incremental stability approach, [32], [29] with a convergent systems approach, and [35], [25] with a contraction approach, the following result provides a condition under which  $\delta$ GAS ensures entrainment.

**Theorem 2 (Entrainment by periodic inputs)** *Assume that Assumption 1 holds and that (1) is  $\delta$ GAS. If there exists a continuous function  $\sigma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that, given any  $x_0 \in \mathcal{C}$  and any periodic  $u \in \mathcal{U}$ , the solution of (1) satisfies*

$$\limsup_{t \rightarrow \infty} \|x(t)\|_{\mathcal{F}} \leq \sigma\left(\sup_{t \geq 0} \|u(t)\|_{\mathcal{F}}\right), \quad (5)$$

<sup>1</sup>When  $T = 0$ ,  $u$  is called  $T$ -periodic if it is constant.

then (1) is entrained by its input  $u$ .

The proof, provided in [11], relies on the fact that  $\delta\text{GAS}$  ensures entrainment provided that a compact set, whose size may depend on the input magnitude, attracts all solutions.

### III. DELAYED NEURAL FIELDS

#### A. Considered dynamics

We now focus on a particular class of spatiotemporal dynamics for which the entrainment property constitutes an interesting feature. Delayed neural fields are integro-differential equations representing the spatiotemporal activity of neuronal populations and accounting for the non-instantaneous communication between neurons. They read:

$$\begin{aligned} \tau_i \frac{\partial z_i}{\partial t}(r, t) &= -z_i(r, t) + \\ S_i \left( \sum_{j=1}^n \int_{\Omega} w_{ij}(r, r') z_j(r', t - d_j(r, r')) dr' + I_i^{\text{ext}}(r, t) \right), \end{aligned} \quad (6)$$

for each  $i \in \{1, \dots, n\}$ .  $\Omega$  is a compact set of  $\mathbb{R}^q$ ,  $q \in \{1, 2, 3\}$ , representing the physical support of the populations.  $r, r' \in \Omega$  are the space variables, whereas  $t \in \mathbb{R}_{\geq 0}$  is the time variable.  $z_i(r, t) \in \mathbb{R}$  represents the neuronal activity of population  $i$  at position  $r$  and at time  $t$ .  $\tau_i > 0$  is the time decay constant of the activity of population  $i$ .  $w_{ij}(r, r')$  describes the synaptic strength between location  $r'$  in population  $j$  and location  $r$  in population  $i$ ; it is assumed that  $w_{ij} \in L_2(\Omega \times \Omega, \mathbb{R})$ .  $I_i^{\text{ext}} : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  is a continuous function describing the external input of population  $i$ , arising either from the influence of exogenous cerebral structures or from an artificial stimulation device. The function  $d_j : \Omega \times \Omega \rightarrow [0; \bar{d}]$ ,  $\bar{d} \geq 0$ , is continuous;  $d_j(r, r')$  represents the axonal, dendritic and synaptic delays between a pre-synaptic neuron at position  $r'$  in population  $j$  and a post-synaptic neuron at position  $r$ . It typically depends on the distance separating the two considered positions, namely  $|r - r'|$ .  $S_i : \mathbb{R} \rightarrow \mathbb{R}$  is a globally Lipschitz function, known as the activation function of the neural population  $i$ .

Neural fields have been introduced in [1], originally without considering delays. Since then, they have been employed in a number of computational neuroscience studies, ranging from the understanding of brain functions to the development of neuromorphic architectures [5]. A reason explaining this success probably stands in the good compromise they offer between richness of possible behaviors and mathematical tractability. In particular, a Lyapunov framework has been developed to study the stability of stationary patterns [14], [15]. Qualitative behavior can also be assessed through bifurcation theory [4], [38]. Delayed neural fields have also been the subject of stability and robustness studies, using linearization techniques [3] and a spatiotemporal extension of the Lyapunov-Krasovskii approach [16], [39], [7]. This mathematical background constitutes a fertile ground for the analysis of incremental stability of delayed neural fields, and particularly their ability to be entrained by a periodic input.

#### B. Delay-independent condition for $\delta\text{GAS}$

Let  $[x(t)](\cdot) := z(\cdot, t)$ ,  $[u(t)](\cdot) := I^{\text{ext}}(\cdot, t)$ , and  $f = (f_1, \dots, f_n)^T$  where each  $f_i$ ,  $i \in \{1, \dots, n\}$ , is defined for all  $\phi \in \mathcal{C}$  and all  $v \in \mathcal{F}$  as

$$\begin{aligned} f_i(\phi, v) &:= \frac{1}{\tau_i} \left[ -\phi(0) \right. \\ &\left. + S_i \left( \sum_{j=1}^n \int_{\Omega} w_{ij}(\cdot, r) [\phi_j(-d_j(\cdot, r))](r) dr + v_i \right) \right]. \end{aligned} \quad (7)$$

Then the delayed neural fields (6) take the form (1). Applying Theorem 1 to this particular dynamics leads to the following explicit condition for the  $\delta\text{GAS}$  of delayed neural fields.

**Theorem 3 ( $\delta\text{GAS}$  of delayed neural fields)** *For each  $i \in \{1, \dots, n\}$ , let  $S_i$  be globally Lipschitz with Lipschitz constant  $\ell_i$ . Then, under the condition*

$$\sum_{i,j=1}^n \ell_i^2 \int_{\Omega} \int_{\Omega} w_{ij}(r, r')^2 dr' dr < 1, \quad (8)$$

*the delayed neural fields (6) are  $\delta\text{GAS}$ .*

To the best of our knowledge, this result is new even in the non-delayed case ( $\bar{d} = 0$ ). Roughly speaking, condition (8) can be interpreted as follows:  $\delta\text{GAS}$  holds if the  $L_2$  norm of the synaptic gains distributions is below the inverse of the Lipschitz constant of the activation functions. It therefore imposes a trade-off between synaptic strength (as measured through  $w_{ij}$ ) and excitability (as measured through  $\ell_i$ ).

The exact same condition was proposed in [16] to establish the existence of a globally asymptotically stable equilibrium configuration for any given constant input. In view of Theorem 2, the above statement allows to go beyond that conclusion by considering periodic inputs.

**Corollary 1 (Entrainment of neural fields)** *Assume that the assumptions Theorem 3 hold and that the activation functions  $S_i$ ,  $i \in \{1, \dots, n\}$ , are bounded. Then the delayed neural fields (6) are entrained by their input  $I^{\text{ext}}$ .*

The proof of this result, provided in [11], relies on Theorems 2 and 3 by observing that the boundedness of  $S_i$  ensures ultimate boundedness of solutions.

### IV. EXAMPLE

In order to illustrate our theoretical findings, we make use of the model introduced in [12]. This model involves two brain structures: the subthalamic nucleus (STN), mostly excitatory, and the external part of the globus pallidus (GPe), mostly inhibitory. These two structures are believed to be involved in the generation of beta oscillations (13-30Hz), the intensity of which correlates with the severity of parkinsonian motor symptoms [21]. More precisely, [27] and [28] showed that too strong coupling between STN and GPe may yield beta oscillations through an instability process. Here, we advocate that another mechanism might be at work that relies on entrainment: the preferential amplification of

exogenous oscillations in the beta band. The spatiotemporal dynamics of these two structures can be grasped by:

$$\tau_1 \frac{\partial z_1}{\partial t}(r, t) = -z_1(r, t) + \quad (9a)$$

$$S_1 \left( \int_{\Omega} w_{12}(r, r') z_2(r', t - d_2(r, r')) dr' + I_1^{\text{ext}}(r, t) \right),$$

$$\tau_2 \frac{\partial z_2}{\partial t}(r, t) = -z_2(r, t) + \quad (9b)$$

$$S_2 \left( \int_{\Omega} w_{21}(r, r') z_1(r', t - d_1(r, r')) dr' + \int_{\Omega} w_{22}(r, r') z_2(r', t - d_2(r, r')) dr' + I_2^{\text{ext}}(r, t) \right),$$

where  $I_1^{\text{ext}}$  denotes the cortical input, whereas  $I_2^{\text{ext}}$  represents inputs from the striatum. The physical space  $\Omega = [0, 15]$  mm is divided in two regions:  $\Omega_1 = [0, 2.5]$  mm corresponding to STN, and  $\Omega_2 = [12.5, 15]$  mm corresponding to GPe. The time constants are picked as  $\tau_1 = 6$  ms and  $\tau_2 = 14$  ms. The activation functions  $S_i$ ,  $i \in \{1, 2\}$ , are taken as sigmoids, namely  $S_i(x) = m_i b_i / (b_i + (m_i - b_i) e^{-4x/m_i})$ , where  $m_1 = 300$ ,  $b_1 = 17$ ,  $m_2 = 400$  and  $b_2 = 75$ . These functions are bounded and globally Lipschitz with Lipschitz constants  $\ell_i = 1$ . The delays  $d_i$  are given by  $d_i(r, r') = \frac{|r - r'|}{c_i}$ , for all  $r, r' \in \Omega$ , where  $c_i$  denotes the conductance velocity for neurons projecting from population  $i$  ( $c_1 = 2.5$  m/s and  $c_2 = 1.4$  m/s). The synaptic weights are taken as

$$\begin{aligned} w_{12}(r, r') &= -g_{12}(|r - r' - \mu_2|), \quad \forall r \in \Omega_1, r' \in \Omega_2 \\ w_{21}(r, r') &= g_{21}(|r - r' - \mu_1|), \quad \forall r \in \Omega_2, r' \in \Omega_1, \\ w_{22}(r, r') &= -|r - r'| g_{22}(|r - r'|), \quad \forall r, r' \in \Omega_2, \end{aligned}$$

and zero anywhere else, where  $\mu_1 = 1.25$  mm and  $\mu_2 = 13.25$  mm correspond the centers of STN and GPe respectively, and the functions  $g_{ij}$  are Gaussian functions:  $g_{ij}(x) := k_{ij} \exp(-x^2/2\sigma_{ij})$  with  $\sigma_{12} = \sigma_{21} = 0.03$ , and  $\sigma_{22} = 0.015$ . All these parameters are spatiotemporal extensions of those in [27], that were derived based on experimental evidence: please refer to [12] for more details.

By picking  $I_1^{\text{ext}}(r, t) = I_1^*(r)$  and  $I_2^{\text{ext}}(r, t) = I_2^*(r) + v(r, t)$ , where  $I_1^*$  and  $I_2^*$  are chosen in such a way that the system exhibits an equilibrium at the point where the slopes of  $S_1$  and  $S_2$  are maximum, this dynamics can be reformulated, after a change of variables, as

$$\tau_1 \frac{\partial z_1}{\partial t} = -z_1 + S_1 \left( \int_{\Omega} w_{12}(r, r') z_2(r', t - d_2) dr' \right), \quad (10a)$$

$$\begin{aligned} \tau_2 \frac{\partial z_2}{\partial t} &= -z_2 + S_2 \left( \int_{\Omega} w_{21}(r, r') z_1(r', t - d_1) dr' \right. \\ &\quad \left. + \int_{\Omega} w_{22}(r, r') z_2(r', t - d_2) dr' + v(r, t) \right), \quad (10b) \end{aligned}$$

where some variables have been omitted for the sake of notation compactness. For each  $i \in \{1, 2\}$ ,  $S_i(x) := S_i(x - a_i) - S_i(a_i)$ , where  $a_i$  is the point at which the slope of  $S_i$  is maximum. For simulation purposes, we discretize the space  $\Omega$  in 60 segments of equal length. By picking  $k_{12} = 7$ ,

$k_{21} = 10.5$  and  $k_{22} = 3.0$ , the quantity

$$\bar{w} := \int_{\Omega} \int_{\Omega} \left( w_{12}(r, r')^2 + w_{21}(r, r')^2 + w_{22}(r, r')^2 \right) dr' dr,$$

equals 0.97, thus making (8) fulfilled. Theorem 3 then ensures that (10) is  $\delta$ GAS and Corollary 1 guarantees that the dynamics are entrained by the striatal input  $v$ . We let  $v(r, t) = U \sin(\omega t)$  and we compute the ratio  $\bar{z}_2(\omega)/U$  in dB, for various frequencies  $\omega$  between 1 and 1000 rad/s, where  $\bar{z}_2(\omega)$  denotes the steady-state magnitude of the oscillations of the spatial average of the GPe activity  $\sqrt{\frac{1}{\#\Omega_2} \int_{\Omega_2} z_2(r, t)^2 dr}$ , where  $\#\Omega_2 = 2.5$  denotes the measure of  $\Omega_2$ . We obtain the frequency profiles depicted in Figure 1 (solid lines). Varying the amplitude  $U$  of the striatal input, different frequency profiles are obtained due to the nonlinear nature of the dynamics. For  $U = 10$ , a slight resonance appears at around 150 rad/s, corresponding to the beta band. Increasing the synaptic coupling ( $k_{12} = 21.5$ ,  $k_{21} = 22.5$ , and  $k_{22} = 20.5$ ) simulations indicate that (10) remains entrained by its input although  $\bar{w} = 6 > 1$  thus violating condition (8). The corresponding frequency profiles appear in dashed lines, for the same input magnitudes as before. The beta resonance is much more pronounced.

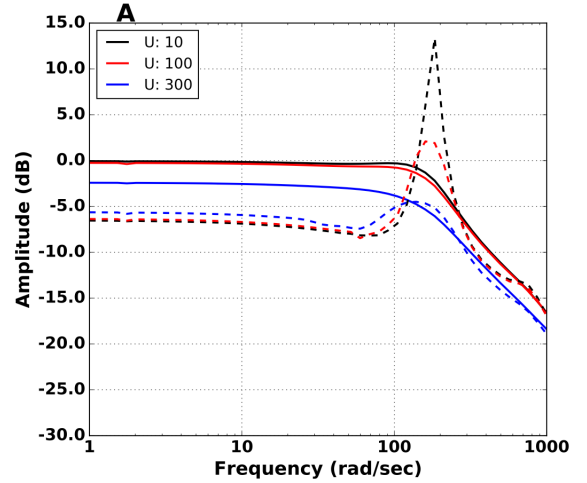


Fig. 1. Frequency responses of (10) for  $\bar{w} = 0.97$  (solid lines) and  $\bar{w} = 6$  (dashed lines), and for different input amplitudes  $U$ .

When synaptic coupling is increased even more ( $k_{12} = 23.5$ ,  $k_{21} = 26$ , and  $k_{22} = 20.5$ ), corresponding to  $\bar{w} = 7.5$ , the spatiotemporal response of Figure 2 shows that endogenous oscillations take place (still in the beta band) even though the applied input is constant ( $v(r, t) = 50$ ). Consequently, the system is no longer entrained by its input and, in view of Theorem 2,  $\delta$ GAS does not hold anymore.

## V. CONCLUSION

Incremental stability is thus a powerful tool to ensure that a system is entrained by its input, and to derive frequency profiles, even for spatiotemporal delayed dynamics. The sufficient condition proposed for  $\delta$ GAS of delayed neural

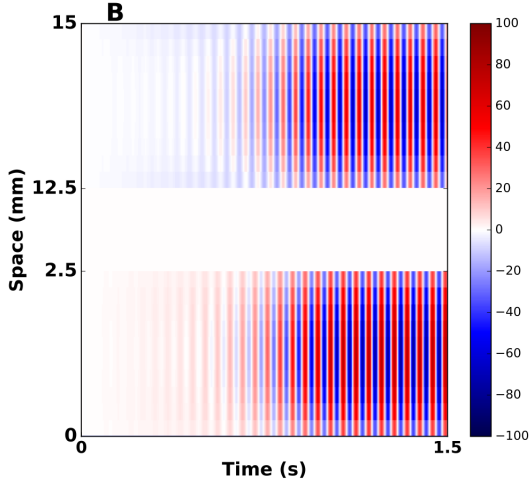


Fig. 2. spatiotemporal evolution of (10) with constant input for  $\bar{w} = 7.5$ .

fields can easily be tested based on the system parameters. However, simulations indicate that this condition is not tight. In the future, deriving delay-dependent condition for  $\delta$ GAS of spatiotemporal dynamics may help reducing this conservatism. The results of this paper also plead for the development of control tools to tune the frequency profiles of the system, for instance to reduce amplification in a targeted frequency band.

## VI. PROOF OF THEOREM 3

First observe that Assumption 1 is satisfied since the functions  $S_i$ ,  $i \in \{1, \dots, n\}$ , are globally Lipschitz. We consider the functional  $V$  defined, for all  $\phi, \varphi \in \mathcal{C}$ , as

$$V(\phi, \varphi) = \frac{1}{2} \sum_{i=1}^n \tau_i \|\phi_i(0) - \varphi_i(0)\|_{\mathcal{F}}^2 + \sum_{i=1}^n \int_{\Omega} \gamma(r) \int_{\Omega} \int_{-d_i(r, r')}^0 \left( [\phi_i(\theta)](r') - [\varphi_i(\theta)](r') \right)^2 d\theta dr' dr,$$

where  $\gamma$  is a bounded function to be chosen latter. Given any continuous  $I^{\text{ext}}$ , let  $u \in \mathcal{U}$  be defined as  $u(t) := I^{\text{ext}}(t, \cdot)$ . Consider any  $x_0, y_0 \in \mathcal{C}$  and let  $x(\cdot)$  and  $y(\cdot)$  denote the corresponding solutions of (6). Elementary computations show that

$$\underline{c} \|x(t) - y(t)\|_{\mathcal{F}}^2 \leq V(x_t, y_t) \leq \bar{c} \|x_t - y_t\|_{\mathcal{C}}^2,$$

where  $\underline{c} := \frac{1}{2} \min_{i=1, \dots, n} \tau_i$  and  $\bar{c} := \max_{i=1, \dots, n} \tau_i + \#\Omega \max_{r \in \Omega} \gamma(r)$ . This establishes (3). In view of [16, Theorem 3.2.1],  $x(\cdot)$  and  $y(\cdot)$  are uniquely defined and continuously differentiable on  $\mathbb{R}_{\geq 0}$ . It follows that the map  $\mathcal{V} : t \mapsto V(x_t, y_t)$  is continuously differentiable on  $\mathbb{R}_{\geq 0}$ . Consequently, its upper-right Dini derivative coincides with its classical derivative, which reads

$$\dot{V}(t) = \sum_{i=1}^n \dot{V}_i(t) + \dot{W}_i(t), \quad (11)$$

with

$$\mathcal{V}_i(t) := \frac{\tau_i}{2} \|x_i(t) - y_i(t)\|_{\mathcal{F}}^2$$

$$\mathcal{W}_i(t) := \int_{\Omega} \gamma(r) \int_{\Omega} \int_{t-d_i}^t \left( [x_i(\theta)](r') - [y_i(\theta)](r') \right)^2 d\theta dr' dr,$$

where the spatial dependency of the delays  $d_i$  has been omitted in the notation. In order to lighten the notation, we let  $e_i := x_i - y_i$ . The derivative of  $\mathcal{V}_i$  reads

$$\begin{aligned} \dot{\mathcal{V}}_i &= \frac{d}{dt} \left( \frac{\tau_i}{2} \int_{\Omega} |[x_i(t)](r) - [y_i(t)](r)|^2 dr \right) \\ &\leq - \int_{\Omega} [e_i(t)](r)^2 dr + \int_{\Omega} [e_i(t)](r) \times \\ &\quad \left[ S_i \left( \sum_{j=1}^n \int_{\Omega} w_{ij} [x_j(t-d_j)](r') dr' + [u(t)](r) \right) \right. \\ &\quad \left. - S_i \left( \sum_{j=1}^n \int_{\Omega} w_{ij} [y_j(t-d_j)](r') dr' + [u(t)](r) \right) \right] dr. \end{aligned}$$

Using the fact that  $|S_i(a-b)| \leq \ell_i |a-b|$  for all  $a, b \in \mathbb{R}$  and Cauchy-Schwarz inequality, it follows that

$$\begin{aligned} \dot{\mathcal{V}}_i &\leq - \|e_i(t)\|_{\mathcal{F}}^2 \\ &\quad + \ell_i \int_{\Omega} |[e_i(t)](r)| \left| \sum_{j=1}^n \int_{\Omega} w_{ij} [e_j(t-d_j)](r') dr' \right| dr \\ &\leq - \|e_i(t)\|_{\mathcal{F}}^2 \\ &\quad + \ell_i \|e_i(t)\|_{\mathcal{F}} \sqrt{\int_{\Omega} \left( \sum_{j=1}^n \int_{\Omega} w_{ij} [e_j(t-d_j)](r') dr' \right)^2 dr} \\ &\leq - \frac{1}{2} \|e_i(t)\|_{\mathcal{F}}^2 + \frac{\ell_i^2}{2} \int_{\Omega} \left( \sum_{j=1}^n \int_{\Omega} w_{ij} [e_j(t-d_j)](r') dr' \right)^2 dr, \end{aligned}$$

where we used the fact that  $ab \leq (a^2 + b^2)/2$  to get the last bound. Using again Cauchy-Schwarz inequality, we get that

$$\begin{aligned} \dot{\mathcal{V}}_i &\leq - \frac{1}{2} \|e_i(t)\|_{\mathcal{F}}^2 \\ &\quad + \frac{\ell_i^2}{2} \int_{\Omega} \left( \sum_{j=1}^n \sqrt{\int_{\Omega} w_{ij}^2 dr'} \int_{\Omega} [e_j(t-d_j)](r')^2 dr' \right)^2 dr \\ &\leq - \frac{1}{2} \|e_i(t)\|_{\mathcal{F}}^2 \\ &\quad + \frac{\ell_i^2}{2} \int_{\Omega} \left( \sum_{j=1}^n \int_{\Omega} w_{ij}^2 dr' \right) \left( \sum_{j=1}^n \int_{\Omega} [e_j(t-d_j)](r')^2 dr' \right) dr \\ &\leq - \frac{1}{2} \|e_i(t)\|_{\mathcal{F}}^2 + \frac{1}{2} \int_{\Omega} \bar{w}_i(r) \left( \sum_{j=1}^n \int_{\Omega} [e_j(t-d_j)](r')^2 dr' \right) dr, \end{aligned}$$

where  $\bar{w}_i(r) := \ell_i^2 \sum_{j=1}^n \int_{\Omega} w_{ij}(r, r')^2 dr'$ . It follows that

$$\begin{aligned} \sum_{i=1}^n \dot{\mathcal{V}}_i(t) &\leq - \frac{1}{2} \|e(t)\|_{\mathcal{F}}^2 \\ &\quad + \frac{1}{2} \int_{\Omega} \left( \sum_{i=1}^n \bar{w}_i(r) \right) \left( \sum_{j=1}^n \int_{\Omega} [e_j(t-d_j)](r')^2 dr' \right) dr. \end{aligned} \quad (12)$$

Similarly, the derivative of  $\mathcal{W}_i$  reads

$$\begin{aligned}\dot{\mathcal{W}}_i(t) &:= \frac{d}{dt} \int_{\Omega} \gamma(r) \int_{\Omega} \int_{t-d_i}^t [e_i(\theta)](r')^2 d\theta dr' dr \\ &= \int_{\Omega} \gamma(r) \int_{\Omega} \left( [e_i(t)](r')^2 - [e_i(t-d_i)](r')^2 \right) dr' dr \\ &= \|e_i(t)\|_{\mathcal{F}}^2 \int_{\Omega} \gamma(r) dr - \int_{\Omega} \gamma(r) \int_{\Omega} [e_i(t-d_i)](r')^2 dr' dr.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{i=1}^n \dot{\mathcal{W}}_i(t) &= \|e(t)\|_{\mathcal{F}}^2 \int_{\Omega} \gamma(r) dr \\ &\quad - \int_{\Omega} \gamma(r) \left( \sum_{i=1}^n \int_{\Omega} [e_i(t-d_i)](r')^2 dr' \right) dr.\end{aligned}\quad (13)$$

Combing (11), (12), and (13), we obtain that

$$\begin{aligned}\dot{\mathcal{V}}(t) &\leq - \left( \frac{1}{2} - \int_{\Omega} \gamma(r) dr \right) \|e(t)\|_{\mathcal{F}}^2 \\ &\quad - \int_{\Omega} \left[ \left( \gamma(r) - \sum_{i=1}^n \frac{\bar{w}_i(r)}{2} \right) \int_{\Omega} [e_j(t-d_j)](r')^2 dr' \right] dr.\end{aligned}$$

So the assumptions of Theorem 1 are fulfilled provided that  $\int_{\Omega} \gamma(r) < 1/2$  and  $\gamma(r) \geq \sum_{i=1}^n \frac{\bar{w}_i(r)}{2}$ . By picking  $\gamma(r) = \sum_{i=1}^n \frac{\bar{w}_i(r)}{2}$ , the first condition becomes  $\sum_{i,j=1}^n \ell_i^2 \int_{\Omega} \int_{\Omega} w_{ij}(r, r')^2 dr' dr < 1$ , which corresponds to condition (8).

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