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Towards a Decay Rate Assignment Based Design for Time-Delay Systems with Multiple Spectral Values

Islam Boussaada[1] and Silviu-Iulian Niculescu[2] and Karim Trabelsi[3]

***Abstract*— Recent results on maximal multiplicity induced-dominancy for spectral values in reduced-order Time-Delay Systems naturally apply in controllers design. As a matter of fact, the approach is merely a delayed-output-feedback where the candidates' delays and gains result from the manifold defining the maximal multiplicity of a real spectral value, then, the dominancy is shown using the argument principle. Various reduced order examples illustrate the applicative perspectives of the approach.**

I. Introduction

The present study centers on stabilizing-controllers design for linear time-invariant retarded time-delay systems. The investigation of conditions on the equation parameters that guarantee the exponential stability of solutions is a question of ongoing interest. In particular, an efficient way to study a solution's stability is the frequency domain approach since in the Laplace domain, where a number of effective methods have been proposed, the stability analysis amounts to studying the distribution of the characteristic quasipolynomial function's roots, see for instance [1], [2], [3], [4], [8], [5], [6], [7]. The idea of exploiting the delay effect in controllers design was first introduced in [9] where it is shown that the conventional proportional controller equipped with an appropriate time-delay performs an averaged derivative action and thus can replace the proportional-derivative controller, see also [10]. Furthermore, it was stressed in [11] that time-delay has a stabilizing effect in the control design. Indeed, the closed-loop stability is guaranteed precisely by the existence of the delay. In the context of mechanical engineering problems, the effect of time-delay was emphasized in [4] where concrete applications are studied, such as the machine tool vibrations and robotic systems.

In recent works, the characterization of multiple spectral values for time-delay systems of retarded type were established using a Birkhoff/Vandermonde-based approach; see for instance [12], [13], [14], [15]. In particular, in [13], it is shown that the admissible multiplicity of the zero spectral value is bounded by the generic *Polya and Szegö bound* denoted PS_B, which is merely the *degree* of the corresponding quasipolynomial [1], see for instance [16]. In [12], it is shown that a given crossing imaginary root with a non vanishing frequency never reaches PS_B and a sharper bound for its admissible multiplicities is established.

Moreover, in [15], the manifold corresponding to a multiple root for scalar time-delay equations defines a stable manifold for the steady state. An example of a scalar retarded equation with two delays is studied in [12] where it is shown that the multiplicity of real spectral values may reach the PS_B. In addition, the corresponding system has some further interesting properties: (i) it is asymptotically stable, (ii) its spectral abscissa (rightmost root) corresponds to this maximal allowable multiple root located on the imaginary axis. Such observations enhance the outlook of further exhibiting the existing links between the maximal allowable multiplicity of some negative spectral value reaching the quasipolynomial degree and the stability of the trivial solution of the corresponding dynamical system. This interesting property induced by multiplicity appears also in optimization problems since such a multiple spectral value is indeed the rightmost root, see also [17]. Also notice that the property was already observed in [18], where a tuning strategy is proposed for the design of a delayed Proportional-Integral controller by placing a triple real dominant root for the closed-loop system. However, the dominancy is only checked using a Mikhailov curve and QPmR toolbox, see for instance [19].

It is worth noting that the rightmost root for quasipolynomial function corresponding to stable time-delay systems is actually the exponential decay rate of its time-domain solution, see for instance [20] for an estimate of the decay rate for stable linear delay systems. To the best of our knowledge, the first time an analytical proof of the dominancy of a spectral value for the scalar equation with a single delay was presented in [21]. The dominancy property is further explored and analytically shown in scalar delay equations in [15], then in second-order systems controlled by a delayed proportional is proposed in [22], [23] where its applicability in damping active vibrations for a piezo-actuated beam is proved. An extension to the delayed proportional-derivative controller case is studied in [24] where the dominancy property is parametrically characterized. We emphasize that the idea of using roots assignment for controller-design

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[1]The quasipolynomial degree is exactly the number of the involved polynomials plus their degree minus one

for time-delay system is not new. As a matter of fact, an analytical/numerical stabilization method for retarded time-delay systems related to the classical pole-placement method for ordinary differential equations is proposed in [25], see also [26] for further insights on pole-placement methods for retarded time-delays systems with proportional-integral-derivative controller-design.

This work provides an overview of those recent results on the dominancy criterion for scalar and second-order systems and it further explores the applicability of such a criterion in a third-order model describing the Mach number regulation in a *wind tunnel*. Roughly speaking, the Mach number regulation in a wind tunnel is based on Navier-Stokes equations for unsteady flow and contains control laws for temperature and pressure regulation. Here, the model we consider consists of a system of three state equations with a delay in one of the state variables.

II. Prerequisites

Consider the generic second-order system with a single time delay:

$$\dot{\chi} = A_0\chi(t) + A_1\chi(t-\tau), \tag{1}$$

where $\chi = (\chi_1, \ldots, \chi_n) \in \mathbb{R}^n$ is the state-vector, under appropriate initial conditions belonging to the Banach space of continuous functions $\mathcal{C}([-\tau_N, 0], \mathbb{R}^n)$. Here τ is a positive constant delay and $A_j \in \mathcal{M}_n(\mathbb{R})$ for $j = 0 \ldots 1$ are real valued matrices. It is well known that the asymptotic behavior of the solutions of (1) is determined from the spectrum $\aleph$ designating the set of the roots of the associated characteristic function (denoted $\Delta(s,\tau)$ in the sequel). Namely, the characteristic function corresponding to system (1) is a quasipolynomial $\Delta : \mathbb{C} \times \mathbb{R}_+ \to \mathbb{C}$ of the form:

$$\Delta(s,\tau) = \det\left(s\, I - A_0 - A_1\, e^{-\tau s}\right). \tag{2}$$

To start with, let us recall a generic result on the location of spectral values corresponding to (2). The proof of the proposition below can be found in [5].

Proposition 1. *If s is a characteristic root of system* (1)*, then it satisfies*

$$|s| \leq ||A_0 + A_1\, e^{-\tau s}||_2. \tag{3}$$

The above proposition combined with the triangular inequality provides a generic *envelope curve* around the characteristic roots corresponding to system (1).

In particular, the present work is focused on time-delay systems characterized by the quasipolynomial function of the form

$$\Delta(s,\tau) = P_0(s) + P_1(s)\, e^{-\tau s}, \tag{4}$$

where $\deg(P_0) > \deg(P_1)$. We shall consider the problem of the analytical characterization of its rightmost root.

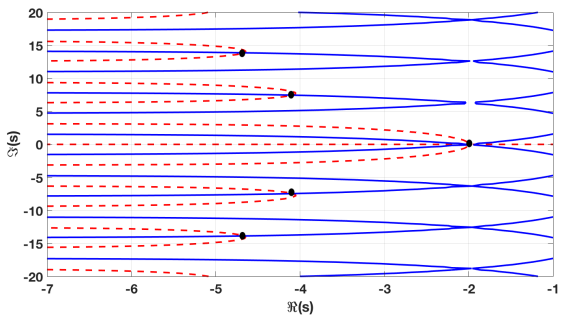


Fig. 1. The distribution of the spectrum corresponding to equation (7) and system (6) for $a_0 = \tau = 1$. The roots' distribution is illustrated using QPmR toolbox from [19]

III. Exponential Decay Rate for a scalar equation with a single delay

The starting point of this work in progress and the first analytical proof of the multiplicity-induced dominancy was proposed in [15]. Indeed, a simple scalar differential equation with one delay representing a biological model describing the dynamics of a vector disease model was considered. In its linearized version, the infected host population $\xi(t)$ is governed by:

$$\dot{\xi}(t) + a_0\,\xi(t) + a_1\,\xi(t-\tau) = 0, \tag{5}$$

where $a_1 > 0$ designates the contact rate between infected and uninfected populations assuming that the infection of the host recovery proceeds exponentially at a rate $-a_0 > 0$. It was shown that for a given positive delay, equation (5) admits a double spectral value at $s = s_0$ if, and only if,

$$s_0 = -\frac{a_0\tau + 1}{\tau} \quad \text{and} \quad a_1 = \frac{\mathrm{e}^{-a_0\tau - 1}}{\tau}. \tag{6}$$

Furthermore, it was stressed that s_0 is the corresponding rightmost root and if $s_0 < 0$ then the zero solution of system (5) is asymptotically stable.

One knows that $s = s_0$ is a spectral value of (5) if, and only if, s_0 is a root of the characteristic equation

$$\Delta(z,\tau) = s + a_0 + a_1\, e^{-s\tau} = 0. \tag{7}$$

The main ingredient of the dominancy proof of s_0 is an integral equation which cannot be satisfied for any spectral value s with $\Re(s) > s_0$. Namely, it was shown that if a_1 satisfies (6), then the characteristic function reads:

$$\Delta(s,\tau) = (s - s_0)\left(1 - \int_0^1 e^{-\tau(s-s_0)\,t}\, dt\right). \tag{8}$$

As a matter of fact, if $s_1 = \zeta + j\,\eta \neq s_0$ is a root of (8) then s_1 is a root of its second factor. Hence, we obtain

$$1 = \int_0^1 e^{-\tau(\zeta - s_0)\,t} dt. \tag{9}$$

But, $e^{-\tau(\zeta-s_0)\,t} < 1$ for $\zeta - s_0 > 0$ and $0 < t < 1$, thereby exhibiting the dominancy of s_0.

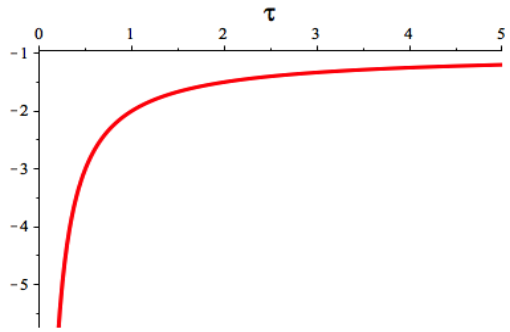


Fig. 2. The rightmost root corresponding to equation (7) and system (6) as a function of the delay τ (red solid line) for a fixed value of $a_0 = 1$.

Remark 1. The rightmost root s_0 corresponding to equation (7) where system (6) is satisfied varies in the interval $s_0 \in]-\infty, -a_0[$. Figure 2 illustrates the behavior of the rightmost root with respect to the time-delay variation.

IV. Second-order Systems

Second-order linear systems capture the dynamic behavior of many natural phenomena and have found wide applications in a variety of fields, such as vibration and structural analysis. In the sequel, we recall some hints, recent results and examples motivating the use of delay in controller-design for stabilizing the steady state solution corresponding to such a class of systems. In its generic form, equation (4) is written as:

$$\Delta(s,\tau) = s^2 + c_1 s + c_0 + (\beta_0 + s\beta_1)\, e^{-\tau s}. \tag{10}$$

The case $\beta_1 = 0$ yields a rightmost root with maximal multiplicity as characterized in [22].

Consider the standard linear change of variables

$$s = \frac{c_1\,\lambda}{2}, \tag{11}$$

leading to the normalized characteristic function

$$\begin{cases} \tilde{\Delta}(\lambda,\tilde{\tau}) = \lambda^2 + 2\,\lambda + a_0 + \alpha\, e^{-\lambda\tilde{\tau}}, \ \text{ where} \\ \qquad \alpha = \dfrac{4}{c_1^2}\beta_0,\ \tilde{\tau} = \dfrac{c_1}{2}\,\tau \text{ and } a_0 = 4\,\dfrac{c_0}{{c_1}^2}. \end{cases} \tag{12}$$

If $\alpha = 0$, the spectral abscissa is minimized at $a_0 = 1$ which corresponds to to the rightmost root located at $\lambda_0 = -1$, see for instance [27]. By exploiting the delay effect, the following proposition proved in [22] asserts that the solution's decay rate can be further improved by decreasing the corresponding rightmost root. Assume that $a_0 > 1$, then the following proposition holds.

Proposition 2.

i) *The multiplicity of any given root of the quasipolynomial function* (12) *is bounded by* 3.

ii) *The quasipolynomial* (12) *admits a real spectral value at* $\lambda_0 = -1 - \frac{1}{\tilde{\tau}}$ *with algebraic multiplicity* 3 *if, and only if,*

$$\tilde{\tau} = \sqrt{\frac{1}{a_0 - 1}} \text{ and } \alpha = -\frac{2\,e^{-(1+\tilde{\tau})}}{\tilde{\tau}^2}. \tag{13}$$

iii) *If equations* (13) *are satisfied then* $\lambda = \lambda_0$ *is the rightmost root of function* (12).

Remark 2. If equations (13) are satisfied then the trivial solution of the second order equation $\ddot{x}(t) + 2\,\dot{x}(t) + a_0\,x(t) + \alpha\,x(t-\tilde{\tau}) = 0$ is asymptotically stable with λ_0 as the corresponding exponential decay.

A. Multiple spectral values for time-delay systems are not necessarily dominant

The problem of stabilization of a chain of integrators is considered in [28] where a single integrator can be stabilized by a single delay state-feedback. Indeed, a positive gain guarantees the closed-loop stability of the system free of delay, and, by continuity, there exists a (sufficiently small) delay in the output preserving the stability of the closed-loop system. However, the situation is completely different for a chain of integrators of order n when $n > 1$. For instance, consider the time-delay system characterized by the following quasipolynomial function:

$$\Delta(s,\tau) = s^2 + \alpha\, e^{-\tau s}. \tag{14}$$

It can be checked that the maximal admissible multiplicity is 2 and it can be attained if, and only if,

$$\alpha = -4\,\frac{\mathrm{e}^{-2}}{\tau^2},\ s = -\frac{2}{\tau}. \tag{15}$$

However, the main result from [28] asserts that either n distinct delays or a proportional+delay compensator with $n-1$ distinct delays are sufficient to stabilize a chain including n integrators. In [29], a like assertion is shown to be also necessary to stabilize the chain of n integrators. Hence, in our case, either 2 distinct delays or a proportional+delay are necessary and sufficient to stabilize the double integrator. In conclusion, there exists at least a spectral value for (14) with a positive real part. As a result, $s_0 = -\frac{2}{\tau}$, while being a multiple root cannot be dominant. Indeed, consider (14)-(15) with $\tau = 1$, that is

$$\Delta(s,1) = s^2 - 4e^{-(s+2)}. \tag{16}$$

As illustrated in Figure 3, the dominancy property is lost since $s_1 \approx 0.557$ is a root of function (16). This is justified by the sparsity of (16).

B. Stabilizing a delayed proportional-derivative controller for generic second order systems

Let us consider again the quasipolynomial function (10):

$$\Delta(s,\tau) = s^2 + c_1 s + c_0 + (\beta_0 + s\beta_1)\, e^{-\tau s}.$$

The following result generalizes Proposition 2 which is restricted to $\beta_1 = 0$.

Proposition 3. *Considering equation* (10)*, the following assertions hold:*

i) *The multiplicity of any given root of the quasipolynomial function* (10) *is bounded by* 4*, it can be attained only on the real axis.*

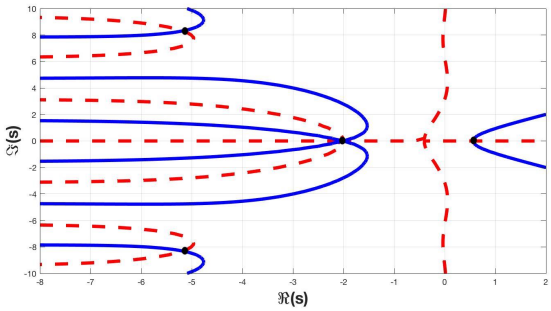


Fig. 3. Sparsity-induced loss of dominancy for the multiple spectral value. Each intersection between the blue/red curves corresponds to a spectral value of function (16).

ii) *The quasipolynomial* (10) *admits a real spectral value at* $s = s_\pm$ *with algebraic multiplicity* 4 *if, and only if, either*

$$\begin{cases} s_+ = \dfrac{-2+\sqrt{-2+c_0\tau^2}}{\tau}, \\ \beta_0 = 2\,\dfrac{\mathrm{e}^{-2+\sqrt{-2+c_0\tau^2}}\left(-5+\sqrt{-2+c_0\tau^2}\right)}{\tau^2}, \\ \beta_1 = -2\,\dfrac{\mathrm{e}^{-2+\sqrt{-2+c_0\tau^2}}}{\tau},\ c_1 = -2\,\dfrac{\sqrt{-2+c_0\tau^2}}{\tau}, \end{cases} \tag{17}$$

or

$$\begin{cases} s_- = \dfrac{-2-\sqrt{-2+c_0\tau^2}}{\tau}, \\ \beta_0 = 2\,\dfrac{\mathrm{e}^{-2-\sqrt{-2+c_0\tau^2}}\left(-5-\sqrt{-2+c_0\tau^2}\right)}{\tau^2}, \\ \beta_1 = -2\,\dfrac{\mathrm{e}^{-2-\sqrt{-2+c_0\tau^2}}}{\tau}, c_1 = 2\,\dfrac{\sqrt{-2+c_0\tau^2}}{\tau}, \end{cases} \tag{18}$$

where τ *is arbitrarily chosen satisfying* $c_0\tau^2 \geq 2$.

iii) *If either* (17) *or* (18) *is satisfied, then* $s = s_\pm$ *is the rightmost root of* (4).

A complete proof of the main result will be presented in an extended version of the paper; its sketch is summarized below.

Proof: The degree of the quasipolynomial function is equal to 4 as defined above. First, the vanishing of the quasipolynomial Δ yields the elimination of the exponential term as a rational function in s. The substitution of the obtained equality in the first three derivatives gives a system of algebraic equations. Solving them, one obtains the two solutions (17) and (18). Next using *the argument principle* one shows the dominancy of $s_\pm$; see Figure 6. Further explanation can be found in the next section. For an effective implementation in complex integral computations we refer the reader to [30].

Remark 3. It is worth noting that including information on the acceleration in the control loop allows to a time-delay system of neutral type characterized by the following quasipolynomial function of degree 5:

$$\Delta(s,\tau) = s^2 + c_1 s + c_0 + (\beta_0 + s\beta_1 + s\beta_2)\,e^{-\tau\,s}. \tag{19}$$

Since we are dealing with the asymptotic stability analysis, one assumes that $|\beta_2| \neq 1$, see for instance [8]. More precisely, if one assumes that $-1 < \beta_2 < 0$ then function (19) admits a negative root at $s_0 = \frac{\ln(-\beta_2)}{\tau}$ with multiplicity 5 if, and only if, the function parameters satisfy:

$$\begin{cases} \beta_0 = \dfrac{\beta_2\left(12 - 6\,\ln(-\beta_2) + (\ln(-\beta_2))^2\right)}{\tau^2}, \\ \beta_1 = 2\,\dfrac{\beta_2\,(3-\ln(-\beta_2))}{\tau}, \\ c_0 = \dfrac{6\,\ln(-\beta_2) + (\ln(-\beta_2))^2 + 12}{\tau^2}, \\ c_1 = 2\,\dfrac{-3-\ln(-\beta_2)}{\tau}. \end{cases}$$

Furthermore, the spectrum distribution of function (19) consists of a chain of roots with real parts close to s_0. However, the dominancy of multiple spectral values for neutral type remains an open question.

V. A Parameterized Dominancy Analysis in Delayed-Feedback Undamped Oscillators

Roughly speaking, sparsity of a quasipolynomial may preclude a given spectral value to attain the maximal admissible multiplicity, which is indeed the degree of the quasipolynomial. This section is devoted to the analysis of the parameters' effect on the admissible multiplicity as well as the dominancy of spectral values.

Consider the undamped oscillator controlled by a delayed proportional-derivative controller

$$\begin{cases} \ddot{\xi}(t) + \gamma\xi(t) = u(t), \\ u(t) = -\tilde{\beta}\xi(t-\tau) - \tilde{\alpha}\dot{\xi}(t-\tau), \end{cases} \tag{20}$$

where γ is a real parameter, $\tilde{\alpha}$ and $\tilde{\beta}$ are the gains of the delayed proportional-derivative controller. The corresponding quasipolynomial function is given by:

$$\Delta(s,\tau) = s^2 + \gamma + (\beta + \alpha\,s)\,\mathrm{e}^{-s\,\tau}. \tag{21}$$

If $\gamma = 0$, then the control problem (20) reduces to the stabilization of the double integrator using a delayed proportional-derivative controller. Otherwise, using a linear transformation, it is sufficient to study the two cases $\gamma = 1$ and $\gamma = -1$ to get a complete picture of the effect of the parameter γ on the dominancy of admissible multiple roots.

A. The double integrator stabilized by a delayed proportional-derivative controller

A result from [28] and [29], mentioned in Section IV-A, asserts that a delayed proportional controller (with a single delay) is not able to stabilize a double integrator. In [24] investigate the effect of the additional derivative term equipped with the same delay is investigated its stabilizing effect through the multiplicity induced-dominancy property is emphasized. Consider the quasipolynomial function

$$\Delta(s,\tau) = s^2 + (\beta + \alpha\,s)\,\mathrm{e}^{-s\,\tau}, \tag{22}$$

where $\alpha \neq 0$.

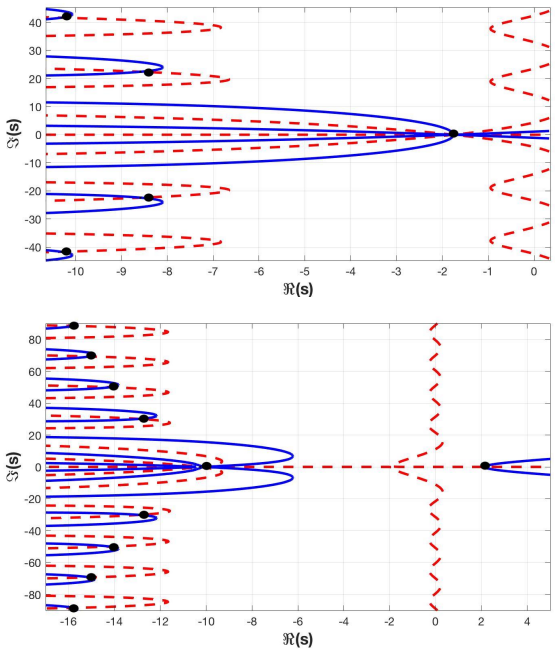


Fig. 4. (Up) Spectrum distribution corresponding to function (22) for $\tau = 1/3$ with system (24) is satisfied. (Down) Spectrum distribution corresponding to function (22) for $\tau = 1/3$ with system (23) is satisfied, the dominancy property is lost.

Proposition 4. *The following assertions hold for function* (22)*:*

i) *The multiplicity of any given root of the quasipolynomial function* (22) *is bounded by* 3*, it can be attained only on the real axis.*

ii) *The quasipolynomial* (22) *admits a real spectral value at* $s = s_{\pm}$ *with algebraic multiplicity* 3 *if, and only if,*

$$\begin{cases} \alpha = 2\,\dfrac{\left(-1-\sqrt{2}\right)\mathrm{e}^{-2-\sqrt{2}}}{\tau}, \\ \beta = 2\,\dfrac{\mathrm{e}^{-2-\sqrt{2}}\left(-7-5\sqrt{2}\right)}{\tau^2}, \\ s_- = \dfrac{-2-\sqrt{2}}{\tau}, \end{cases} \tag{23}$$

or

$$\begin{cases} \alpha = 2\,\dfrac{\left(\sqrt{2}-1\right)\mathrm{e}^{-2+\sqrt{2}}}{\tau}, \\ \beta = 2\,\dfrac{\mathrm{e}^{-2+\sqrt{2}}\left(-7+5\sqrt{2}\right)}{\tau^2}, \\ s_+ = \dfrac{-2+\sqrt{2}}{\tau}, \end{cases} \tag{24}$$

iii) *A spectral value of function* (22) *with maximal multiplicity (equal to* 3*) is dominant if, and only if,* $s = s_+$.

Proof: The degree of the quasipolynomial function is equal to 4 as defined in Section 2. First, the vanishing of the quasipolynomial Δ yields the elimination of the exponential term as a rational function in s. The substitution of the obtained equality in the first three derivatives gives a system of algebraic equations. Solving them, one shows that the solutions set is empty. Thus, the maximal multiplicity is less than or equal to 3. Solving the two first derivatives yields solutions (23) and (24).

The dominancy of s_+ proof follows the same steps as that of Proposition 3. First, using Proposition 1, one establishes a generic supremum bound for the real and imaginary parts of roots of function (22) such that system (24) is satisfied. Then define an integration contour $\gamma = \cup_{k=1}^{6} C_k$ which is taken as a counterclockwise closed curve, hence an integral over γ is defined as the sum of the integrals over the directed smooth curves that make γ up, as illustrated in Figures 6. Elementary calculations give a parametrization of γ on each C_k. Since Δ is analytic then *the argument principle* asserts:

$$\frac{1}{2i\pi}\oint_{\gamma}\frac{\partial_s\Delta(s,\tau)}{\Delta(s,\tau)}ds = \mathcal{Z}, \tag{25}$$

where $\mathcal{Z}$ designates the number of the quasipolynomial roots enclosed by γ. Furthermore, the left-hand side of (31) gives:

$$\begin{aligned}\oint_{\gamma}\frac{\partial_s\Delta(s,\tau)}{\Delta(s,\tau)}ds = \lim_{\epsilon\to 0}\sum_{k=1,k\neq 4}^{6}\int_0^1 \dot{s}^k(t)\frac{\partial_s\Delta(s^k(t),\tau)}{\Delta(s^k(t),\tau)}dt \\ + \lim_{\epsilon\to 0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{s}^4(t)\frac{\partial_s\Delta(s^4(t),\tau)}{\Delta(s^4(t),\tau)}dt.\end{aligned} \tag{26}$$

where $s^k(t)$ designates the parametrization of s along C_k for $k \in \{1,\ldots,6\}$. Some tedious but elementary computations lead to $\mathcal{Z} = 0$ in case (24), but $\mathcal{Z} = 1$ in case (23). Finally, Figure 4 illustrates the result.

B. Harmonic Oscillator stabilized by a delayed proportional-derivative controller

Consider the problem of stabilization of a classical harmonic oscillator using proportional-derivative controller:

$$\Delta(s,\tau) = s^2 + 1 + (\beta + \alpha\, s)\,\mathrm{e}^{-s\,\tau}. \tag{27}$$

Proposition 5. *Consider the quasipolynomial function* (27) *for which the following assertions hold:*

i) *The multiplicity of any given root of the quasipolynomial function* (10) *is bounded by* 4*, it can be attained only on the real axis.*

ii) *The quasipolynomial* (10) *admits a real spectral value at* $s = -\sqrt{2}$ *with algebraic multiplicity* 4 *if, and only if,*

$$\alpha = -\sqrt{2}\mathrm{e}^{-2},\ \beta = -5\,\mathrm{e}^{-2},\ \tau = \sqrt{2}. \tag{28}$$

iii) *If* (28) *is satisfied then* $s = -\sqrt{2}$ *is the rightmost root of* (4).

Proof: Following Proposition 3, $s_0 = -\sqrt{2}$ is a root of the quasipolynomial function (27) of multiplicity 4 with parameters values:

$$\alpha_0 = -\sqrt{2}\mathrm{e}^{-2}, \beta_0 = -5\,\mathrm{e}^{-2},\ \tau_0 = \sqrt{2} \tag{29}$$

Furthermore, s_0 is the rightmost root of (27)-(29). The dominancy proof follows the same steps as that of Proposition 3. First, using Proposition 1, one establish a generic supremum

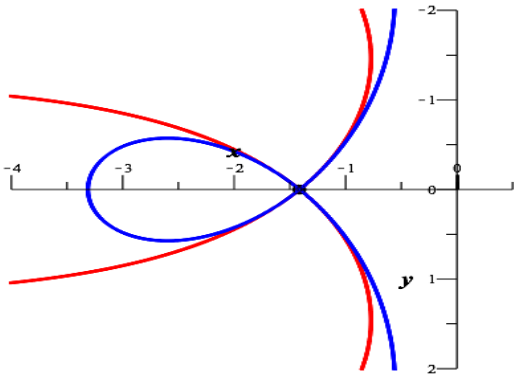


Fig. 5. Zoom on the dominant non resonant spectral value corresponding to (27)-(29) located at $\lambda = -\sqrt{2}$. The solid blue line is the zero modulus manifold and the solid red line is the zero imaginary-part manifold of (27)-(29).

bound for the real part as well as the imaginary part of roots of (27)-(29). Then define an integration contour $\gamma = \cup_{k=1}^{6} C_k$ which is taken as a counterclockwise closed curve, then an integral over γ is defined as the sum of the integrals over the directed smooth curves that make γ up, as illustrated in Figure 6. Elementary calculations leads to a parametrization of γ on each C_k:

$$\begin{aligned}
C_1: &\quad s^1(t) = \left(2+\sqrt{2}\right) t - \sqrt{2} - 7\,i \quad \text{where} \quad 0 \le t \le 1 \\
C_2: &\quad s^2(t) = 2 + 7\,i(2t-1) \quad \text{where} \quad 0 \le t \le 1 \\
C_3: &\quad s^3(t) = \left(-2-\sqrt{2}\right) t + 2 + 7\,i \quad \text{where} \quad 0 \le t \le 1 \\
C_4: &\quad s^4(t) = -\sqrt{2} + \epsilon\, \mathrm{e}^{it} \quad \text{where} \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2} \\
C_5: &\quad s^5(t) = -\sqrt{2} + i\left((\epsilon - 7)\, t + 7\right) \quad \text{where} \quad 0 \le t \le 1 \\
C_6: &\quad s^6(t) = -\sqrt{2} + i\left((\epsilon - 7)\, t - \epsilon\right) \quad \text{where} \quad 0 \le t \le 1
\end{aligned} \tag{30}$$

Since Δ is an analytic function then *argument principle* asserts:

$$\frac{1}{2i\pi} \oint_{\gamma} \frac{\partial_s \Delta(s,\tau)}{\Delta(s,\tau)} ds = \mathcal{Z}, \tag{31}$$

where $\mathcal{Z}$ designates the number of the quasipolynomial roots enclosed by γ. Furthermore, the left-hand side of (31) gives:

$$\begin{aligned}
\oint_{\gamma} \frac{\partial_s \Delta(s,\tau)}{\Delta(s,\tau)} ds = &\lim_{\epsilon \to 0} \sum_{k=1, k\neq 4}^{6} \int_0^1 \dot{s}^k(t) \frac{\partial_s \Delta(s^k(t),\tau)}{\Delta(s^k(t),\tau)} dt \\
&+ \lim_{\epsilon \to 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dot{s}^4(t) \frac{\partial_s \Delta(s^4(t),\tau)}{\Delta(s^4(t),\tau)} dt.
\end{aligned} \tag{32}$$

Some tedious but elementary computations lead to $\mathcal{Z} = 0$. Figures 5 and 6 illustrate the distribution of the spectrum of function (27)-(29).

C. Controlling an unstable second-order system via a delayed PD Controller

Consider the problem of stabilization of the second-order sparse polynomial (having two real roots with different

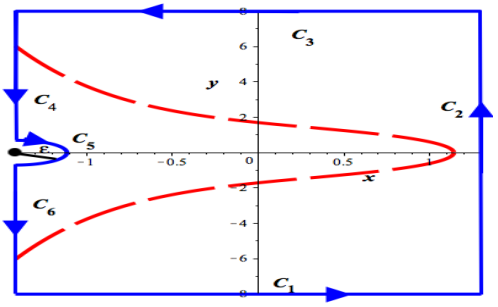


Fig. 6. The simplified contour used for applying the argument principle to investigate the dominancy of the multiple root in blue solid line. The dashed red line gives the generic spectrum envelope established in Proposition 1.

signs):

$$\Delta(s,\tau) = s^2 - 1 + (\beta + \alpha\, s)\, \mathrm{e}^{-s\,\tau}. \tag{33}$$

Proposition 6. *Consider the quasipolynomial function* (33) *for which the following assertions hold:*

i) *The multiplicity of any given root of the quasipolynomial function* (33) *is bounded by* 3*, it can be attained only on the real axis.*

ii) *The quasipolynomial* (33) *admits a real spectral value at* $s = s_{\pm}$ *with algebraic multiplicity* 3 *if and only if either*

$$\begin{cases}
s_+ = \dfrac{-2 + \sqrt{\tau^2+2}}{\tau}, \\
\beta = 2\,\dfrac{\mathrm{e}^{-2+\sqrt{\tau^2+2}} \left(-7 + 5\sqrt{\tau^2+2} - \tau^2\right)}{\tau^2}, \\
\alpha = 2\,\dfrac{\left(-1+\sqrt{\tau^2+2}\right) \mathrm{e}^{-2+\sqrt{\tau^2+2}}}{\tau},
\end{cases} \tag{34}$$

or

$$\begin{cases}
s_- = \dfrac{-2 - \sqrt{\tau^2+2}}{\tau}, \\
\alpha = 2\,\dfrac{\left(-1-\sqrt{\tau^2+2}\right) \mathrm{e}^{-2-\sqrt{\tau^2+2}}}{\tau}, \\
\beta = 2\,\dfrac{\mathrm{e}^{-2-\sqrt{\tau^2+2}} \left(-7 - 5\sqrt{\tau^2+2} - \tau^2\right)}{\tau^2},
\end{cases} \tag{35}$$

Throughout the above result, one can illustrate various scenarios. In the first one, the multiple root is dominant and stable, for instance when condition (34) is satisfied and $\tau = 1$, see Figure 7 (Up). In the second, the multiple root is dominant and unstable which occurs when condition (34) is satisfied and $\tau = 2$, see Figure 7 (Down). In the last, the multiple spectral value is not dominant as exposed in Figure 8. Finally, Figure 9 illustrates the behavior of the rightmost root corresponding to function (33) with respect to the delay value in both cases (34) and (35).

VI. A third-order Mach number regulation in a wind tunnel model

Transonic flows analysis is still a challenging problem in compressible fluid dynamic. In a stationary transonic flow, subsonic and supersonic regions live at the same time

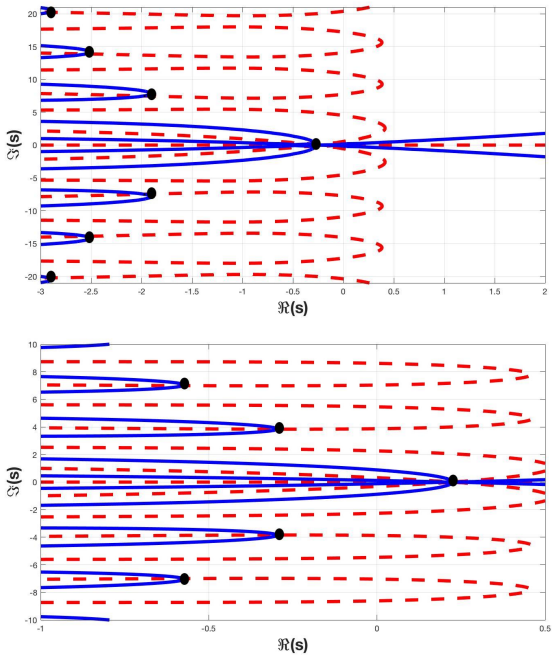


Fig. 7. (Up) The spectral distribution of function (33) under the conditions (34) with $\tau = 1$. The multiple spectral value is dominant and stable. (Down) The spectral distribution of function (33) under the conditions (34) with $\tau = 2$. The multiple spectral value is dominant but unstable.

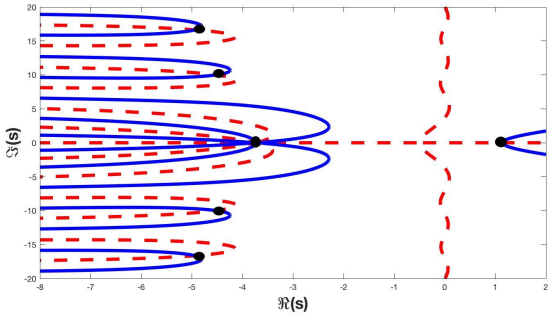


Fig. 8. The spectral distribution of the quasipolynomial function (33) under conditions (35) with $\tau = 1$. The multiple spectral value is not dominant.

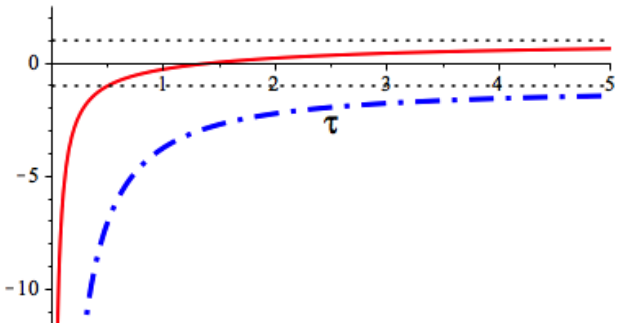


Fig. 9. The dominant root corresponding to equation (33) and system (35) as a function of the delay τ (red solid line). The dominant root corresponding to equation (33) and system (34) as a function of the delay τ (blue dashed line).

and are respectively governed by elliptical and hyperbolic equations. Furthermore, these two types of partial differential equations require completely different approaches, which often preclude solutions that are valid in the entire region.

In particular, the Mach number regulation in a wind tunnel is based on the Navier-Stokes equations for unsteady flow and contains control laws for temperature and pressure regulation. The following simplified model of Mach number regulation described in [31] consists of a system of three state equations with a delay in one of the state variables. It is stressed that in steady-state operating conditions, the dynamic response of the Mach number perturbations ξ_1 to small perturbations in the guide vane angle actuator ξ_2 are governed by:

$$\begin{cases} \dot{\xi}_1(t) = -a\xi_1(t) + k\,a\,\xi_2(t-\tau) \\ \dot{\xi}_2(t) = \xi_3(t) \\ \dot{\xi}_3(t) = -\omega^2\,\xi_2(t) - 2\zeta\omega\xi_3(t) + \omega^2 u(t) \end{cases} \tag{36}$$

where a, ω, ζ, k and τ are parameters depending on the operating point and presumed constant when the perturbations ξ_i are small. Moreover, following the experimental parameter values of the wind tunnel developed at NASA Langley Research Center, the parameters a, ω, ζ, τ are positive.

In [31], a feedback consisting of a linear combination of state variables and weighted integrals of some of the state variables over a period equal to the time delay, where the spectrum of the closed-loop system is finite (consists of three eigenvalues). However, our method does not render the closed-loop system finite dimensional but only involves controlling its rightmost root. Consider the control law: $u(t) = -\frac{\alpha}{\omega^2}\xi_2(t) - \frac{\beta_0}{\omega^2}\xi_2(t-\tau) - \frac{\beta_1}{\omega^2}\xi_3(t-\tau)$. In our case, the corresponding quasipolynomial function is given by:

$$\Delta(s,\tau) = (s+a)\left((s\beta_1+\beta_0)e^{-s\tau} + s^2 + 2\,s\,\zeta\,\omega + \omega^2 + \alpha\right). \tag{37}$$

Since a is a positive parameter, our aim is to establish conditions on parameters such that the rightmost root of the second factor of (37) has a negative real part. Interestingly, the analysis of the second factor can be deduced directly from the result in Section IV-B. As a matter of fact, by denoting

$$\omega^2 + \alpha = c_0 \quad \text{and} \quad 2\,\zeta\,\omega = c_1, \tag{38}$$

one may directly exploit condition (18) from Proposition 3 to guarantee the exponential stability of the trivial solution by assigning its rightmost root as a stable root. Note that condition (17) is not convenient as it imposes that $c_1 < 0$ which it cannot be applied here. Since the delay is intrinsic to the model, then the first step consists in finding the gain α such that

$$c_1 = 2\,\frac{\sqrt{-2 + c_0\tau^2}}{\tau}$$

which gives:

$$\alpha = \frac{2 + \zeta^2\omega^2\tau^2 - \omega^2\tau^2}{\tau^2}.$$

Substituting this last equality in (38) one obtains

$$\left\{ c_0 = \frac{2 + \zeta^2\omega^2\tau^2}{\tau^2} \quad \text{and} \quad c_1 = 2\,\zeta\,\omega, \right. \tag{39}$$

which satisfies the condition $c_0\tau^2 \geq 2$. The gains β_0 and β_1 are easily computed from system (18) and owing to the first equality from (39). Finally,

$$s_- = \frac{-2 - \zeta\,\omega\,\tau}{\tau}$$

is the rightmost root of the second factor of function (37), which insures the stability of the steady state solution.

VII. Concluding Remarks

Recent results by the authors on maximal multiplicity induced-dominancy for spectral value of time-delay systems of retarded type are overviewed. This note emphasizes a delayed controller-design based on the trivial solution's decay rate assignment. To illustrate the corresponding steps, a parameterized analysis of the dominancy property validity is established for generic second order oscillators. Finally, to demonstrate its concrete applicability, the regulation of the Mach number in a wind tunnel is considered. In future works, a generalization of the approach to arbitrary order dynamical systems will be studied and further applications treated.

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