Intermediate calculation steps from (7) to small perturbation formulation

Starting from (7)

$$\dot{X} = -\left(\frac{1}{Q_x} + \frac{1}{4}\alpha_x X^2\right)\frac{X}{2} + \frac{1}{2\Omega}(f_{cosx} + n_{cosx})$$

$$\dot{Y} = -\left(\frac{1}{Q_y} + \frac{1}{4}\alpha_y Y^2\right)\frac{Y}{2} + \frac{1}{2\Omega}(f_{cosy} + n_{cosy})$$

$$\dot{\phi} = -\frac{1}{\Omega}\left(\epsilon + \frac{3}{8}\gamma_x X^2 - \frac{3}{8}\gamma_y Y^2\right) + \frac{1}{2\Omega X}(f_{sinx} + n_{sinx}) - \frac{1}{2\Omega Y}(f_{siny} + n_{siny})$$

the full Jacobian matrix with respect to the state of the system without any assumption on the coupling and sustaining forces is:

$$\boldsymbol{J_s} = \begin{pmatrix} -\frac{1}{2}\left(\frac{1}{Q_x} + \frac{3}{4}\alpha_x X^2\right) + \frac{1}{2\Omega}\frac{\partial f_{cosx}}{\partial X} & \frac{1}{2\Omega}\frac{\partial f_{cosx}}{\partial Y} & \frac{1}{2\Omega}\frac{\partial f_{cosx}}{\partial \phi} \\ \frac{1}{2\Omega}\frac{\partial f_{cosy}}{\partial X} & -\frac{1}{2}\left(\frac{1}{Q_y} + \frac{3}{4}\alpha_y Y^2\right) + \frac{1}{2\Omega}\frac{\partial f_{cosy}}{\partial Y} & \frac{1}{2\Omega}\frac{\partial f_{cosy}}{\partial \phi} \\ -\frac{1}{X}\left(\frac{3}{4\Omega}\gamma_x X^2 + \frac{f_{sinx}}{2\Omega X}\right) + \frac{1}{2\Omega X}\frac{\partial f_{sinx}}{\partial X} - \frac{1}{2\Omega Y}\frac{\partial f_{siny}}{\partial X} & \frac{1}{Y}\left(\frac{3}{4\Omega}\gamma_y Y^2 + \frac{f_{siny}}{2\Omega Y}\right) + \frac{1}{2\Omega X}\frac{\partial f_{sinx}}{\partial Y} - \frac{1}{2\Omega Y}\frac{\partial f_{siny}}{\partial Y} & \frac{1}{2\Omega X}\frac{\partial f_{sinx}}{\partial \phi} - \frac{1}{2\Omega Y}\frac{\partial f_{siny}}{\partial \phi} \end{pmatrix}$$

In the case of a MILO, for example, the coupling and sustaining forces only depend on ϕ, so that this reduces to

$$\boldsymbol{J_s} = \begin{pmatrix} -\frac{1}{2}\left(\frac{1}{Q_x} + \frac{3}{4}\alpha_x X^2\right) & 0 & \frac{1}{2\Omega}\frac{\partial f_{cosx}}{\partial \phi} \\ 0 & -\frac{1}{2}\left(\frac{1}{Q_y} + \frac{3}{4}\alpha_y Y^2\right) & \frac{1}{2\Omega}\frac{\partial f_{cosy}}{\partial \phi} \\ -\frac{1}{X}\left(\frac{3}{4}\gamma_x X^2 + \frac{1}{2\Omega X}f_{sinx}\right) & \frac{1}{Y}\left(\frac{3}{4}\gamma_y Y^2 + \frac{1}{2\Omega Y}f_{siny}\right) & \frac{1}{2\Omega X}\frac{\partial f_{sinx}}{\partial \phi} - \frac{1}{2\Omega Y}\frac{\partial f_{siny}}{\partial \phi} \end{pmatrix}$$

Now, if we consider the MILO with $\theta = 90°$, at phase $\phi = 90°$, we have

$$f_{sinx}\left(= \frac{F_x}{\pi}(\cos\theta + \cos(\theta + \phi))\right) = -\frac{F_x}{\pi}$$

$$f_{cosx}\left(= \frac{F_x}{\pi}(\sin\theta + \sin(\theta + \phi))\right) = \frac{F_x}{\pi}$$

$$f_{siny}\left(= \frac{F_y}{\pi}(\cos\theta - \cos(\theta - \phi))\right) = -\frac{F_y}{\pi}$$

$$f_{cosy}\left(= \frac{F_y}{\pi}(\sin\theta - \sin(\theta - \phi))\right) = \frac{F_y}{\pi}$$

$$\frac{\partial f_{sinx}}{\partial \phi} = \frac{\partial f_{siny}}{\partial \phi} = 0$$

$$\frac{\partial f_{cosx}}{\partial \phi} = -\frac{F_x}{\pi}$$

$$\frac{\partial f_{cosy}}{\partial \phi} = \frac{F_y}{\pi}$$

so that the Jacobian becomes

$$\boldsymbol{J_s} = \begin{pmatrix} -\frac{1}{2}\left(\frac{1}{Q_x} + \frac{3}{4}\alpha_x X^2\right) & 0 & -\frac{1}{2\Omega}\frac{F_x}{\pi} \\ 0 & -\frac{1}{2}\left(\frac{1}{Q_y} + \frac{3}{4}\alpha_y Y^2\right) & \frac{1}{2\Omega}\frac{F_y}{\pi} \\ -\frac{1}{X}\left(\frac{3}{4\Omega}\gamma_x X^2 - \frac{1}{2\Omega X}\frac{F_x}{\pi}\right) & \frac{1}{Y}\left(\frac{3}{4\Omega}\gamma_y Y^2 - \frac{1}{2\Omega Y}\frac{F_y}{\pi}\right) & 0 \end{pmatrix}$$

From (7), we also have the following relations:

$$\Omega\left(\frac{1}{Q_x} + \frac{1}{4}\alpha_x X^2\right)X = f_{cosx} = \frac{F_x}{\pi}$$

$$\Omega\left(\frac{1}{Q_y} + \frac{1}{4}\alpha_y Y^2\right)Y = f_{cosy} = \frac{F_y}{\pi}$$

so that the Jacobian may be rewritten as:

$$\boldsymbol{J_s} = \begin{pmatrix} -\frac{1}{2}\left(\frac{1}{Q_x} + \frac{3}{4}\alpha_x X^2\right) & 0 & -\frac{1}{2}\left(\frac{1}{Q_x} + \frac{1}{4}\alpha_x X^2\right)X \\ 0 & -\frac{1}{2}\left(\frac{1}{Q_y} + \frac{3}{4}\alpha_y Y^2\right) & \frac{1}{2}\left(\frac{1}{Q_y} + \frac{1}{4}\alpha_y Y^2\right)Y \\ -\frac{1}{X}\left(\frac{3}{4\Omega}\gamma_x X^2 - \frac{1}{2}\left(\frac{1}{Q_x} + \frac{1}{4}\alpha_x X^2\right)\right) & \frac{1}{Y}\left(\frac{3}{4\Omega}\gamma_y Y^2 - \frac{1}{2}\left(\frac{1}{Q_y} + \frac{1}{4}\alpha_y Y^2\right)\right) & 0 \end{pmatrix}$$

Regarding the Jacobian with respect to parametric fluctuations or additive noise components, these are more straightforward to derive. For example, considering only fluctuations of parameter ϵ and additive noise components $n_{sinx}, n_{siny}, n_{cosx}$ and n_{cosy}, as we do in most of our paper, we have:

$$\boldsymbol{J_p} = \frac{1}{\Omega} \times \begin{pmatrix} 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \\ -1 & 1/2X & -1/2Y & 0 & 0 \end{pmatrix}$$

Note that only $\boldsymbol{J_s}$ changes from one WCR architecture to the other (or from one steady state solution to another).

How to use the Simulink simulation files

The provided files are:

- "MILO_NL". It simulates a MILO with $\theta = 90°$. The simulation is set to run with the following parameters: $Q = 100, \gamma = 0.1,$ and F slowly varying from 10^{-3} to 10^{-2}. Above

$$A_{Duff} = \sqrt{\frac{2}{3\gamma Q}} = 0.25$$

the mode with $\phi = 90°$ becomes unstable, and the system starts oscillating with $\phi = -90°$.

- "MOLO_NL". It simulates a MOLO with $\theta = 90°$, as considered in our paper. The simulation is set to run with the following parameters: $Q = 100, \gamma = 0.1, \kappa = 0.1$ and F slowly varying from 10^{-2} to 10^{-1}. Above

$$A_{Duff} = \sqrt{\frac{4\kappa}{3\gamma}} = 1.15$$

the phase opposition mode with $R = 1$ becomes unstable, and the system starts oscillating with $R \neq 1$, but still in phase opposition. The coupling spring may be nonlinear: setting the coupling force to "u+100*u^3" for example shows that nonlinear coupling increases the effective coupling stiffness, and thus the range of stability.

- "noisy_MILO". This is the simulation file used for generating the results in Fig. 7, for a MILO with $\theta = 45°$. Note that the precise value of θ is set by the transfer functions in the "Phase-shift" block. In the case Ω_0 is significantly different from 1 (this may be assessed by the formulas in the paper), one should modify these transfer functions to $1/(1 + s/\Omega_0)$. The values of Q, γ and F are set by parameters "Q", "gama" and "f".