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This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2016.2614888, IEEE Transactions on Automatic Control

Immersion and Invariance Stabilization of Nonlinear Systems via Virtual and Horizontal Contraction

Lei Wang[1], Fulvio Forni[2], Romeo Ortega[3], Zhitao Liu[1] and Hongye Su[1]

***Abstract*—The main objective of this paper is to revisit one of the key steps of immersion and invariance stabilizing controller design. Namely, the one that ensures attractivity of the manifold whose internal dynamics contains a copy of the desired system behavior. Towards this end we invoke *contraction theory* principles and propose two alternative procedures to carry out this step: (i) to replace attractivity of the manifold by *virtual contraction* of the off–the–manifold coordinate and (ii) to ensure the attractivity of the manifold rendering it *horizontally contractive*. This makes more systematic the design with more explicit degrees of freedom to accomplish the task. Several examples, including the classical case of systems in feedback form, are used to illustrate the proposed design.**

I. INTRODUCTION

Immersion and invariance (I&I) is a controller design technique to stabilize non-linear systems proposed in [1]—see also [2] where many practical applications are presented. The I&I approach captures the desired behavior of the system to be controlled by introducing a target dynamical system. Then, a suitable stabilizing control law is designed to guarantee that the controlled system asymptotically behaves like the target system. More precisely, the I&I methodology relies on generating a manifold in the plant state–space that can be rendered *invariant and attractive* by feedback control, such that (i) on the manifold, the closed loop dynamics behaves like the desired dynamics (ii) away from the manifold, the control law steers the state of the system towards the manifold. The usual way to carry out the latter step is to define an extended dynamical system given by a copy of the plant and by a new *error* dynamics that describes the behavior of the off–the–manifold coordinate. Then, a feedback law must be designed to ensure, on one hand, *boundedness* of the plant state while, on the other hand, guaranteeing *convergence to zero* of the off–the–manifold coordinate. The main stabilization result in I&I states that the evaluation of this control law on the manifold defines an asymptotically stabilizing controller for the system.

Given its unusual specifications no systematic procedure to design this feedback law is available in the literature—hampering the application of I&I in several practical applications. The main objective of this paper is to propose to carry out this step by exploiting *contraction theory* principles. More precisely, two kinds of contraction theories, *virtual contraction* [5] and *horizontal contraction* [3], are applied to deal with this problem. In the first case we replace attractivity of the manifold by virtual contraction of the off–the–manifold coordinate, while in the second one we ensure attractivity of the manifold rendering it horizontally contractive. The main advantage is to make more systematic the last step of I&I controller designs—widening its use in applications. We anticipate that the stabilization of the extended system of I&I is replaced by the stabilization of the prolonged system, defined by the plant and its linearization.

The reduction theory elaborated in [7], [8] provides an alternative framework to complete the I&I design. Indeed, in reduction theory it is asked whether a point on a closed set can be rendered asymptotically stable if all solutions on the closed set approach that point and all other solutions at least approach the closed set. Clearly, this is closely related to the issues addressed in the present paper and has the important advantage that, in contrast with contraction theory, it avoids the construction of Lyapunov functions, which was a motivating argument for the introduction of I&I.

The paper is organized as follows. Section II recalls the main stabilization result of I&I control, including an additional assumption, unfortunately, overlooked in in the original works of I&I [1], [2]. The novel designs based on virtual and horizontal contraction are presented in Sections III and IV, respectively. The application of horizontal contraction is illustrated with the classical

*This work is partially supported by National Basic Research Program of China (973 Program 2013CB035406) ; National Natural Science Foundation of China (NSFC: 61134007, 61223004, 61433013). (*Corresponding author: Zhitao Liu.*)

[1]L. Wang, Z. Liu and H. Su are with State Key Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, 310027, Hangzhou, P.R.China and CASY. lwang[ztliu]@iipc.zju.edu.cn

[2]F. Forni is with the University of Cambridge, Department of Engineering, Trumpington Street, CB2 1PZ, Cambridge, UK f.forni@eng.cam.ac.uk.

[3]R. Ortega is with Laboratoire des Signaux et Systèmes, CNRS UMR 8506, Supélec, 91192 Gif sur Yvette, France ortega@lss.supelec.fr

example of systems in feedback form in Section V. Concluding remarks are given in Section VI. Proofs of the main propositions are given in the appendices.

Notation For $x \in \mathbb{R}^n$ we denote the Euclidean norm $|x|^2 := x^\top x$. $\mathbb{R}^{n\times n}_{>0}$ is the set of $n \times n$ positive definite matrices. All the functions in the paper are assumed to be continuous and sufficiently smooth. Given a function $f : \mathbb{R}^n \to \mathbb{R}$ we define the differential operators $\nabla f := \left[\frac{\partial f}{\partial x}\right]^\top$, $\nabla_{x_i} f := \left[\frac{\partial f}{\partial x_i}\right]^\top$, where $x_i \in \mathbb{R}^p$ is an element of the vector x. When clear from the context the subindex of the operator ∇ and the arguments of the functions will be omitted.

II. I&I Stabilization Procedure

Consider the system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

with state $x \in \mathbb{R}^n$, control $u \in \mathbb{R}^m$, and an assignable equilibrium point $x_* \in \{x \in \mathbb{R}^n \mid g^\perp(x)f(x) = 0\}$ to be *stabilized*, where $g^\perp : \mathbb{R}^n \to \mathbb{R}^{(n-m)\times n}$ is a full–rank left annihilator of $g(x)$. Stabilization is achieved in I&I fulfilling the following four steps [2].

Proposition 1: Assume that there exist mappings

$$\alpha : \mathbb{R}^p \to \mathbb{R}^p, \quad \pi : \mathbb{R}^p \to \mathbb{R}^n, \quad c : \mathbb{R}^n \to \mathbb{R}^m,$$
$$\phi : \mathbb{R}^n \to \mathbb{R}^{n-p}, \quad v : \mathbb{R}^n \times \mathbb{R}^{n-p} \to \mathbb{R}^m,$$

with $p < n$, such that the following hold.

(A1) *(Target system)* The system

$$\dot{\xi} = \alpha(\xi)\ , \tag{2}$$

has a globally asymptotically stable equilibrium at $\xi_* \in \mathbb{R}^p$ and $x_* = \pi(\xi_*)$.

(A2) *(Manifold invariance condition)* For all $\xi \in \mathbb{R}^p$,

$$f(\pi(\xi)) + g(\pi(\xi))c(\pi(\xi)) = \nabla\pi(\xi)\alpha(\xi)\ . \tag{3}$$

(A3) *(Implicit manifold description)* The following set identity holds

$$\mathcal{M} := \{x \in \mathbb{R}^n | x = \pi(\xi)\} = \{x \in \mathbb{R}^n | \phi(x) = 0\}\ . \tag{4}$$

(A4) *(Manifold attractivity and trajectory boundedness)* Consider the system

$$\dot{x} = F(x,z) \tag{5}$$
$$\dot{z} = \Phi(x,z), \tag{6}$$

where we defined

$$F(x,z) := f(x) + g(x)v(x,z) \tag{7}$$
$$\Phi(x,z) = \nabla\phi(x)[f(x) + g(x)v(x,z)], \tag{8}$$

with the initial condition constraint

$$z(0) = \phi(x(0)), \tag{9}$$

and $v(\cdot,\cdot)$ and $c(\cdot)$ verifying

$$v(\pi(\xi), 0) = c(\pi(\xi)),\ \forall\xi \in \mathbb{R}^p. \tag{10}$$

All trajectories of the system are *bounded* and satisfy $\lim_{t\to\infty} z(t) = 0$.

Then, x_* is a *globally asymptotically stable* (GAS) equilibrium of the closed–loop system

$$\dot{x} = f(x) + g(x)v(x, \phi(x))\ . \tag{11}$$

⌟

We stress that, in comparison to the results presented in [1], [2], the initial condition constraint (9) and the requirement (10) have been added. The first condition ensures that $z(t) = \phi(x(t)),\ \forall t \geq 0$, while the second one guarantees that the x–system behaves like the ξ–system when restricted to the manifold $\mathcal{M}$. If these conditions are not imposed it is possible to show that the claim of Proposition 1 is false. Indeed, if the extra condition (10) is not imposed, it is not guaranteed that the actual dynamics on the desired manifold (*i.e.*, when $x = \pi(\xi)$) verifies (3), since these dynamics is given by $f(\pi(\xi)) + g(\pi(\xi))v(\pi(\xi), 0)$, which is different from $\nabla\pi(\xi)\alpha(\xi)$ if $v(\pi(\xi), 0) \neq c(\pi(\xi))$.

The counterexample below shows that I&I without the conditions (9) and (10) may lead to a closed–loop where x_* is not even an equilibrium.

Example 1: Consider the two–input two–state system

$$\dot{x}_1 = -x_1 + x_2 + a + u_1,\ \dot{x}_2 = u_2\ ,$$

with $a \neq 0$, $x_* = 0$, target dynamics $\dot{\xi} = -\xi$ and the mapping $\pi(\xi) = \mathrm{col}(\xi, 0)$, which clearly verify (A1) and (A2) with $c(\pi(\xi)) = \mathrm{col}(-a, 0)$. Condition (4) of Assumption (A3) is verified with $\phi(x) = x_2$. The extended dynamics (5), (6) reads

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + a + v_1(x,z) \\ \dot{x}_2 &= v_2(x,z) \\ \dot{z} &= v_2(x,z). \end{aligned} \tag{12}$$

The feedback law $v(x,z) = \mathrm{col}(0, -z)$, ensures x is bounded and $z(t) \to 0$. However, the closed–loop system (11)

$$\dot{x}_1 = -x_1 + x_2 + a,\ \dot{x}_2 = -x_2,$$

has a GAS equilibrium at $(a, 0)$ and not at the origin. ⌟

Remark 1: Notice that $\dot{z} = \dot{\phi}$ in the example, however, their trajectories (for the extended system (12)) are different, because the initial condition constraint (9) is not satisfied. Moreover, $v(\pi(\xi), 0) = \mathrm{col}(0,0) \neq c(\pi(\xi)) = \mathrm{col}(-a, 0)$, violating condition (10). ⌟

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III. The I&I Virtual Contraction Procedure

In this section we propose to replace the step (A4) in Proposition 1 by a virtual contraction based design.The new design is based on the following technical lemma.

Lemma 1: Consider the system (5), (6) together with

$$\frac{d}{dt}\delta z = \nabla_z \Phi(x,z)\delta z, \tag{13}$$

verifying the following conditions.

(i) $\Phi(x,0)=0$, uniformly in x.

(ii) (5) is forward complete for any bounded signal z.

(iii) There exists a mapping $P : \mathbb{R}^n \times \mathbb{R}^{n-p} \to \mathbb{R}^{(n-p)\times(n-p)}_{>0}$ and two positive constants κ, λ such that the Finsler-Lyapunov function

$$V(x,z,\delta z) := \delta z^\top P(x,z)\delta z, \tag{14}$$

verifies

$$\begin{aligned} V(x,z,\delta z) &\geq \kappa|\delta z|^2 && (15)\\ \dot V(x,z,\delta z) &\leq -\lambda V(x,z,\delta z) && (16)\end{aligned}$$

Then, $\lim_{t\to\infty} z(t) = 0$. ⌟

Equipped with Lemma 1 we can reformulate Proposition 1 as follows.

Proposition 2: Given the conditions (A1)–(A3) in Proposition 1 together with:

(A4') *(Manifold attractivity via virtual contraction)* Conditions (i)-(iii) in Lemma 1 hold, condition (10) is verified and the trajectories $x(\cdot)$ of the system (5), (6) are *bounded.*

Then, x_* is a GAS equilibrium of (11). ⌟

Remark 2: We notice that the application of reduction theory to replace Assumption (A4) requires the additional condition that $\dot x = F(x,0)$ is GAS—stemming from (i) in Theorem 10 of [7]. ⌟

Example 2: Consider the system

$$\dot x_1 = -x_1 + x_1^2 + x_1x_2,\ \dot x_2 = x_3,\ \dot x_3 = u, \tag{17}$$

with $x_* = 0$, target dynamics $\dot\xi = -\xi$ and $\pi(\xi) = \mathrm{col}(\xi,-\xi,\xi)$ that, with $c(\pi(\xi)) = -\xi$, verify conditions (A1) and (A2). Condition (A3) is verified with

$$\phi(x) = \mathrm{col}(x_1 + x_2,\ x_2 + x_3) \tag{18}$$

The dynamics of the off–the–manifold coordinate is given by

$$\begin{aligned} \dot z_1 &= -z_1 + x_1z_1 + z_2\\ \dot z_2 &= z_2 - z_1 + x_1 + u. \end{aligned} \tag{19}$$

Designing a control law that will verify condition (A4) for (17) and (19) in the standard I&I approach seems to be far from obvious. Therefore, we will try instead to satisfy condition (A4') of Proposition 2.

First, we compute the prolongation system (13)

$$\frac{d}{dt}\delta z = \begin{bmatrix} -1+x_1 & 1\\ -1+\nabla_{z_1}v & 1+\nabla_{z_2}v\end{bmatrix}\delta z.$$

Then, we make the observation that the only plant state that appears in the z and δz dynamics is x_1. This suggests to select the matrix P in the Finsler-Lyapunov function (14) depending only on this coordinate as

$$P(x_1) = \begin{bmatrix} 1+2x_1^2 & x_1\\ x_1 & 1\end{bmatrix},$$

which satisfies $P(x_1) \geq \frac{1}{2}I_2$ ensuring (15) with $\kappa = \frac{1}{2}$.

Now, the time derivative of V is given by

$$\begin{aligned}\dot V &= \delta z^\top[\dot P(x_1) + \nabla_z\Phi^\top(x_1,z)P(x_1)\\ &\quad + P(x_1)\nabla_z\Phi(x_1,z)]\delta z\,.\end{aligned}$$

Setting $\lambda = 2$ in (16) we obtain—after some lengthy, but straightforward, calculations—the simple equations

$$\begin{aligned}\nabla_{z_1}v(x_1,z_1,z_2) &= x_1 - 2x_1^2 - x_1z_1\\ \nabla_{z_2}v(x_1,z_1,z_2) &= -2 - x_1\,,\end{aligned}$$

where we have selected $v = v(x_1,z_1,z_2)$. Integrating the equations above we get

$$v(x_1,z_1,z_2) = x_1z_1 - 2x_1^2z_1 - \frac{1}{2}x_1z_1^2 - 2z_2 - x_1z_2 - x_1\,, \tag{20}$$

where we have added the last right hand term, that is $-x_1$, to ensure that (10) is satisfied. Notice that condition (i) of Lemma 1 is also satisfied.

To conclude that $\lim_{t\to\infty} z(t) = 0$ it only remains to prove condition (ii) of Lemma 1. Actually, we are going to prove not just that solutions exist, but that they are bounded completing the proof of the GAS claim. For, we combine (17) with (18)—and recall that $z = \phi(x)$—to rewrite the dynamics of x_2 as $\dot x_2 = -x_2 + z_2$, proving that x_2 is bounded for all bounded z_2. The proof of boundedness of x_1 and x_3 follows from (18) and boundendness of z_1.

The derivations above prove that the control law

$$\begin{aligned} v(x_1,\phi_1(x),\phi_2(x)) &= -\frac{5}{2}x_1^3 - 3x_1^2x_2 - \frac{1}{2}x_1x_2^2\\ &\quad + x_1^2 - x_1x_3 - x_1 - 2x_2 - 2x_3\,,\end{aligned}$$

applied to the system (17) ensures GAS of the zero equilibrium. ⌟

IV. The I&I Horizontal Contraction Procedure

In the Proposition below we propose a second alternative to replace the step (A4) in Proposition 1, this time invoking *horizontal* contraction principles. The approach adopted here is radically different from Proposition 2: the

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convergence to the desired sub-manifold is guaranteed by the design of a state-feedback control law enforcing contraction of a suitable horizontal metric [3, Theorem 3], without any use of the extended dynamics z.

Proposition 3: Given the conditions (A1)–(A3) in Proposition 1 together with the following.

(A4”) *(Manifold attractivity via horizontal contraction)* Assume there exist mappings

$$\begin{array}{rcl} P & : & \mathbb{R}^n \to \mathbb{R}^{(n-p)\times(n-p)}_{>0},\ R : \mathbb{R}^n \to \mathbb{R}^{(n-p)\times n} \\ \beta & : & \mathbb{R}^n \to \mathbb{R}^m,\ \rho : \mathbb{R}^n \to \mathbb{R}_{>0}, \end{array}$$

such that the following holds.

(A) $R(x)$ is full rank, uniformly in x, and

$$R(\pi(\xi)) = \nabla\phi(\pi(\xi)),\ \forall \xi \in \mathbb{R}^p. \tag{21}$$

(B) For all $\xi \in \mathbb{R}^p$ $\beta(\pi(\xi)) = c(\pi(\xi))$.

(C) The Finsler-Lyapunov function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ given by

$$V(x, \delta x) := \delta x^\top R^\top(x) P(x) R(x) \delta x, \tag{22}$$

satisfies

$$\dot{V}(x, \delta x) \leq -\rho(x) V(x, \delta x) \tag{23}$$

along the trajectories of the prolonged closed–loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\beta(x) &(24)\\ \frac{d}{dt}\delta x &= \Psi(x)\delta x, &(25) \end{aligned}$$

where we defined $\Psi(x) := \nabla[f(x) + g(x)\beta(x)]$.

(D) The trajectories of (24) are *bounded*.

Then, x_* is a GAS equilibrium of (24). ⌟

Remark 3: A natural choice for $R(x)$ is $\nabla\phi(x)$, provided that $\nabla\phi(x)$ is full rank. Notice also that, in contrast with classical I&I, Proposition 3 directly provides the static state–feedback controller $\beta(x)$. ⌟

Remark 4: Proposition 3 can be formulated in a similar way for any forward invariant region $\mathcal{C} \subseteq \mathbb{R}^n$—in which case we get regional stability. ⌟

Example 3: Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 + x_1 x_2 + x_1 x_3 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -x_3 + u \end{aligned} \tag{26}$$

with $x_* = 0$, target dynamics $\dot{\xi} = -\xi$. Choosing $c(\pi(\xi)) = 0$ and $\pi(\xi) = \text{col}(0, \xi, -\xi)$ the manifold invariance condition (A2) is satisfied. Condition (A3) is verified with $\phi(x) = \text{col}(x_1,\ x_2 + x_3)$, yielding the off–the–manifold coordinate dynamics

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_1^2 + z_1 z_2 \\ \dot{z}_2 &= u\,. \end{aligned} \tag{27}$$

Similarly to Example 2 designing a control law that will verify condition (A4) is far from obvious. Instead, we proceed to verify (A4”) of Proposition 3.

First, we compute the variational dynamics (25) as

$$\Psi(x) = \begin{bmatrix} -1 + 2x_1 + x_2 + x_3 & x_1 & x_1 \\ 0 & 0 & 1 \\ \nabla_1\beta & \nabla_2\beta & -1 + \nabla_3\beta \end{bmatrix}.$$

Second, since $\nabla\phi(x)$ is full rank, we set

$$R = \nabla\phi(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and then choose $P(x) = \Theta^\top(x)\Theta(x)$, with

$$\Theta(x) = \begin{bmatrix} (x_1+1)e^{x_1+x_2+x_3} & x_1 e^{x_1+x_2+x_3} \\ 1 & 1 \end{bmatrix}.$$

The derivative of the Finsler-Lyapunov function (22) yields

$$\dot{V} = 2\delta x^\top R^\top \Theta^\top(x)[\dot{\Theta}(x)R + \Theta(x)R\Phi(x)]\delta x\,.$$

To satisfy (23) we fix $\rho(x) = 2$ that yields the identity

$$\dot{\Theta}(x)R + \Theta(x)R\Phi(x) = -\Theta(x)R\,,$$

which is satisfied with

$$\beta(x) = -x_2 - x_3 - x_1^2 - x_1 x_2 - x_1 x_3\,, \tag{28}$$

It is easy to see that $\beta(\pi(\xi)) = 0$, hence condition (ii) is also satisfied.

To prove that the origin of the system (26) in closed–loop with (28) is GAS it only remains to verify the boundedness condition (D) of Assumption (A4”). Some simple calculations show that, introducing the partial coordinate $\eta := x_1 + x_2 + x_3$, the closed–loop dynamics may be written as

$$\dot{\eta} = -\eta,\ \dot{x}_1 = -x_1 + x_1\eta,\ \dot{x}_2 = -x_2 - x_1 + \eta,$$

whose trajectories are bounded, completing the proof. ⌟

V. Application to Systems in Feedback Form

Consider the class of systems in feedback form described by the equations

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2)\,, \\ \dot{x}_2 &= u\,, \end{aligned} \tag{29}$$

with $x := \text{col}(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}$, and $u \in \mathbb{R}$. Consistent with the standard backstepping scenario [4] assume there exists a mapping $\pi_2 : \mathbb{R}^n \to \mathbb{R}$ such that the system

$$\dot{x}_1 = f(x_1, \pi_2(x_1))$$

has a GAS equilibrium at the origin. A sensible choice of the target dynamics is then given by $\dot{\xi} = f(\xi, \pi_2(\xi))$, and this implies that $\pi(\xi) = \text{col}(\xi,\ \pi_2(\xi))$. To verify

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Assumptions (A2) and (A3) of Proposition 1 we can choose

$$\begin{aligned} c(\xi, \pi_2(\xi)) &= \nabla\pi_2(\xi) f(\xi, \pi_2(\xi)) &(30)\\ \phi(x) &= x_2 - \pi_2(x_1), &(31)\end{aligned}$$

which clearly satisfy (3) and (4).

The differential relation of the system (29) in closed–loop with the control $\beta(x)$ is given by (25) with

$$\Psi(x) = \begin{bmatrix} \nabla_{x_1} f(x) & \nabla_{x_2} f(x) \\ \nabla_{x_1}\beta(x) & \nabla_{x_2}\beta(x) \end{bmatrix}.$$

Fixing $R(x) = \nabla\phi(x)$ and $P(x) = I$ in (22) yields

$$V(x, \delta x) = \delta x^\top M(x_1)\delta x\,,$$

where we defined

$$M(x_1) \;:=\; \begin{bmatrix} \nabla\pi_2(x_1)[\nabla\pi_2(x_1)]^\top & -\nabla\pi_2(x_1) \\ -[\nabla\pi_2(x_1)]^\top & 1 \end{bmatrix}$$

Fixing $\rho(x) = k > 0$ condition (23) becomes

$$\dot M(x_1) + M(x_1)[\Psi(x) + \frac{k}{2}I] + [\Psi^\top(x) + \frac{k}{2}I]M(x_1) \le 0. \quad (32)$$

We have the following as a direct corollary of Proposition 3 and the derivations above.

Proposition 4: Consider the system (29) and suppose there exist $\pi_2 : \mathbb{R}^n \to \mathbb{R}$ and $\beta : \mathbb{R}^{(n+1)} \to \mathbb{R}$ such that the following holds.

(i) $\dot x_1 = f(x_1, \pi_2(x_1))$ has a GAS equilibrium at zero.
(ii) The inequality (32) is satisfied for some $k > 0$.
(iii) $\beta(\xi, \pi_2(\xi)) = \nabla\pi_2(\xi) f(\xi, \pi_2(\xi))$.
(iv) The trajectories of (29) with $u = \beta(x)$ are bounded.

Then, (29) with $u = \beta(x)$ has a GAS equilibrium at zero. ⌟

Example 4: To illustrate the result in Proposition 4, consider the two-dimensional system

$$\dot x_1 = -x_1 + \mu x_1^3 x_2,\ \dot x_2 = u, \quad (33)$$

with $\mu > 0$. Condition (i) is satisfied with $\pi_2(x_1) = -x_1^2$. To check condition (ii) we compute

$$\begin{aligned} \phi(x) &= x_1^2 + x_2,\ M(x) = \begin{bmatrix} 4x_1^2 & 2x_1 \\ 2x_1 & 1 \end{bmatrix} \\ \Psi(x) &= \begin{bmatrix} -1 + 3\mu x_1^2 x_2 & \mu x_1^3 \\ \nabla_{x_1}\beta(x) & \nabla_{x_2}\beta(x) \end{bmatrix}.\end{aligned}$$

Some lengthy, but straightforward calculations, show that

$$\beta(x) = -\frac{1}{2}(k-4)x_1^2 - \frac{1}{2}kx_2 - 2\mu x_1^4 x_2 \quad (34)$$

solves (32) with identity. Condition (iii) holds because

$$\beta(\xi, \pi_2(\xi)) = 2\xi^2(1 + \mu\xi^4) = \underbrace{(-2\xi)}_{\pi_2'(\xi)}\underbrace{[-\xi + \mu\xi^3(-\xi^2)]}_{f(\xi,\pi_2(\xi))}.$$

We proceed now to verify the boundedness condition. With the definition of $\phi(x)$ given above we see that the z-dynamics takes the form $\dot z = -\frac{k}{2}z$, while the x_1 dynamics is $\dot x_1 = -\mu x_1^5 - x_1 + \mu x_1^3 z$, hence x_1 is bounded. Finally, since $x_2 = z - x_1^2$, we have that x_2 is also bounded, completing the proof. ⌟

VI. Conclusions

Two alternative procedures to complete the design of I&I controllers for stabilization of nonlinear systems have been proposed. The central idea is to replace the stabilization step of the extended dynamics (5), (6) required by condition (A4) of the I&I procedure by two contraction–based designs. The main advantage of the contraction–based approach is to render more systematic the design and to give more degrees of freedom for its accomplishment. The key steps of the novel design are the use of virtual and horizontal Finsler–Lyapunov functions [3] that—in the spirit of classical Lyapunov theory—decay along the trajectories of the prolonged system.

The approaches based on contraction replace the stabilization of the off-manifold coordinate z of I&I with the stabilization of the linearization along trajectories. The virtual approach looks at a direct characterization of the contraction of the z coordinates by considering its dynamics as an open system driven by an exogeneous signal x. In a similar way, the horizontal approach stabilizes the linearization of the system along suitable directions of its tangent space, thus providing a local and intrinsic feedback design procedure that does not require any a-priori definition of the off-manifold coordinate z. The advantage is a more general design method, with the generality directly encoded into the conditions of Proposition 3: the z coordinate of classical I&I is replaced at local level by the matrix $R(x)$, which is one of the free parameters to be selected in the formulation of the partial differential equation that needs to be solved.

Of course, similarly to all constructive procedures for the design of nonlinear controllers or observers, for the successful application of the novel designs proposed in the paper it is necessary to solve partial differential equations.

Appendix

A. Proof of Lemma 1

Consider the dynamics of (6) as a non autonomous system with state z and an exogenous input signal $x(t)$, that is, $\dot z = \Phi(x(t), z)$. We show that $\lim_{t\to\infty} z(t) = 0$ for any initial condition $z(0)$ by adapting the argument of Theorem 1 in [3].

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2016.2614888, IEEE Transactions on Automatic Control

Consider any initial condition $z(0)$, take any differentiable curve $\gamma(\cdot) : [0,1] \to \mathbb{R}^{n-p}$ such that $\gamma(0) = 0$ and $\gamma(1) = z(0)$, and recall that the length of the curve γ is given by $\ell(\gamma(\cdot)) = \int_0^1 |\frac{d}{ds}\gamma(s)|ds$.

For any given $s \in [0,1]$ and $t \geq 0$, the quantity $\psi_t^x(\gamma(s))$ denotes the state reached by $\dot{z} = \Phi(x(t), z)$ at time t from the initial condition $\gamma(s)$. We show that $\lim_{t\to\infty} \ell(\psi_t^x(\gamma(\cdot))) = 0$.

For instance,

$$
\begin{aligned}
&\lim_{t\to\infty} \ell(\psi_t^x(\gamma(\cdot))) \; = \lim_{t\to\infty} \int_0^1 \left| \frac{d}{ds}\psi_t^x(\gamma(s)) \right| ds \\
&\leq \lim_{t\to\infty} \int_0^1 \left(\frac{V\left(x(t), \psi_t^x(\gamma(s)), \frac{d}{ds}\psi_t^x(\gamma(s))\right)}{k} \right)^{\frac{1}{2}} ds \\
&\leq \lim_{t\to\infty} \int_0^1 \left(\frac{e^{-\lambda t}}{k} V\left(x(0), \gamma(s), \frac{d}{ds}\gamma(s) \right) \right)^{\frac{1}{2}} ds \\
&\leq \lim_{t\to\infty} \left(\frac{e^{-\lambda t}}{k} \max_{s\in[0,1]} V\left(x(0), \gamma(s), \frac{d}{ds}\gamma(s) \right) \right)^{\frac{1}{2}} \int_0^1 ds \\
&= 0 \; .
\end{aligned}
$$

The first inequality above follows from (15). The second inequality above follows from (16) and the comparison lemma. To see this, define $A(x(t), z) := \nabla_z \Phi(x(t), z)$. Then, for any $s \in [0,1]$,

$$
\begin{aligned}
\frac{d}{dt}\frac{d}{ds}\psi_t^x(\gamma(s)) &= \frac{d}{ds}\frac{d}{dt}\psi_t^x(\gamma(s)) \\
&= \frac{d}{ds} f(\psi_t^x(\gamma(s))) \\
&= A(x(t), \psi_t^x(\gamma(s))) \frac{d}{ds}\psi_t^x(\gamma(s)) \; .
\end{aligned}
$$

Therefore, for any $s \in [0,1]$, the curve $t \mapsto \frac{d}{ds}\psi_t^x(\gamma(s))$ is a trajectory of the variational dynamics (13). It follows that (16) guarantees $\frac{d}{dt}V(x(t), \psi_t^x(\gamma(s)), \frac{d}{ds}\psi_t^x(\gamma(s))) \leq -\lambda V(x(t), \psi_t^x(\gamma(s)), \frac{d}{ds}\psi_t^x(\gamma(s)))$.

Combining $\lim_{t\to\infty} \ell(\psi_t^x(\gamma(s))) = 0$ with the fact that $0 = \Phi(x(t), 0)$ we get that any trajectory of $\dot{z} = \Phi(x(t), z)$ converges to 0. Note that the limit is well defined by forward completeness of (5).

B. Proof of Proposition 2

By Lemma 1, (A4') guarantees $\lim_{t\to\infty} z(t) = 0$. Therefore, by boundedness of the trajectories, global asymptotic stability of the equilibrium x^* for (11) follows by the same argument of the proof of Theorem 1 in [1] (see also [2]).

C. Proof of Proposition 3

The proof is divided in four parts. The first two parts state relevant technical facts which are used in the last two parts to establish global attractivity and stability of the equilibrium point x_*. Part of the proof is adapted from the proof of Theorem 3 in [3].

I. Horizontal "distance" from the desired manifold $\mathcal{M}$: take $|\delta x|_x := \sqrt{V(x, \delta x)}$. Given any differentiable curve $\gamma : [0,1] \to \mathbb{R}^n$ define the horizontal length $\ell(\gamma(\cdot)) := \int_0^1 |\dot{\gamma}(s)|_{\gamma(s)} ds$. Note that $\ell(\gamma(\cdot)) \neq 0$ iff $R(\gamma(s))\dot{\gamma}(s) \neq 0$ for some $s \in [0,1]$, which implies that $\ell(\gamma(\cdot)) \neq 0$ whenever $\gamma(s) \notin \mathcal{M}$ for some $s \in [0,1]$. For instance, suppose $\gamma(s) \in \mathcal{M}$ for all $s \in [0,1]$. Then $R(\gamma(s))\dot{\gamma}(s) = \nabla\phi(\gamma(s))\dot{\gamma}(s) = 0$, by (21), which implies $|\dot{\gamma}(s)|_{\gamma(s)} = 0$, thus $\ell(\gamma(\cdot)) = 0$. In a similar way, consider any $\gamma(0) \in \mathcal{M}$ and $\gamma(1) \notin \mathcal{M}$. $R(\cdot)$ is full rank and differentiable thus there exists a measurable subset of $\mathcal{I} \subset [0,1]$ such that $|\dot{\gamma}(s)|_{\gamma(s)} \neq 0$. It follows that $\ell(\gamma(\cdot)) > 0$.

II. The differential characterization of a curve $\gamma(\cdot)$ along the system semiflow $\psi_t(\cdot)$: consider any differentiable curve $\gamma(\cdot) : [0,1] \to \mathbb{R}^n$. For any given $s \in [0,1]$ and $t \geq 0$, the quantity $\psi_t(\gamma(s))$ denotes the state reached by $\dot{x} = f(x) + g(x)\beta(x)$ at time t from the initial condition $\gamma(s)$. Note that

$$
\begin{aligned}
\frac{d}{dt}\frac{d}{ds}\psi_t(\gamma(s)) &= \frac{d}{ds}\frac{d}{dt}\psi_t(\gamma(s)) \\
&= \frac{d}{ds}[f(\psi_t(\gamma(s))) + g(f(\psi_t(\gamma(s)))\beta(f(\psi_t(\gamma(s)))] \\
&= \Psi(\psi_t(\gamma(s))) \frac{d}{ds}\psi_t(\gamma(s)) \; .
\end{aligned}
$$

Therefore, for any $s \in [0,1]$, the curve $t \mapsto \left(\psi_t(\gamma(s)), \frac{d}{ds}\psi_t(\gamma(s))\right)$ is a trajectory of (24), (25).

III. Global attractivity: By Item I, to show asymptotic convergence to the manifold $\mathcal{M}$ we have to show that $\lim_{t\to\infty} \ell(\psi_t(\gamma(\cdot))) = 0$ for any given curve $\gamma : [0,1] \to \mathbb{R}^n$ such that $\ell(\gamma(\cdot)) \neq 0$ and $\gamma(0) \in \mathcal{M}$.

By boundedness of trajectories, for any $\gamma : [0,1] \to \mathbb{R}^n$ there exists a compact set $\mathcal{K}$ such that, $\psi_t(\gamma(s)) \in \mathcal{K}$ for each $s \in [0,1]$ and $t \geq 0$; and there exists $\lambda \leq \rho(x)$ for all $x \in \mathcal{K}$. Also, for any fixed s, the curve $t \mapsto \left(\psi_t(\gamma(s)), \frac{d}{ds}\psi_t(\gamma(s))\right)$ is a trajectory of (24), (25), as shown in Item II. Therefore, (23) guarantees that

$$
V\left(\psi_t(\gamma(s)), \frac{d}{ds}\psi_t(\gamma(s))\right) \leq e^{-\lambda t} V\left(\gamma(s), \frac{d}{ds}\gamma(s)\right),
$$

which implies that

$$
\left| \frac{d}{ds}\psi_t(\gamma(s)) \right|_{\psi_t(\gamma(s))} \leq e^{-\frac{\lambda t}{2}} \left| \frac{d}{ds}\gamma(s) \right|_{\gamma(x)} .
$$

Thus, $\ell(\psi_t(\gamma(\cdot))) \leq e^{-\frac{\lambda t}{2}} \ell(\gamma(\cdot))$.

Suppose now that $\gamma(0) \in \mathcal{M}$ and $\gamma(1) \notin \mathcal{M}$. By (A2), $\psi_t(\gamma(0)) \in \mathcal{M}$ for all $t \geq 0$ (manifold invariance). Thus, combining $\lim_{t\to\infty} \ell(\psi_t(\gamma(\cdot))) = 0$

with boundedness of trajectories and with (A2), any trajectory of the closed loop system converges to the manifold $\phi(x) = 0$. Moreover, by (A1) and (A2), the manifold is invariant and internally asymptotically stable, hence all trajectories of the closed loop system converge to the equilibrium x_*. [1]

IV. Stability.

By (A1) and (A2), the manifold is invariant and internally asymptotically stable. At the equilibrium x_*, $\dot{V}(x_*, \delta x) \leq -\rho(x_*)V(x_*, \delta x)$ which guarantees local exponential stability of the offset dynamics $e := \phi(x)$ near the equilibrium x_*.For instance, by construction, the linearization on the fixed point x^* reads $\delta e = \nabla\phi(x_*)\delta x$ and $V(x_*, \delta x) = \delta x^\top R^\top(x_*)P(x_*)R(x_*)\delta x = \delta x^\top \nabla\phi(x_*)^T P(x_*)\nabla\phi(x_*)\delta x = \delta e^\top P(x_*)\delta e$, where the second identity follows from (21). Furthermore, (23) guarantees that $\frac{d}{dt}\delta e^\top P(x_*)\delta e = \dot{V}(x^*, \delta x) \leq -\rho(x^*)V(x, \delta x) = -\rho(x^*)\delta e^\top P(x_*)\delta e$. Thus, the linearization of the offset coordinates δe near the fixed point is exponentially stable.

The dynamics on the manifold $\dot{\xi} = \alpha(\xi)$ is asymptotically stable. Hence stability of x^* follows by center manifold theory [6, Appendix B].

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[1] As in the proof of Theorem 2.1 of [2].