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Abstract

In this paper, we present a Lyapunov-based homogeneous controller for the stabilization of a perturbed chain of integrators of arbitrary order $r \geq 1$. The proposed controller is based on homogeneous controller for stabilization of pure chain of integrators. The control of homogeneity degree is also introduced and various controllers are designed using this concept, namely a bounded-controller with minimum amplitude of discontinuous control and a controller with globally fixed-time convergence. The performance of the controller is validated through simulations.

I. Introduction

The problem of finite-time stabilization of a perturbed chain of integrators arises in many control applications. For example, electromechanical systems such as motorized actuators or robotic arms are modeled as perturbed double integrators [1], [2], [3]. Another application is in Higher Order Sliding Mode Control (HOSM) [4], which can be formulated as the stabilization of an auxiliary system arising as a perturbed chain of integrators built from the output and its higher time derivatives [5]. The finite-time stability problem was addressed in relation with homogeneous systems in [6], [7], and homogeneity concept was used for stabilization of linear systems in [8]. In [9], the link between finite-time stabilization and homogeneity of a system was established, and

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it was shown that a homogeneous system is finite-time stable if and only if it is asymptotically stable and has a negative homogeneity degree. This result has, since then, been used for the development of many controllers for pure and perturbed chain of integrators. A homogeneous nonsmooth proportional-derivative controller for robot manipulators (double integrator system) was developed in [2]. This work was generalized for an arbitrary-length of a chain of integrators in [10]. In [11] and [12], negative homogeneity was used for the finite-time stabilization of a class of nonlinear systems that includes perturbations at each integrator link.

Among Sliding Mode techniques, the homogeneity approach was used in [13], [14], to demonstrate finite-time stabilization of the arbitrary order sliding mode controllers for Single Input Single Output (SISO) systems [15]. A robust Multi Input Multi Output (MIMO) HOSM controller was also presented in [16], using a constructive algorithm with geometric homogeneity based finite-time stabilization of a chain of integrators. A controller, which stabilizes a perturbed chain of integrators of arbitrary length using only the signs of state variables, was presented in [17]. A Lyapunov-based approach for arbitrary HOSMC controller design was first presented in [18]. In these works, it was shown that a class of homogeneous controllers that satisfies certain conditions, could be used to stabilize perturbed chain of integrators.

In this paper, we present a continuation of [18], and develop a Lyapunov-based robust controller for the finite-time stabilization of a perturbed chain of integrators of arbitrary order, with bounded uncertainty. The main focus of this paper is to obtain various properties in the controller by controlling the degree of homogeneity. The homogeneous controller for perturbed chain of integrators is developed from a discontinuous Lyapunov-based controller for pure chain of integrators. It is then demonstrated that the homogeneity degree can be controlled in the neighborhood of zero, such that the amplitude of discontinuous control is kept to its minimum possible value when the states have converged. It is also shown that the recently developed "Fixed-Time" stability notion can be achieved by changing the homogeneity degree. Globally fixed-time stability was first introduced in [19]; this term refers to the finite-time stabilization of systems with uniform convergence, i.e. the convergence time is bounded and independent of the system's initial state. In [20] and [21], uniform convergence to a neighborhood of the origin was demonstrated for second order systems and arbitrary order respectively. In [22], globally fixed-time convergence controllers were developed for linear systems, insuring guaranteed convergence exactly to zero. Based on the control of homogeneity degree, the controller presented in this paper ensures globally fixed-time convergence to zero of a perturbed chain of integrators.

The paper is organized as follows: the problem formulation as well as the motivation and contributions of the paper are discussed in Section 2. The controller design is presented in Section 3 and its special cases are demonstrated in Section 4. Simulation results are shown in Section 5 and concluding remarks are given in Section 6.

II. Problem formulation, motivation and contribution

The mathematical formulation of the perturbed chain of integrators problem is developed first. Then, the motivation behind using homogeneity based controllers and the contribution of this paper are presented.

A. Problem Formulation

Let us consider an uncertain nonlinear system:

$$\begin{cases} \dot{x}(t) = f(x,t) + g(x,t)u, \\ y(t) = s(x,t), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}$ is the input control. The sliding variable s is a measured smooth output-feedback function and $f(x,t)$ and $g(x,t)$ are uncertain smooth functions. It is assumed that the relative degree, r of the system [23] is globally well defined, uniform and time invariant [5] and the associated zero dynamics are asymptotically stable. For autonomous systems, r is the minimum order of time derivatives of the output $y(t)$ in which the control input u appears explicitly. This means that, for suitable functions $\tilde{\varphi}(x,t)$ and $\tilde{\gamma}(x,t)$, we obtain

$$y^{(r)}(t) = \tilde{\varphi}(x(t),t) + \tilde{\gamma}(x(t),t)u(t). \tag{2}$$

The functions $\tilde{\gamma}(x,t)$ and $\tilde{\varphi}(x,t)$ are assumed to be bounded by positive constants $\gamma_m \leq \gamma_M$ and $\bar{\varphi}$, such that, for every $x \in \mathbb{R}^n$ and $t \geq 0$,

$$0 < \gamma_m \leq \tilde{\gamma}(x,t) \leq \gamma_M, \quad |\tilde{\varphi}(x,t)| \leq \bar{\varphi}. \tag{3}$$

Defining $s^{(i)} := \frac{d^i}{dt^i}y$; the goal of r^{th} order sliding mode control is to arrive at, and keep the following manifold in finite-time:

$$s^{(0)}(x,t) = s^{(1)}(x,t) = \cdots = s^{(r-1)}(x,t) = 0. \tag{4}$$

To be more precise, let us introduce $z = [z_1\ z_2\ ...z_r]^T := [s\ \dot{s} ...\ s^{(r-1)}]^T$. Then (4) is equivalent to $z = 0$. Since the only available information on $\tilde{\varphi}(x,t)$ and $\tilde{\gamma}(x,t)$ are the bounds (3), it is natural to consider a more general control system instead of System (2), such as

$$\dot{z}_i = z_{i+1}, i = 1, \cdots, r-1, \quad \dot{z}_r = \varphi(t) + \gamma(t)u, \tag{5}$$

where the new functions φ and γ are arbitrary measurable functions that verify the condition

$$(H1) \quad \varphi(t) \in [-\bar{\varphi}, \bar{\varphi}], \quad \gamma(t) \in [\gamma_m, \gamma_M]. \tag{6}$$

The objective of this paper is to design controllers which stabilize System (5) to the origin in finite-time . Since these controllers are to be discontinuous feedback laws $u = U(z)$, solutions of (5) with $u = U(z)$ will fall in the realm of differential inclusions and need to be understood here in Filippov sense, i.e. the right hand vector set is enlarged at the discontinuity points of (5) to the convex hull of the set of velocity vectors obtained by approaching z from all the directions in $\mathbb{R}^r$, while avoiding zero-measure sets [24].

B. Definitions and technical results

We need the following definitions to state our results.

Definition 1. *[10], [25] The family of dilations ζ_ε^p, $\varepsilon > 0$, are the linear maps defined on $\mathbb{R}^r$ given by*

$$\zeta_\varepsilon^p(z_1, \cdots, z_r) = (\varepsilon^{p_1} z_1, \cdots, \varepsilon^{p_r} z_r),$$

where $p = (p_1, \cdots, p_r)$ with the homogeneity weights $p_i > 0$, for $i = 1, \cdots, r$.

A function $\Omega(z)$ is homogeneous of degree $a > 0$ with respect to the family of dilations ζ_ε^p where a is a positive real number if, for every $z \in \mathbb{R}^r$ and $\varepsilon > 0$, $\Omega(\varepsilon^{p_1} z_1, \cdots, \varepsilon^{p_r} z_r) = \varepsilon^a \Omega(z_1, \cdots, z_r)$.

A set function $F : \mathbb{R}^r \rightrightarrows \mathbb{R}^r$ is said to be a homogeneous differential inclusion of degree $\kappa \in \mathbb{R}$ with respect to the family of dilation ζ_ε^p if it satisfies $F(\zeta_\varepsilon^p(z)) = \varepsilon^\kappa \zeta_\varepsilon^p(F(z))$.

A differential equation $\dot{z} = f(z)$ *(a differential inclusion* $\dot{z} \in F(z)$*) is said to be homogeneous of degree* $\kappa \in \mathbb{R}$ *with respect to the family of dilations* $\zeta_\varepsilon(z)$ *if the vector field* $f : \mathbb{R}^r \to \mathbb{R}^r$ *(the set function* $F : \mathbb{R}^r \rightrightarrows \mathbb{R}^r$ *is).*

Definition 2. *The sign function is a multi-valued function defined on* $\mathbb{R}$ *by* $sign(z) = z/|z|$ *if* $z \neq 0$ *and* $sign(0) = [-1,1]$*. Moreover, if* $\alpha \geq 0$ *and* $a \in \mathbb{R}$*, we use* $\lfloor a \rceil^\alpha$ *to denote* $|a|^\alpha sign(a)$*. Then, for* $a, b \in \mathbb{R}$ *and* $\alpha > 0$*, it holds* $sign(\lfloor a \rceil^\alpha - \lfloor b \rceil^\alpha) = sign(a - b)$*.*

Proposition 1 ([26])**.** *Let* Ω *be a positive definite* C^1 *function, homogeneous of degree a with respect to* ζ_ε^p*. Then, for all* $i = 1, \cdots, r$*;* $\partial\Omega/\partial z_i$ *is homogeneous of degree* $(a - p_i)$*.*

Consider the differential system

$$\dot{z} = f(t, z), \tag{7}$$

where $z \in \mathbb{R}^r$ is the vector system states, $f : \mathbb{R}_+ \times \mathbb{R}^r \to \mathbb{R}^r$ is a time-varying vector field.

Definition 3. *[22], [27], [28] Assume that* $f(\cdot, 0) \equiv 0$*, i.e., the point* $z = 0$ *is an equilibrium point of System* (7)*. Then* $z = 0$ *is said to be locally* ***finite-time*** *stable in a neighborhood* $\hat{U} \subset \mathbb{R}^r$ *of 0 if (i) it is asymptotically stable in* $\hat{U}$*; (ii) it is finite-time convergent in* $\hat{U}$*, i.e., there exists a settling function* $T : \mathbb{R}^r \to \mathbb{R}_{>0}$ *called the settling-time function such that, for any initial condition* z_0*,* $z(t, z_0) = 0, \forall t \geq T(z_0)$*. The equilibrium point* $z = 0$ *is globally finite-time stable if* $\hat{U} = \mathbb{R}^r$ *and it is* ***globally fixed-time*** *stable if (i) it is globally finite-time stable; (ii) the settling-time function is bounded by a constant* T_{max}*, i.e.* $\exists T_{max} > 0 : \forall z_0 \in \mathbb{R}^r, T(z_0) \leq T_{max}$*.*

Definition 4. *[22], [28] The set S is said to be globally* ***finite-time*** *attractive for* (7)*, if for any initial condition* z_0*, the corresponding trajectory starting at* z_0 *achieves S in finite-time* $T(z_0)$*. Moreover, the set S is said to be* ***globally fixed-time*** *attractive for* (7)*, if (i) it is globally finite-time stable; (ii) the settling-time function is bounded by a constant* T_{max}*.*

Lemma 1. *[9], [27] Suppose there exists a positive definite* C^1 *function V defined on a neighborhood* $\hat{U} \subset \mathbb{R}^r$ *of the equilibrium point* $z = 0$ *and real numbers* $C > 0$ *and* $\alpha \geq 0$*, such that the following condition is true for every non trivial trajectory z of System* (7)*,*

$$\dot{V} + CV^\alpha(z(t)) \leqslant 0, \; \textit{if } z(t) \in \hat{U}, \tag{8}$$

where $\dot{V}$ is the time derivative of $V(z(t))$. (Here for $\alpha = 0$, Equation (8) *means $\dot{V} \leq -C$ if $z(t) \in \hat{U} \setminus \{0\}$.) Then all trajectories of System* (7) *which stay in $\hat{U}$ converge to zero. If $\hat{U} = \mathbb{R}^r$ and V is radially unbounded, then System* (7) *is globally stable with respect to the equilibrium point $z = 0$.*

Depending on the value α, we have different types of convergence: if $0 \leq \alpha < 1$, the equilibrium point $z = 0$ is finite-time stable ([27]), if $\alpha = 1$, it is exponentially stable and if $\alpha > 1$ the equilibrium point $z = 0$ is asymptotically stable equilibrium and, for every $\varepsilon > 0$, the set $\mathbf{B}(0,\varepsilon) = \{z \in \hat{U} : V(z) < \varepsilon\}$ is fixed-time attractive.

Proof. The argument is obvious (cf. [27] for $0 < \alpha < 1$). Let us just note that for $\alpha > 1$ with initial condition $V(z(0)) = V_0$, an integration of $\dot{V} + CV^{\alpha} \leq 0$ shows that every trajectory enters the neighborhood defined by $V(z) \leq \varepsilon$ in a fixed time less than or equal to $\frac{1}{C(\alpha-1)\varepsilon^{\alpha-1}}$ for any initial condition. □

The following lemmas are used in the course of some subsequent arguments.

Lemma 2. *For $\alpha \geq 1$, define on $\mathbb{R}^2$ the functions $w(b,a) = \lfloor b \rceil^{\alpha} - \lfloor a \rceil^{\alpha}$ and $W(b,a) = \int_a^b w(s,a)ds$. Then the function $g(b,a) = \frac{W(b,a)}{|w(b,a)|^{\frac{\alpha+1}{\alpha}}}$ is continuous on $\mathbb{R}^2 \setminus \{(0,0\}$ and homogeneous of degree zero with respect to $\zeta_{\varepsilon}^{(1,1)}$. In particular, for every $\beta > 0$, the function g^{β} is uniformly bounded over $\mathbb{R}^2 \setminus \{(0,0\}$.*

Proof. The only fact non trivial to establish is that g is well-defined on $\mathbb{R}^2 \setminus \{(0,0\}$. One can assume with no loss of generality that both a and b are positive real numbers. Set $B := b/a$. Then $g(b,a) = g(B,1)$ and we are left to prove that $g(B,1)$ is continuous at $B = 1$. It is immediate to see that the latter fact hold true by taking the Taylor's expansions of both $W(B,1)$ and $w(B,1)$ in a neighborhood of $B = 1$. □

Lemma 3. *For every $\theta > 0$, positive integer i and non negative real numbers $a_1, \cdots, a_i$, one has that $(\sum_{j=1}^{i} a_j)^{\theta} \leq \max(1, i^{\theta-1}) \sum_{j=1}^{i} a_j^{\theta}$.*

Proof. The result is immediate for either $i = 1$ or $\theta \geq 1$, since it follows from the convexity of the function $x \mapsto x^{\theta}$ defined on $\mathbb{R}_+$. Assume now that $i > 1$ and $\theta < 1$. Let $\Delta_i := \{a = (a_1, \cdots, a_i) \in (\mathbb{R}_+)^i,\ \sum_{j=1}^{i} a_j = 1\}$, f_i be the real-valued function given by $f_i(a) = \sum_{j=1}^{i} a_j^{\theta}$ and C_i be the minimum of f_i over Δ_i, which is well defined since Δ_i is compact and f_i is continuous. We want to prove that $C_i = 1$ for every $i \geq 1$. Since Δ_i can be seen as a face of Δ_{i+1}, one gets that

$C_{i+1} \leq C_i \leq C_1 = 1$. We now prove the resullt by induction on i. If C_{i+1} is reached at an interior point $\bar{a}$ of Δ_{i+1}, then a trivial application of Lagrange's theorem shows that all the coordinates of $\bar{a}$ are equal to $1/i+1$ implying that $C_{i+1} = i+1^{1-\theta} > 1$, which is not possible. Then, C_{i+1} is reached at a boundary point $\bar{a}$ of Δ_{i+1}, i.e., $C_{i+1} = C_i = 1$. □

C. Motivation behind Homogeneity based control

The main motivation behind this paper is to develop a Lyapunov-based universal homogeneous controller for an arbitrary order perturbed chain of integrators represented by System (5). The insistence upon homogeneity based control is due to the fact that varying the controller's homogeneity degree produces different interesting results.

Let us present some observations that were made in [19]. Considering the following one-dimensional differential equation

$$\dot{z} = \omega(z) = -c\lfloor z \rceil^{\alpha}, \tag{9}$$

where $\alpha \geq 0$, $c > 0$. This system is stable for all $c > 0$ and $\alpha \geq 0$. However, different characteristics can be obtained in the system, depending upon the value of α:

- $\alpha = 0$: the convergence to zero occurs in finite-time. The controller $\omega(z)$ is uniformly bounded for $z \in \mathbb{R}$ but discontinuous at $z = 0$;
- $0 < \alpha < 1$: the convergence to zero occurs in finite-time. The controller $\omega(z)$ is unbounded and tends to zero as $|z| \to 0$;
- $\alpha > 1$: the convergence to zero is asymptotic, however the convergence time to the sphere $\mathbf{B}(0,1) = \{z \in \mathbb{R} : \|z\| < 1\}$ is uniformly bounded by a constant. The controller $\omega(z)$ is unbounded. the set $\mathbf{B}(0,\varepsilon) = \{z \in \hat{U} : V(z) < \varepsilon\}$ is fixed-time attractive.

Let us now consider a perturbed integrator, i.e. (9) is replaced by the following differential inclusion:

$$\dot{z} \in [-\bar{\varphi}, +\bar{\varphi}] + u(z)[\gamma_m, \gamma_M], \tag{10}$$

where $\gamma_m \leq \gamma_M$ and $\bar{\varphi}$ are arbitrary positive constants. The observations mentioned above can be extended to System (10) by applying a control of the form $u = \frac{1}{\gamma_m}(\omega(z) + \bar{\varphi} sign(\omega(z)))$ [18]. In addition, manipulation of homogeneity degree κ leads us to controllers with the following properties:

- globally fixed-time unbounded controller: switch from $\kappa_1 > 0$ to $\kappa_2 < 0$ when z reaches the sphere $\mathbf{B}(0,1)$.
- uniformly bounded controller, with reduced amplitude of discontinuous control as z converges to zero: switch from $\kappa_1 = -1$ to $-1 < \kappa_2 < 0$.

D. Contribution

In this work, we extend the above observations related to homogeneous controllers for a single integrator to the stabilization of perturbed chain of integrators of arbitrary order $r \geq 1$, based on a controller which stabilizes pure chain of integrators. It is shown that, for particular choice of homogeneity degree, a bounded Lyapunov-based arbitrary order controller can be designed, which is similar in structure to Levant's well-known homogeneous controller [15]. The existence of the Lyapunov function provides the added advantage of analytical tuning of controller parameters. It is also demonstrated that a bounded controller is synthesized using a change of homogeneity degree, such that the controller has a reduced amplitude at $z = 0$. Then a globally fixed-time controller is obtained, also by controlling the homogeneity degree.

III. Controller design

We will now develop the controller in two steps. The stabilization of a pure chain of integrators will be considered first. Then the study is extended to the case of a perturbed chain of integrators.

A. Stabilization of a pure chain of integrators

For r a positive integer, consider the following pure chain of integrators:

$$\dot{z}_i = z_{i+1},\ i = 1,...,r-1, \quad \dot{z}_r = u. \tag{11}$$

The following Hong's controller guarantees the stabilization of (11).

Theorem 1. *[10] Let r be the order of the pure chain of integrators given in* (11). *For $\kappa \in [-1/r, 1/r]$, set $p_i = 1 + (i-1)\kappa, \quad i = 1,\cdots,r$. Then there exist constants $l_i > 0,\ i = 1,\cdots,r$, independent of κ, such that the feedback control law*

$$u = \omega_\kappa^H(z) := v_r, \tag{12}$$

where v_r is defined inductively by:

$$v_0 = 0,\ v_i = -l_i \lfloor \lfloor z_i \rceil^{\beta_{i-1}} - \lfloor v_{i-1} \rceil^{\beta_{i-1}} \rceil^{(p_i+\kappa)/(p_i\beta_{i-1})}, \tag{13}$$

stabilizes System (11)*, where* β_i *are defined by*

$$\beta_0 = p_2, \qquad (\beta_i + 1)p_{i+1} = \beta_0 + 1 > 0, \quad i = 1, ..., r-1. \tag{14}$$

There also exists a homogeneous Lyapunov function $L_{\kappa,r}(z)$ *for the closed-loop system* (11) *with the state-feedback* u*, that satisfies* $\dot{L}_{\kappa,r} \leq -CL_{\kappa,r}^{(2+2\kappa)/(2+\kappa)}$*, for some positive constant* C*, independent of* κ*.*

Remark 1. *Theorem 1 has been proved by Hong [10] for* $\kappa \in (-1/r, 0)$*, following the same proof the theorem can be extended easily to* $\kappa \in [-1/r, 1/r]$*.*

Remark 2. *The remarkable feature of the above result lies in the explicit construction of both the controller and the Lyapunov function that we recall next. For* $1 \leq i \leq r$*, define*

$$\begin{aligned} w_i(z_1, \cdots, z_i) &= \lfloor z_i \rceil^{\beta_{i-1}} - \lfloor v_{i-1} \rceil^{\beta_{i-1}}, \\ W_i(z_1, \cdots, z_i) &= \int_{v_{i-1}(z_1,...,z_{i-1})}^{z_i} w_i(z_1, ..., z_{i-1}, s)\, ds, \\ &= \frac{1}{\beta_{i-1}+1}\left(|z_i|^{\beta_{i-1}+1} - |v_{i-1}|^{\beta_{i-1}+1}\right) - \lfloor v_{i-1} \rceil^{\beta_{i-1}} (z_i - v_{i-1}). \end{aligned}$$

Then the Lyapunov function $L_{\kappa,r}$ *is defined by*

$$L_{\kappa,r}(z) = \textstyle\sum_{i=1}^{r} W_i(z_1, \cdots, z_i). \tag{15}$$

We present next a modified version of Hong's controller denoted ω_κ^{HM} which also guarantees the stabilization of (11).

Theorem 2. *Let* r *be the order of the pure chain of integrators given in* (11)*. For* $\kappa \in [-1/r, 0]$*, set* $p_i = 1 + (i-1)\kappa, \quad i = 1, \cdots r$*, and let* c *be a positive constant such that* $c \geq \max(p_1, \cdots, p_r)$*. Then there exist constants* $l_i > 0,\ i = 1, \cdots, r$*, independent of* κ*, such that the feedback control law*

$$u = \omega_\kappa^{HM}(z) := v_r, \tag{16}$$

where v_r *is defined inductively by:*

$$v_0 = 0,\ v_i = -l_i N_i,\ i = 1, \cdots, r, \tag{17}$$

stabilizes System (11)*, where* $N_i = \left\lfloor \lfloor z_i \rceil^{c/p_i} - \lfloor v_{i-1} \rceil^{c/p_i} \right\rceil^{(p_i+\kappa)/c}$*. There also exists a homogeneous Lyapunov function* $V_{\kappa,r}(z)$ *for the closed-loop system* (11) *under* u*, that satisfies* $\dot{V}_{\kappa,r} \leq -CV_{\kappa,r}^{(c+1+\kappa)/(c+1)}$*, for some positive constant* C *independent of* κ*.*

Proof. The argument largely follows the lines of [10]. For $1 \leq i \leq r$, we define

$$
\begin{aligned}
w_i(z_1,\cdots,z_i) &:= \lfloor z_i \rceil^{c/p_i} - \lfloor v_{i-1} \rceil^{c/p_i}, \\
W_i(z_1,\cdots,z_i) &= \int_{v_{i-1}}^{z_i} \lfloor s \rceil^{\frac{c}{p_i}} - \lfloor v_{i-1} \rceil^{\frac{c}{p_i}} ds, \\
&= \frac{|z_i|^{\frac{c}{p_i}+1} - |v_{i-1}|^{\frac{c}{p_i}+1}}{\frac{c}{p_i}+1} - \lfloor v_{i-1} \rceil^{\frac{c}{p_i}} (z_i - v_{i-1}).
\end{aligned} \tag{18}
$$

It can be seen that W_i is positive definite function with respect to $v_{i-1} - z_i$, homogeneous with respect to ξ_p^ε of degree $(c+p_i)$. We introduce $\bar{W}_i := W_i^{\delta_i}$, where $\delta_i = (c+1)/(c+p_i)$, so that all functions $\bar{W}_i$ are homogeneous of the same homogeneity degree $(c+1)$. We proceed to prove the theorem by induction on r.

Step 1: Consider $\dot{z}_1 = u$. For any $l_1 > 0$, taking $u = \omega_\kappa^{HM}(z_1) = -l_1 \lfloor z_1 \rceil^{(p_1+\kappa)/p_1}$ stabilizes the closed-loop system. The Lyapunov function $V_{\kappa,1} = W_1 = |z_1|^{1+c}/(1+c)$ is homogeneous of degree $c+1$ and $\dot{V}_{\kappa,1} = -l_1 |z_1|^{c+p_2} \leq -\eta_1 V_{\kappa,1}^{(c+1+\kappa)/(c+1)}$, for some constant $\eta_1 > 0$.

Step i: Assume that the conclusion holds true till $i-1$. Define the Lyapunov function $V_{\kappa,i}$ by $V_{\kappa,i} = V_{\kappa,i-1} + \bar{W}_i = \sum_{j=1}^{i} \bar{W}_j$. Then,

$$
\begin{aligned}
\dot{V}_{\kappa,i} &= \sum_{j=1}^{i-1} \frac{\partial \bar{W}_i}{\partial z_j} z_{j+1} + w_i v_i W_i^{\frac{-(i-1)\kappa}{c+p_i}} + \dot{V}_{\kappa,i-1} + \frac{\partial V_{\kappa,i-1}}{\partial z_{i-1}} (z_i - v_{i-1}), \\
&= \sum_{j=1}^{i-1} \frac{\partial \bar{W}_i}{\partial z_j} z_{j+1} - l_i |w_i|^{\frac{c+p_i+\kappa}{c}} \bar{W}_i^{\frac{-\kappa(i-1)}{c+p_1}} + \dot{V}_{\kappa,i-1} + \frac{\partial V_{\kappa,i-1}}{\partial z_{i-1}} (z_i - v_{i-1}).
\end{aligned} \tag{19}
$$

Using Lemma 2, one gets firstly that there exists $k_i > 0$ such that for every non zero $(z_1,\cdots,z_i)$, one has $\bar{W}_i(z_1,\cdots,z_i)/|w_i(z_1,\cdots,z_i)|^{(c+1)/c} \leq k_i$ and secondly

$$
-l_i |w_i|^{\frac{c+p_i+\kappa}{c}} \bar{W}_i^{\frac{-\kappa(i-1)}{c+p_1}} \leq -l_i \frac{\bar{W}_i^{\frac{c+p_i+\kappa}{c+p_1}}}{k_i^{\frac{c+1+\kappa}{c+1}}}.
$$

The fact that $\bar{W}_i$ are homogeneous with respect to ζ_ε^p of degree $(c+1)$ for each $i = 1,\cdots,r$, implies that V_i are homogeneous of degree $(c+1)$ with respect to ζ_ε^p as well. In addition, according to Proposition 1, $\dot{V}_i$ are homogeneous of degree $(c+1-\kappa)$ with respect to ζ_ε^p. Then without loss of generality, the study can be restricted to the unit sphere $S_{i,c}$ defined by

$$
S_{i,c} = \{z \in \mathbb{R}^i : \Gamma_{i,c}(z) = 1\},
$$

where $\Gamma_{i,c}(z)$ for $z \in \mathbb{R}^i$ is the homogeneous norm given by

$$
\Gamma_{i,c}(z) \equiv \Gamma_{i,c}(z_1,\cdots,z_i) = \left(\sum_{j=1}^{i} |z_j|^{c/p_j} \right)^{1/c}.
$$

Set

$$V_i^0(z_1,\cdots,z_i) := \sum_{j=1}^{i-1} \frac{\partial \bar{W}_i}{\partial z_j} z_{j+1} + \frac{\partial V_{\kappa,i-1}}{\partial z_{i-1}} (z_i - v_{i-1}),$$

and define $S_i^+ = \{z \in S_{i,c} \;\; V_i^0(z_1,\cdots,z_i) \geq 0\}$. The key point is that $\min_{z\in S_i^+, \kappa\in[-1/r,0]} \bar{W}_i^{\frac{c+p_2}{c+p_1}} > 0$. One can then choose $l_i > 0$ independent of $\kappa \in [-1/r, 0]$ such that, by setting $\eta_i := l_i/2k_i^{\frac{c+p_i+\kappa}{c+1}}$, we get $\dot{V}_i \leq -\sum_{j=1}^{i} \eta_j \bar{W}_j^{\frac{c+1+\kappa}{c+1}}$. At the final step, all parameters l_i are determined, by $V_{\kappa,r}(z) = \sum_{j=1}^{r} \bar{W}_j$ and $\dot{V}_{\kappa,r}(z) \leq -\sum_{j=1}^{i} \eta_j \bar{W}_j^{\frac{c+1+\kappa}{c+1}} \leq -\eta \sum_{j=1}^{i} \bar{W}_j^{\frac{c+1+\kappa}{c+1}}$, where $\eta := \min_{1\leq i\leq r} \eta_i$. Using Lemma 3, one gets that

$$\sum_{j=1}^{i} \bar{W}_j^{\frac{c+1+\kappa}{c+1}} \geq \left(\sum_{j=1}^{i} \bar{W}_j\right)^{\frac{c+1+\kappa}{c+1}}.$$

Finally we get $\dot{V}_{\kappa,r} \leq -\eta V_{\kappa,r}^{(c+1+\kappa)/(c+1)}$. □

Remark 3. *The controller ω_r^{HM} is only defined for $\kappa \in [\frac{-1}{r}, 0]$ as one can see from Eq.* (19).

B. Stabilization of an r-perturbed chain of integrator

From the controllers $\omega_\kappa^H(z)$ and $\omega_\kappa^{HM}(z)$ obtained in Theorem 1 and Theorem 2, we now proceed to the stabilization of the perturbed chain of integrators presented in System (5). The extension of Theorem 1 to the case of System (5) is based on the following result.

Theorem 3. *Let $\omega(z)$ and $V(z)$ be respectively, a state-feedback control law stabilizing System* (11) *and a Lyapunov function for the closed-loop system, which satisfy Lemma 1 and obey the following additional conditions: for every $z \in \hat{U}$,*

$$\tfrac{\partial V}{\partial z_r}(z)\omega(z) \leq 0 \text{ and } \omega(z) = 0 \Rightarrow \tfrac{\partial V}{\partial z_r}(z) = 0. \tag{20}$$

Then, for arbitrary constants $P, Q \geq 1$, the following control law stabilizes System (5)*:*

$$u(z) = P(\omega(z) + Q\bar{\varphi} sign(\omega(z)))/\gamma_m. \tag{21}$$

The function $V(z)$ remains a Lyapunov function for the closed-loop system and satisfies Condition (8)*. If $\hat{U} = \mathbb{R}^r$ and $V(z)$ is radially unbounded, then the closed-loop system is globally stable with respect to the origin.*

Proof. This theorem is a generalization of Theorem 2 of [18], where it has been proven for $P=Q=1$, and is established in the same way. □

Remark 4. *The controllers* $\omega_\kappa^H(z)$ *and* $\omega_\kappa^{HM}(z)$ *defined in Equation* (12) *and* (16)*, satisfy the geometric condition* (20) *imposed in Theorem 3. Indeed, one gets for* $z \in \mathbb{R}^r$*,*

$$\frac{\partial L_{\kappa,r}}{\partial z_r}\ \omega_\kappa^H = -l_r|z_i^{\beta_{i-1}} - v_{i-1}^{\beta_{i-1}}|^{2(1+\kappa)/p_i\beta_{i-1}}, \quad \frac{\partial V_{\kappa,r}}{\partial z_r}\ \omega_\kappa^{HM} = -l_r|z_j^{c/p_j} - v_{i-1}^{c/p_j}|^{1+(p_i+\kappa)/c}.$$

Remark 5. *The controller* $u(z)$ *presented in Equation* (21) *is clearly discontinuous. However, its absolute value* $|u(z)|$ *is equal to* $P(|\omega(z)|+Q\bar{\varphi})\gamma_m$*. Then* $\lim_{\|z\|\to 0}|u(z)|$ *takes its minimal value at the origin if* $\omega(z)$ *vanishes there. In particular when* $P=Q=1$*, it has been claimed in Section 2 of [14] that in order to stabilize the uncertain System* (5) *by a state-feedback controller* $u=u(z)$*, it is necessary that the controller be discontinuous at* $z=0$ *and* $\lim_{\|z\|\to 0}|u(z)| \geq \bar{\varphi}/\gamma_m =: M_{min}$*.*

IV. Discussion of Special Cases

Let us now consider some results that arise for some specific choices of the homogeneity degree. First, a bounded controller with minimum amplitude M_{min} of discontinuous control at $z=0$ is designed. Finally, a controller with globally fixed-time convergence is synthesized.

A. Homogeneous controller with minimum amplitude of discontinuous control at $z=0$

We first notice that, for $\kappa=-1/r$, if $\omega(z)$ denotes one of the controllers presented in Theorem 1 or Theorem 2, then $\omega(z)$ is bounded and the corresponding controller u defined as

$$u = \tfrac{1}{\gamma_m}\left(\omega(z) + \bar{\varphi} sign(\omega(z))\right) \equiv \tfrac{l_r+\bar{\varphi}}{\gamma_m}\mathrm{sign}(\omega(z)), \tag{22}$$

stabilizes System (5) in finite time. Moreover, the above controller is identical to that of Levant [15] for $1 \leq r \leq 2$. The advantage of our controller in these cases is that the Lyapunov function provides an analytical method of tuning the controller parameters, whereas the tuning is empiric in Levant's case. Unfortunately this is not the case as soon as $r \geq 3$ and one needs the delicate analysis developed in [15].

The amplitude of the discontinuous control given in Eq. (22) is equal to $M=(l_r+\bar{\varphi})/\gamma_m$. We shall now see that this amplitude can be reduced to its minimum level $M_{min}=\bar{\varphi}/\gamma_m$ when the state z tends to zero, by changing the degree of homogeneity.

Theorem 4. *For $k \in (-1/r, 0)$ and $A > 0$ satisfying*

$$\max_{\bar{V}_{k,r}(z) \le A} |\bar{\omega}_k(z)| \le l_r, \tag{23}$$

we define the function $U_{k,A}(z) := \begin{cases} \omega_{-1/r}(z) & \text{if } \bar{V}_{k,r}(z) > A, \\ \bar{\omega}_k(z) & \text{if } \bar{V}_{k,r}(z) \le A, \end{cases}$
where $\omega_{-1/r}(z)$ is either equal to $\omega^H_{-1/r}$ or $\omega^{HM}_{-1/r}$, for $k \le 0$, $\bar{\omega}_k(z)$ is either equal to ω^H_k or ω^{HM}_k and $\bar{V}_{k,r}$ is the Lyapunov function associated with $\bar{\omega}_k(z)$. Here ω^H and ω^{HM} are defined in Equation (12) and (16) respectively.
Then the controller $u(z) := \left(U_{k,A}(z) + \bar{\varphi} sign(U_{k,A}(z))\right)/\gamma_m$ stabilizes System (5) in finite time, and $u(z)$ is bounded with minimum amplitude of discontinuous control M_{min} at $z = 0$.

Proof. Consider the following sets

$$\mathbf{S}_1 = \{z \in \mathbb{R}^r : |\bar{\omega}_k(z)| \le l_r\}, \quad \mathbf{S}_2 = \{z \in \mathbb{R}^r : \bar{V}_{k,r}(z) \le A\}.$$

According to Condition (23), we have $\mathbf{S}_2 \subset \mathbf{S}_1$. As $\dot{V}_{-1/r,r}(z) < 0, \forall z \notin \mathbf{S}_2$, then every trajectory of System (5) reaches $\mathbf{S}_2$ in finite-time. Moreover, for $z \in \mathbf{S}_2$, $U_{k,A}(z)$ is equal to $\bar{\omega}_k(z)$, with $|\bar{\omega}_k(z)| \le l_r$. Therefore, as soon as a trajectory reaches $\mathbf{S}_2$, it will stay in it forever since $\dot{\bar{V}}_{k,r}(z) < 0, \forall z \notin \mathbf{S}_1, \forall z \neq 0$. One concludes that every trajectory of System (5) converges to zero in finite-time and $U_{k,A}(z)$ tends to zero as z tends to zero. As a result, $\forall z \in \mathbb{R}^r$, $|u(z)| \le M_{min} + l_r/\gamma_m$. and $\lim_{\|z\| \to 0} |u(z)| = \bar{\varphi}/\gamma_m = M_{min}$. □

B. Globally fixed-time Homogeneous controller

In certain cases, it is required that the controller converges within a fixed interval of time, irrespectively of its initial condition. This can also be achieved by changing the homogeneity degree.

Theorem 5. *For $k_1 \in (0, 1/r)$, $k_2 \in (-1/r, 0)$ and $B > 0$, define*

$$E := \min_{V_{k_2,r}(z) = B} V_{k_1,r}(z) > 0, \tag{24}$$

and the function $U_{k,B}(z) = \begin{cases} \omega^H_{k_1}(z) & \text{if } V_{k_2,r}(z) > B, \\ \bar{\omega}_{k_2}(z) & \text{if } V_{k_2,r}(z) \le B, \end{cases}$

where $\bar{\omega}_{k_2}(z)$ is either equal to $\omega^H_{k_2}$ or $\omega^{HM}_{k_2}$ and $\bar{V}_{k_2,r}$ is the Lyapunov function associated with $\bar{\omega}_{k_2}(z)$. Here ω^H and ω^{HM} are defined in Equation (12) and (16) respectively.
Then the controller $u(z) := \left(U_{k,B}(z) + \bar{\varphi} sign(U_{k,B}(z))\right)/\gamma_m$ stabilizes System (5) in fixed time $T \leq T_u + T_f$ where the values of T_u and T_f are given by

$$T_u = (2+k_1)E^{\frac{k_1}{2+k_1}}/(k_1C), \quad T_f = \begin{cases} (2+k_2)B^{\frac{-k_2}{2+k_2}}/(-k_2C), \text{ if } \bar{\omega}_{k_2}(z) = \omega^H_\kappa(z). \\ (c+1)B^{\frac{-k_2}{c+1}}/(-k_2C), \text{ if } \bar{\omega}_{k_2}(z) = \omega^{HM}_\kappa(z). \end{cases}$$

Proof. The conclusion follows by integrating the differential equation $\dot{V} = -CV^\alpha$ on appropriate time intervals. Consider first the following sets

$$\mathbf{S}_1 = \{z \in \mathbb{R}^r : V_{k_1,r}(z) \leq E\}, \quad \mathbf{S}_2 = \{z \in \mathbb{R}^r : \bar{V}_{k_2,r}(z) \leq B\}.$$

According to Condition (24), we get that $\mathbf{S}_1 \subset \mathbf{S}_2$. Clearly, z will reach $\mathbf{S}_2$ in a fixed-time, bounded by a constant T_u, calculated as follows: for $\alpha = 1 + \frac{k_1}{2+k_1}$, $\int_E^{+\infty} \frac{dV}{V^\alpha} = -C\int_0^{T_u} dt$, then $T_u = (2+k_1)E^{\frac{k_1}{2+k_1}}/(k_1C)$. When z reaches $\mathbf{S}_2$, i.e. $\bar{V}_{k_2,r}(z) = B$, z will converge to zero in a finite-time bounded by T_f, which is calculated as follows: for $\alpha = 1 + \frac{k_2}{2+k_2}$, $\int_B^0 \frac{dV}{V^\alpha} = -C\int_{T_u}^{T=T_u+T_f} dt$, then $T_f = (2+k_2)B^{\frac{-k_2}{2+k_2}}/(-k_2C)$. Finally, for $\alpha = 1 + \frac{k_2}{c+1}$, $\int_B^0 \frac{dV}{V^\alpha} = -C\int_{T_u}^{T=T_u+T_f} dt$, then $T_f = (c+1)B^{\frac{-k_2}{c+1}}/(-k_2C)$ □

Remark 6. *The rate of convergence can be accelerated via time-rescaling (see Theorem 2 of Hong et al. [29]). This is done by replacing the controller $\omega(z_1, z_2, \cdots, z_r)$ by $\bar{\omega}(z_1, z_2, \cdots, z_r) = \tau^r \omega(z_1, \frac{z_2}{\tau}, \cdots, \frac{z_r}{\tau^{r-1}})$ where $\tau > 1$, and taking u as $u = \frac{m}{\gamma_m}(\bar{\omega} + n\bar{\varphi} sign(\bar{\omega}))$. By taking $\bar{t} = \tau t$ and $\bar{z}_i = \tau^{1-i} z_i$, we obtain $\dot{V}(\bar{z}_1, \cdots, \bar{z}_r) \leq -\tau CV(\bar{z}_1, \cdots, \bar{z}_r)$ and the settling time becomes $\bar{T} \leq (T_u + T_f)/\tau$.*

V. Simulation Results

In this section, we illustrate the performance of our proposed controllers using the following perturbed triple integrator defined by: $\dot{z}_1 = z_2$, $\dot{z}_2 = z_3$, $\dot{z}_3 = \varphi + \gamma u$, with $\varphi = \sin(t)$ and $\gamma = 3 + \cos(t)$. Then, we have $\gamma_m = 2$, $\gamma_M = 4$, $\bar{\varphi} = 1$.
The parameters of the controller are chosen as follows: $l_1 = 1$, $l_2 = 3$, $l_3 = 10$.
We start first by fixing the parameter κ for different values $\{1/4, -1/4, -1/3\}$.
For $\kappa > 0$, Figure 1 shows a fast convergence of the states to a neighborhood of zero by an unbounded controller, otherwise the convergence to zero is asymptotic. For $-1/3 < \kappa < 0$, the convergence of the states to zero in finite-time is obtained by an unbounded controller with a

minimum amplitude of the discontinuous control at $z=0$, as shown in Figure 2. The finite-time convergence of the states is also shown in Figure 3 for $\kappa=-1/3$, using a bounded controller with a large discontinuous control at $z=0$.

The performance of a bounded controller which ensures a minimum discontinuous control amplitude at zero is shown in Figure 4 by switching κ in neighborhood of zero, from $-1/3$ to $-1/4$.

The performance of a globally fixed-time controller is shown in Figure 5. Figure 6 shows convergence time versus several initial conditions: $z_1=z_2=z_3=1$, 10, 100, 10^3, ... 10^{10}. It is shown that the convergence time will not exceed 8.5 sec for any initial condition. Globally fixed-time stability is assumed to be established by the time after which, $|z_1|$, $|z_2|$, $|z_3|$ are less than 1×10^{-4}.

VI. Conclusions

In this paper, we presented a Lyapunov-based method for designing finite-time convergent controllers for stabilization of perturbed chain of integrators of arbitrary order. This method consists in appropriate modifications of homogeneous controller stabilizing pure chain of integrators. It was also shown that the properties of minimum discontinuity amplitude of the controller and globally fixed-time convergence can be obtained by changing the homogeneity degree of the controller.

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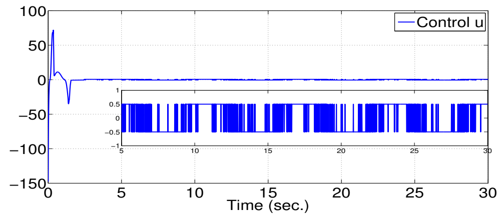
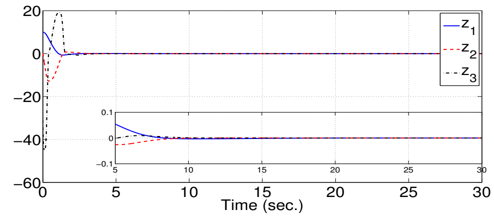
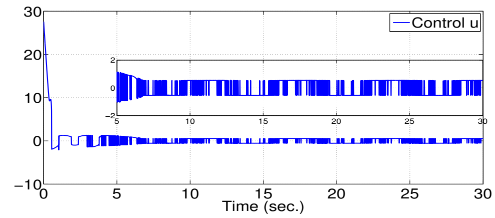


(a) control law u versus time (s).

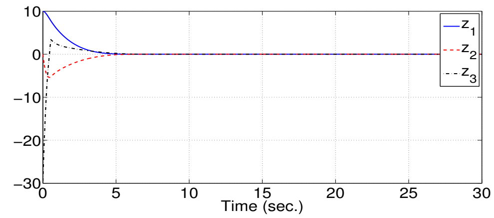


(b) z_1, z_2 and z_3 versus time (s).

Fig. 1. test for $\kappa > 0$

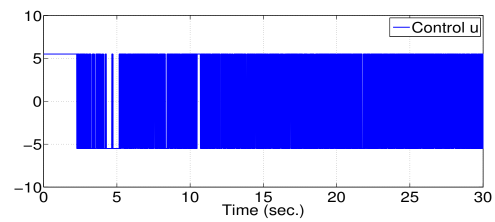


(a) control law u versus time (s).

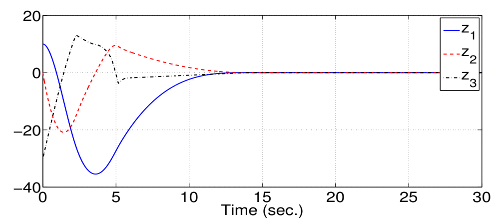


(b) z_1, z_2 and z_3 versus time (s).

Fig. 2. test for $-1/r < \kappa < 0$

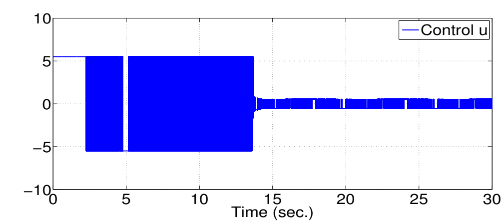


(a) control law u versus time (s).

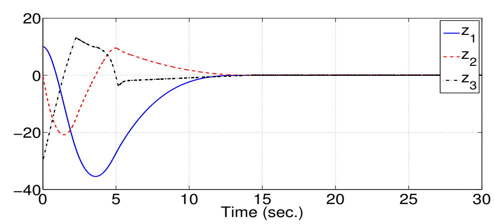


(b) z_1, z_2 and z_3 versus time (s).

Fig. 3. test for $\kappa = -1/r$ (case equivalent of [15])

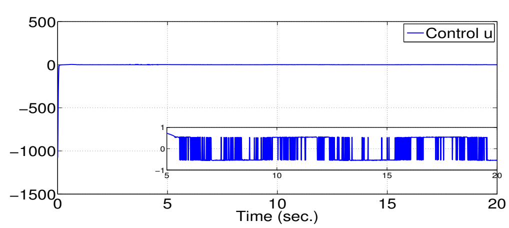


(a) control law u versus time (s).

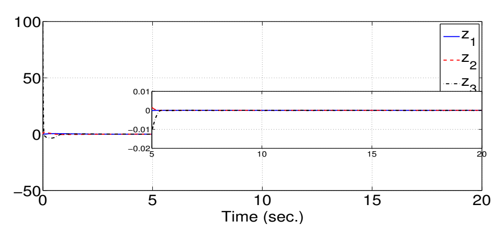


(b) z_1, z_2 and z_3 versus time (s).

Fig. 4. test for κ switching from $-1/r$ to $k \in (-1/r, 0)$



(a) control law u versus time (s).



(b) z_1, z_2 and z_3 versus time (s)

Fig. 5. test for κ switching from $-k$ to k, $k \in (-1/r, 0)$

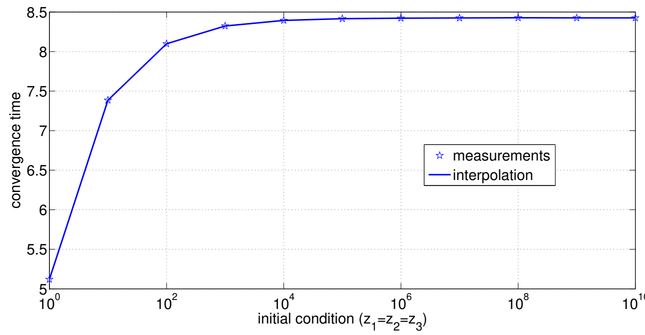


Fig. 6. Convergence time versus initial condition.

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