Horizontal Holonomy for Affine Manifolds

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Abstract

In this paper, we consider a smooth connected finite-dimensional manifold M, an affine connection ∇ with holonomy group H^∇ and Δ a smooth completely non integrable distribution. We define the Δ-horizontal holonomy group H^∇_Δ as the subgroup of H^∇ obtained by ∇-parallel transporting frames only along loops tangent to Δ. We first set elementary properties of H^∇_Δ and show how to study it using the rolling formalism ([9]). In particular, it is shown that H^∇_Δ is a Lie group. Moreover, we study an explicit example where M is a free step-two homogeneous Carnot group and ∇ is the Levi-Civita connection associated to a Riemannian metric on M, and show that in this particular case the connected component of the identity of H^∇_Δ is compact and strictly included in H^∇.

1 Introduction

The purpose of this paper consists of extending the concept of horizontal holonomy of an affine connection in the context of distributions on a manifold i.e., subbundles of the tangent bundle of a manifold. More precisely, consider the triple (M, ∇, Δ) where M is an n-dimensional smooth connected manifold, ∇ is an affine connection on M (one says then that (M, ∇) is an affine manifold) and Δ is a smooth distribution on M. One furthermore assumes that Δ is completely controllable, i.e., every pair of points in M can be connected by an absolutely continuous tangent to the distribution Δ. Recall that the holonomy group H^∇ of ∇ as the subgroup of $\mathrm{GL}(n)$ obtained (up to conjugation) by ∇-parallel transporting frames along absolutely continuous (or piecewise smooth) loops of M.

For every point $x \in M$, we define the subset $H^\nabla_\Delta|_x$ of $H^\nabla|_x$, the holonomy group of ∇ at x, obtained by parallel transporting, with respect to ∇, frames of M along a restricted set of

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absolutely continuous Δ-horizontal loops based at x, namely along loops which are tangent (almost everywhere) to the distribution Δ. Thanks to the hypotheses of connectedness of M and complete controllability of Δ, one can deduce that the sets $H^{\nabla}_{\Delta}|_x$, $x \in M$ are all conjugate to a Lie-subgroup H^{∇}_{Δ} of H^{∇} that we call Δ-horizontal (or simply horizontal) holonomy group of ∇. In the case where ∇ is the Levi-Civita connection associated to some Riemannian metric g on M, one can take both H^{∇}_{Δ} and H^{∇} as subgroups of $\mathrm{O}(n)$ and even $\mathrm{SO}(n)$ if in addition M is assumed to be oriented. Understanding the relationships between H^{∇} and H^{∇}_{Δ} appears to be an interesting challenge. For instance, given an affine manifold (M, ∇), determining necessary and (or) sufficient conditions on a completely controllable distribution Δ of M so that the H^{∇}_{Δ} equals H^{∇} is not an obvious question, besides trivial cases. Another issue to be addressed consists of fixing the pair manifold and distribution i.e., (M, Δ) and then make the connection ∇ vary. One question could be to undestand if there are connections "more adapted or intrinsic" than others (in a sense to be defined) for the pair (M, Δ). Moreover, one could also study the mapping $g \mapsto H^{\nabla^g}_{\Delta}$ where g is a complete Riemannian metric on M and ∇^g the corresponding Levi-Civita connection, for instance describing the range of this mapping. Note that such issues have been already addressed in [13] where the authors consider the case of manifolds of contact type with a distribution arising from an adapted connection.

In this paper, we essentially start this program by defining precisely the Δ-horizontal holonomy group associated to a given admissible triple (M, ∇, Δ). Our first main result besides elementary ones is the following: we prove that if Δ is a constant rank completely controllable distribution, then H^{∇}_{Δ} is a connected Lie subgroup of $\mathrm{GL}(n)$ (or $\mathrm{O}(n)$ if ∇ is the Levi-Civita connection of some Riemannian metric on M). This enables us to study H^{∇}_{Δ} via its differentiable structure. Moreover, we also propose to study Δ-horizontal holonomy groups by recasting them within the framework of rolling manifolds. Indeed, recall that E. Cartan defines holonomy groups in [6] as what is called now affine holonomy group by "developing" a manifold its tangent space at any point. This procedure has been generalized in [9, 14, 29] to an arbitrary pair of Riemannian manifolds of same dimension and it is also called as "rolling a Riemannian manifold onto another one without slipping nor spinning". Yet, that type of rolling was extended in [24] to the case where both manifolds do not have necessarily the same dimension. See also [8] for a historical account as well as applications of the rolling of manifolds.

In the present situation, the rolling framework amounts to define an n-dimensional smooth distribution $\mathscr{D}_R$, called the rolling distribution, on the state space Q defined as the fiber bundle over the product of (M, ∇) and $(\mathbb{R}^n, \hat{\nabla}^n)$ where $\hat{\nabla}^n$ is the Euclidean connection on $\mathbb{R}^n$ and the typical fiber over $(x, \hat{x}) \in M \times \mathbb{R}^n$ is identified with the set of endomorphisms of T_xM. For every $q \in Q$, let $\mathcal{O}_{\mathscr{D}_R}(q)$, be the $\mathscr{D}_R$-orbit through q, i.e., the set of endpoints of the absolutely continuous curves starting at q and tangent to $\mathscr{D}_R$. Then, for every $q \in Q$ and $x' \in M$, the fiber of $\mathcal{O}_{\mathscr{D}_R}(q)$ over x' (if non-empty) is conjugate to a subgroup of $\mathbb{R}^n \rtimes \mathrm{GL}(n)$ whose $\mathrm{GL}(n)$-part is exactly H^{∇}. Moreover, since $\mathcal{O}_{\mathscr{D}_R}(q)$ is an immersed manifold in Q whose tangent space at every $q' \in \mathcal{O}_{\mathscr{D}_R}(q)$ contains the (evaluation at q' of the) Lie algebra generated by vector fields tangent to $\mathscr{D}_R$ (cf. [1, 18, 19]), it is possible to determine elements of the Lie algebra of H^{∇} as Lie brackets of vector fields tangent to $\mathscr{D}_R$. Given now a completely controllable distribution Δ, one can define a subdistribution Δ_R of $\mathscr{D}_R$ on Q so that, for every $q \in Q$ and $x' \in M$ the fiber over x' of $\mathcal{O}_{\Delta_R}(q)$ is conjugate to a subgroup of $\mathbb{R}^n \rtimes \mathrm{GL}(n)$ whose $\mathrm{GL}(n)$-part is now equal to H^{∇}_{Δ}. Since the latter has been proved to be a Lie group, one can determine elements of its Lie algebra by computing Lie brackets of vector fields tangent to Δ_R. Note that, as also mentioned above, $\mathrm{GL}(n)$ can be replaced by $\mathrm{O}(n)$ ($\mathrm{SO}(n)$ respectively) if ∇ is the Levi-Civita connection of some Riemannian metric on M (if in addition M is oriented).

We use that approach to provide our second main result, namely an explicit example for a strict inequality in $\dim(H^{\nabla}_{\Delta}) \le \dim(H^{\nabla})$. More precisely, we consider the triple (M, ∇, Δ) where M is a free step-two homogeneous Carnot group of $m \ge 2$ generators $(X_i)_{1 \le i \le m}$, ∇ is the Levi-Civita associated with the Riemannian metric on M defined in such a way that the X_i's, $1 \le i \le m$ and the Lie brackets $[X_i, X_j]$, $1 \le i < j \le m$ form an orthonormal basis and Δ is the distribution defined by the span of the X_i's, $1 \le i \le m$. In this case M is of dimension $m + n$ with $n = m(m-1)/2$. Then we prove that (M, ∇) has full holonomy i.e., $H^{\nabla} = \mathrm{SO}(n)$, and that the connected component of the identity of H^{∇}_{Δ} is a closed Lie subgroup of $\mathrm{SO}(n)$ of dimension $m + n$.

We close this introduction by describing the structure of the paper. We gather in the second section most of the required notations and we precisely define the (∇, Δ)-holonomy group first using classical concepts and secondly by relying on the rolling framework. In the fourth section, we consider in details the example of the free step-two homogeneous Carnot group of $m \ge 2$ generators and we conclude with an appendix containing a technical result needed in the third section.

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2 Notations

Let M be an n-dimensional smooth connected manifold where n is a positive integer. Let $\mathcal{X}(M)$ be the set of smooth vector fields on M. An affine connection ∇ on M is a $\mathbb{R}$-bilinear map

$$\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M); \quad (X, Y) \mapsto \nabla_X Y,$$

which is $C^\infty(M)$-linear in the first variable and verifies the Leibniz rule over $C^\infty(M)$ in the second variable. The pair (M, ∇) is said to be an affine manifold. If, moreover, the exponential map $\exp^{\nabla}_x$ of (M, ∇) is defined on the whole tangent space $T_x M$ for all $x \in M$, then (M, ∇) is said to be a (geodesically) complete affine manifold. We use ∇^n and ∇^g respectively to denote the Euclidean connection on $\mathbb{R}^n$ and the Levi-Civita connection of a Riemannian manifold (M, g). The notation $[\cdot, \cdot]$ stands for the Lie bracket operation in TM.

We define the curvature tensor R^{∇} and the torsion tensor T^{∇} of a affine connection ∇ as

$$\begin{aligned} R^{\nabla}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \\ T^{\nabla}(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y], \end{aligned}$$

respectively, for smooth vector fields X, Y, Z on M.

If $\gamma : I \to M$ is any absolutely continuous (a.c. for short) curve in M defined on a real interval I containing 0, we use $(P^{\nabla})^t_0(\gamma) T_0$, $t \in I$, to denote the ∇-parallel transport along γ of a tensor

T_0 of rank (m,k) at $\gamma(0)$. It is the unique solution for $T(t)$ (in terms of tensor fields of rank (m,k) defined along γ) to the Cauchy problem

$$\nabla_{\dot\gamma(t)}T(t) = 0, \quad \text{for a.e. } t \in I,\ T(0) = T_0.$$

Let $(\hat M, \hat\nabla)$ be another affine manifold and $f : M \to \hat M$ be a smooth map. we say that f is affine if for any a.c. curve $\gamma : [0,1] \to M$, one has

$$f_\star|_{\gamma(1)} \circ (P^\nabla)_0^1(\gamma) = (P^{\hat\nabla})_0^1(f\circ\gamma) \circ f_\star|_{\gamma(0)}. \tag{1}$$

An a.c. curve $\gamma : [a,b] \to M$ is a loop based at $x \in M$ if $\gamma(a) = \gamma(b) = x$. We denote by $\Omega_M(x)$ the space of all a.c. loops $[0,1] \to M$ based at some given point $x \in M$. Moreover, if $\gamma : [0,1] \to M$ and $\delta : [0,1] \to M$ are two a.c. curves on M such that $\gamma(0) = x$, $\gamma(1) = \delta(0) = y$ and $\delta(1) = z$ where $x,\ y,\ z \in M$, the concatenation $\delta\cdot\gamma$ is the a.c. curve defined by

$$\delta\cdot\gamma : [0,1] \to M, \quad (\delta\cdot\gamma)(t) = \begin{cases} \gamma(2t) & t \in [0,\frac12], \\ \delta(2t-1) & t \in [\frac12, 1]. \end{cases} \tag{2}$$

The previous definitions allow us to state the subsequent definition of holonomy group.

Definition 2.1. For every $x \in M$, the holonomy group $H^\nabla|_x$ at x is defined by

$$H^\nabla|_x = \{(P^\nabla)_0^1(\gamma) \mid \gamma \in \Omega_M(x)\}.$$

For every $x \in M$, $H^\nabla|_x$ is a subgroup of $\mathrm{GL}(T_xM)$, the group of isomorphisms of T_xM, which is clearly isomorphic to $\mathrm{GL}(n)$ the group of $n\times n$ invertible matrices with real entries. Since M is connected, it is well-known that, for any two points $x,y \in M$, $H^\nabla|_x$ and $H^\nabla|_y$ are conjugate subgroups of $\mathrm{GL}(T_xM)$ and thus one can define $H^\nabla \subset \mathrm{GL}(n)$ the holonomy group of the affine connection ∇ (cf. [20]).

We also recall that a smooth distribution Δ on M is a smooth subbundle of TM The flag of Δ is the collection of the distributions Δ^j, $j \geq 1$, where, for every $x \in M$, $\Delta^1|_x := \Delta|_x$ and $\Delta^{s+1}|_x := \Delta^s|_x + [\Delta^1, \Delta^s]|_x$ for $s \geq 1$. We say that the distribution Δ on M is of constant rank $m \leq n$ if $\dim(\Delta|_x) = m$ for every $x \in M$ and verifies the Lie algebraic rank condition (LARC) if, for any $x \in M$, there exists an integer $r = r(x)$ such that $\Delta^r|_x = T_xM$. The number $r(x)$ is called the step of $\Delta|_x$ (cf. [18] for more details).

An a.c. curve $\gamma : I \to M$, I bounded interval in $\mathbb{R}$, is said to be Δ-admissible, or Δ-horizontal, if it is tangent to Δ a.e. on I, i.e., if for a.e. $t \in I$, $\dot\gamma(t) \in \Delta|_{\gamma(t)}$. For $x_0 \in M$, the Δ-orbit through x_0, denoted $\mathcal{O}_\Delta(x_0)$, is the set of endpoints of the Δ-admissible curves of M starting at x_0, i.e.,

$$\mathcal{O}_\Delta(x_0) = \{\gamma(1) \mid \gamma : [0,1] \to M,\ \text{a.c. } \Delta\text{-admissible curve},\ \gamma(0) = x_0\}.$$

By the Orbit Theorem (cf.[18]), it follows that $\mathcal{O}_\Delta(x_0)$ is an immersed smooth submanifold of M containing x_0 so that the tangent space $T_y\mathcal{O}_\Delta(x_0)$ for every $y \in \mathcal{O}_\Delta(x_0)$ contains $Lie_y(\Delta)$, the evaluation at $y \in M$ of the Lie algebra $Lie(\Delta)$ generated by Δ. Furthermore, if a smooth distribution Δ' on M is a subdistribution of Δ (i.e., $\Delta' \subset \Delta$), then $\mathcal{O}_{\Delta'}(x_0) \subset \mathcal{O}_\Delta(x_0)$ for all $x_0 \in M$. A smooth distribution Δ is said to be completely controllable if, for every $x \in M$, $\mathcal{O}_\Delta(x) = M$ i.e. any two points of M can be joined by an a.c. Δ-admissible curve. Recall that, the Lie Algebra Rank Condition (LARC), i.e. $Lie_x(\Delta) = T_xM$, is a sufficient condition for the complete controllability of Δ (cf. [18]) when M is connected, which is what we assume in this paper.

3 Affine Holonomy Group of (M, ∇, Δ)

3.1 Definitions

Consider the triple (M, ∇, Δ) where M is a smooth manifold, ∇ a affine connection on M and Δ a completely controllable smooth distribution on M. In this section, we will restrict Definition 2.1 to the Δ-admissible curves on M. To this end, we will define the set of all Δ-admissible loop based at points of M.

Definition 3.1. We define $\Omega_\Delta(x)$ the set of all a.c. Δ-admissible loops based at x, as

$$\Omega_\Delta(x) := \{\gamma \mid \gamma : [a,b] \to M \textit{ a.c.}, \ \gamma(a) = \gamma(b) = x \text{ and } \dot\gamma(t) \in \Delta\big|_{\gamma(t)} \text{ a.e.}\}$$

The following result is immediate from the definitions.

Proposition 3.2. *The set $\Omega_\Delta(x)$ of all a.c. Δ-admissible loop based at x is not empty and is closed under the operation "$\cdot$" given in* (2).

We define the holonomy group associated with the distribution Δ as follows.

Definition 3.3. For every $x \in M$, the holonomy group associated with Δ at x is defined as

$$H^\nabla_\Delta|_x := \{(P^\nabla)^1_0(\gamma) \mid \gamma \in \Omega_\Delta(x)\}.$$

Proposition 3.4. *For every $x, y \in M$, $H^\nabla_\Delta|_x$ is a subgroup of $H^\nabla|_x$ and $H^\nabla_\Delta|_x$ is conjugate to $H^\nabla_\Delta|_y$. One can thus define $H^\nabla_\Delta \subset H^\nabla \subset \mathrm{GL}(n)$ and we call it the Δ-horizontal holonomy group associated with Δ and the affine connection ∇.*

Proof. Since $\Omega_\Delta(x)$ is a nonempty set for any $x \in M$, then $H^\nabla_\Delta|_x$ is also a nonempty subset of $H^\nabla|_x$. By Definitions 2.2.1 and 2.2.2 of [20], the inverse map of $(P^\nabla)^1_0(\gamma)$ is $(P^\nabla)^1_0(\gamma^{-1})$ and $(P^\nabla)^1_0(\delta) \circ (P^\nabla)^1_0(\gamma)$ is equal to $(P^\nabla)^1_0(\delta \cdot \gamma)$, for any $\gamma : [0,1] \to M$ and $\delta : [0,1] \to M$ belonging to $\Omega_\Delta(x)$. Thus, we get the first statement. Next, taking into account the fact that Δ is completely controllable, one deduces the rest of the proposition. $\square$

Remark 3.5. If g is a Riemannian metric on the smooth manifold M and ∇^g is the Levi-Civita connection associated to g, then the holonomy group $H^{\nabla^g}|_x$ with $x \in M$ is a subgroup of $O(T_xM)$, the set of g-orthogonal transformations of T_xM. If, moreover, M is oriented, one can easily prove that $H^{\nabla^g}|_x$ is a subgroup of $\mathrm{SO}(T_xM)$. One can then define the holonomy group of ∇^g as a subgroup of $O(n)$ ($SO(n)$ respectively) the group of orthogonal transformations of the euclidean n-dimensional space (the subgroup of $O(n)$ with determinant equal to one if M is oriented respectively).

3.2 Holonomy groups associated with distributions using the framework of rolling manifolds

Let M be a smooth n-dimensional manifold and ∇ a connection on M. Set $(\hat{M},\hat{\nabla}) := (\mathbb{R}^n,\hat{\nabla}^n)$ where $\hat{\nabla}^n$ is the Euclidean connection on $\mathbb{R}^n$. We associate to (M,∇) the curvature tensor R^∇ and to the product manifold $(M,\nabla)\times(\mathbb{R}^n,\hat{\nabla}^n)$ the affine connection $\overline{\nabla}$.

3.2.1 Affine Holonomy Group of M

We recall next basic definitions and results stated in [9] and [21].

Definition 3.6. The state space of the development of (M,∇) on $(\mathbb{R}^n,\hat{\nabla}^n)$ is

$$Q := Q(M,\mathbb{R}^n) = \{A \in T_x^*M \otimes \mathbb{R}^n \mid A \in GL(T_xM),\ x \in M\}.$$

A point $q \in Q$ is written as $q = (x,\hat{x};A)$. Note that the word "development" can also be replaced by "rolling".

Definition 3.7. Let $\gamma : [0,1] \to M$ be an a.c. curve on M starting at $\gamma(0) = x_0$. We define the development of γ on $T_{x_0}M$ with respect to ∇ as the a.c. curve $\Lambda^\nabla_{x_0}(\gamma) : [0,1] \to T_{x_0}M$

$$\Lambda^\nabla_{x_0}(\gamma)(t) = \int_0^t (P^\nabla)^0_s(\gamma)\dot{\gamma}(s)ds, \qquad t \in [0,1].$$

The following result can be found from [23].

Proposition 3.8. *Let ∇ be the Levi-Civita connection of a Riemannian metric g. Then for any a.c. curve $\Gamma : [0,1] \to T_{x_0}M$ there exists a maximal $T = T(\Gamma)$ such that $0 < T \le 1$ and an a.c. curve $\gamma : [0,T] \to M$ satisfying*

$$\Lambda^\nabla_{x_0}(\gamma)(t) = \Gamma(t), \quad \forall t \in [0,T].$$

Moreover, one can take $T = 1$ for all such Γs if and only if (M,g) is complete.

By identification of $T_x^*M \otimes \mathbb{R}^n$ as the space of all $\mathbb{R}$-linear maps from the tangent space T_xM at $x \in M$ onto the tangent space of $\mathbb{R}^n$ at $\hat{x} \in \mathbb{R}^n$, one gets the following definitions.

Definition 3.9. Let $(x_0,\hat{x}_0) \in M \times \mathbb{R}^N$, $A_0 \in T^*_{x_0}M \otimes \mathbb{R}^n$ and an a.c. curve $\gamma : [0,1] \to M$ such that $\gamma(0) = x_0$. We define the development of γ onto $\mathbb{R}^n$ through A_0 with respect to $\overline{\nabla}$ as the a.c. curve $\Lambda^{\overline{\nabla}}_{A_0}(\gamma) : [0,T] \to M$ given by

$$\Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t) := (\Lambda^{\hat{\nabla}^n}_{\hat{x}_0})^{-1}(A_0 \circ \Lambda^\nabla_{x_0}(\gamma))(t), \qquad t \in [0,T]$$

with $T = T(\gamma)$ as in the previous definition.

We also define the relative parallel transport of A_0 along γ with respect to $\overline{\nabla}$ to be the linear map

$$(P^{\overline{\nabla}})^t_0(\gamma)A_0 : T_{\gamma(t)}M \to T_{\Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t)}\hat{M}, \text{ such that for } t \in [0,1],$$

$$(P^{\overline{\nabla}})^t_0(\gamma)A_0 := (P^{\hat{\nabla}^n})^t_0(\Lambda^{\overline{\nabla}}_{A_0}(\gamma)) \circ A_0 \circ (P^\nabla)^0_t(\gamma) = A_0 \circ (P^\nabla)^0_t(\gamma).$$

We define the No-Spinning development lift of $(X, \hat{X}) \in T_{(x,\hat{x})}(M \times \mathbb{R}^n)$, the Rolling development lift and the Rolling development distribution of $X \in T_x M$ respectively as follows.

Definition 3.10. Let $q = (x, \hat{x}; A) \in Q$, $(X, \hat{X}) \in T_{(x,\hat{x})}(M \times \mathbb{R}^n)$ and γ (resp. $\hat{\gamma}$) be an a.c. curve on M (resp. on $\mathbb{R}^n$) starting at x (resp. $\hat{x}$) with initial velocity X (resp. $\hat{X}$). The No-Spinning development lift of $(X, \hat{X})$ is the unique vector $\mathscr{L}_{NS}(X, \hat{X})|_q$ of $T_q Q$ at $q = (x, \hat{x}; A)$ given by

$$\mathscr{L}_{NS}(X, \hat{X})|_q := \frac{d}{dt}\Big|_0 (P^{\hat{\nabla}^n})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma) = \frac{d}{dt}\Big|_0 A \circ (P^{\nabla})_t^0(\gamma).$$

If, moreover, the initial velocity of $\hat{\gamma}$ is AX, then we define the Rolling lift $\mathscr{L}_R$ at $q = (x, \hat{x}; A) \in Q$ to be the injective map from $T_x M$ onto $T_q Q$, such that for every $X \in T_x M$,

$$\mathscr{L}_R(X)|_q := \mathscr{L}_{NS}(X, AX)|_q = \frac{d}{dt}\Big|_0 (P^{\hat{\nabla}^n})_0^t(\hat{\gamma}) \circ A \circ (P^{\nabla})_t^0(\gamma) = \frac{d}{dt}\Big|_0 A \circ (P^{\nabla})_t^0(\gamma).$$

The Rolling distribution $\mathscr{D}_R$ at $q = (x, \hat{x}; A) \in Q$ is an n-dimensional smooth distribution defined by

$$\mathscr{D}_R|_q := \mathscr{L}_R(T_x M)|_q.$$

We say that an a.c. curve $t \mapsto q(t) = (\gamma(t), \hat{\gamma}(t); A(t))$ on Q, is a rolling curve if and only if it is tangent to $\mathscr{D}_R$ for a.e. $t \in I$, where I is a bounded interval of $\mathbb{R}$, i.e. if and only if $\dot{q}(t) = \mathscr{L}_R(\dot{\gamma}(t))|_{q(t)}$ for a.e. $t \in I$. For the proof of next proposition, see [9, 23].

Proposition 3.11. *For any $q_0 := (x_0, \hat{x}_0; A_0) \in Q$ and any a.c. curve $\gamma : [0,1] \to M$ starting at x_0, there exist unique a.c. curves $\hat{\gamma}(t) := \Lambda^{\overline{\nabla}}_{A_0}(\gamma)(t)$ and $A(t) := (P^{\overline{\nabla}})_0^t(\gamma) A_0$ such that $A(t)\dot{\gamma}(t) = \dot{\hat{\gamma}}(t)$ and $\overline{\nabla}_{(\dot{\gamma}(t), \dot{\hat{\gamma}}(t))} A(t) = 0$, for all $t \in [0, T]$, for a maximal $T = T(\gamma)$ such that $0 < T \leq 1$. We refer to $t \mapsto q_{\mathscr{D}_R}(\gamma, q_0) := (\gamma(t), \hat{\gamma}(t); A(t))$ as the rolling curve along γ with initial position q_0.*

Moreover, if $(\hat{M}, \hat{g})$ is a complete Riemannian manifold and $\hat{\nabla}$ is the corresponding Levi-Civita connection, then one may take above $T = 1$ for all γ's.

Consider the smooth bundle $\pi_Q : Q \to M \times \mathbb{R}^n$ and $q \in Q$. We define $V|_q(\pi_Q)$ to be the set of all $B \in T|_q Q$ such that the tangent application $(\pi_Q)_*(B) = 0$. The collection of spaces $V|_q(\pi_Q)$, $q \in Q$ defines a smooth submanifold $V(\pi_Q)$ of TQ. We will write an element of $V|_q(\pi_Q)$ at $q = (x, \hat{x}; A) \in Q$ as $\nu(B)|_q$ where $B \in T_x^* M \otimes \mathbb{R}^n$ verifies $B \in A\, \mathfrak{so}(T_x M)$. Intrinsically, to know what it means to take the derivative with respect to $\nu(B)|_q$. Then, for all smooth maps f defined on (an open subset of) Q with values in the manifold of (m,k)-tensors of M, we define

$$\nu(B)\,|_q\,(f) := \tfrac{d}{dt}|_0 f(q + tB),$$

that we call the vertical derivative of f at q in the direction of B.

We next present the main computation tools obtained in Proposition 3.7, Lemma 3.18, Proposition 3.24, Proposition 3.26, Proposition 4.1, Proposition 4.6 [22].

Proposition 3.12. *Let $\mathcal{O} \subset T^* M \otimes \mathbb{R}^n$ be an immersed submanifold, $\overline{Z} = (Z, \hat{Z})$, $\overline{S} = (S, \hat{S}) \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^* M \otimes \mathbb{R}^n})$ be such that for all $q = (x, \hat{x}; A) \in \mathcal{O}$, $\mathscr{L}_{NS}(\overline{Z}(q))|_q$, $\mathscr{L}_{NS}(\overline{S}(q))|_q \in T_q \mathcal{O}$ and*

U, $V \in C^\infty(\pi_{\mathcal{O}}, \pi_{T^*M\otimes\mathbb{R}^n})$, be such that for all $q = (x,\hat{x};A) \in \mathcal{O}$, $\nu(U(q))|_q$, $\nu(V(q))|_q \in T_q\mathcal{O}$. Then, one has

$$\mathscr{L}_{NS}(\overline{Z}(A))|_q\overline{S}(\cdot) := \overline{\nabla}_{\overline{Z}(A)}(\overline{S}(A)), \tag{3}$$

$$\begin{aligned}[][\mathscr{L}_{NS}(\overline{Z}(\cdot)),\mathscr{L}_{NS}(\overline{S}(\cdot))]|_q &= \mathscr{L}_{NS}(\mathscr{L}_{NS}(\overline{Z}(A))|_q\overline{S}(\cdot) - \mathscr{L}_{NS}(\overline{S}(A))|_q\overline{Z}(\cdot))|_q \\ &- \mathscr{L}_{NS}(T^\nabla(Z,S))|_q + \nu(AR^\nabla(Z,S))|_q,\end{aligned} \tag{4}$$

$$[\mathscr{L}_R(Z),\mathscr{L}_R(S)]|_q = \mathscr{L}_R([Z,S])|_q + \mathscr{L}_{NS}(AT^\nabla(Z,S))|_q + \nu(AR^\nabla(Z(q),S(q)))|_q, \tag{5}$$

$$[\mathscr{L}_{NS}(\overline{Z}(\cdot)),\nu(U(\cdot))]|_q = -\mathscr{L}_{NS}(\nu(U(A))|_q\overline{Z}(\cdot))|_q + \nu(\mathscr{L}_{NS}(\overline{Z}(A))|_qU(\cdot))|_q, \tag{6}$$

$$[\nu(U(\cdot)),\nu(V(\cdot))]|_q = \nu(\nu(U(A))|_qV - \nu(V(A))|_qU)|_q. \tag{7}$$

Both sides of the equalities in (3), (4), (5), (6) *and* (7) *are tangent to* $\mathcal{O}$.

We use $\mathrm{Aff}(M)$ to denote the affine group of all invertible affine transformations from the affine manifold M onto itself. In particular, the affine group of $\mathbb{R}^n$ is denoted by $\mathrm{Aff}(n)$. One can extend readily Proposition 3.10 of [11] to get the following result.

Proposition 3.13. *For any $f \in \mathrm{Aff}(M)$, $\hat{f} \in \mathrm{Aff}(n)$ and any $q_0 = (x_0,\hat{x}_0;A_0) \in Q$, define the following smooth right and left actions of* $\mathrm{Aff}(M)$ *and* $\mathrm{Aff}(n)$ *on Q*

$$q_0 \cdot f := (f^{-1}(x_0),\hat{x}_0;A_0\circ f_\star|_{f^{-1}(x_0)}), \qquad \hat{f}\cdot q_0 := (x_0,\hat{f}(\hat{x}_0);\hat{f}_\star|_{\hat{x}_0}\circ A_0).$$

Then, for any a.c. curve $\gamma:[0,1]\to M$ starting at x_0, one has for a.e. $t\in[0,1]$

$$\hat{f}\cdot q_{\mathscr{D}_R}(\gamma,q_0)(t)\cdot f = q_{\mathscr{D}_R}(f^{-1}\circ\gamma,\hat{f}\cdot q_0\cdot f)(t).$$

Proof. By the definition of an affine transformation f on M, we have Eq. (1) for any a.c. curve $\gamma:[0,1]\to M$. This implies that, for a.e. $t\in[0,1]$

$$f_\star|_{\gamma(t)}\circ(P^\nabla)_0^t(\gamma) = (P^\nabla)_0^t(f\circ\gamma)\circ f_\star|_{\gamma(0)}.$$

We have the same conclusion for affine transformations $\hat{f}$ on $\mathbb{R}^n$. Then, since $\mathrm{Aff}(n)$ is a Lie group and by what precedes, one can repeat the steps of the proof of Proposition 3.10 in [11] with the group $\mathrm{Aff}(n)$ instead of isometry groups on M and $\mathbb{R}^n$ to get the claim. □

Recall that if G is a Lie group, then a smooth bundle $\pi:E\to M$ is a principal G-bundle over M if there exists a smooth and free action of G on E which preserves the fibers of π, cf. [20]. Furthermore, we recall that the affine group $\mathrm{Aff}(n)$ is equal to $\mathbb{R}^n \rtimes GL(n)$ and its product group $\diamond$ is given by

$$(v,L)\diamond(u,K) := (Lu+v, L\circ K).$$

Using the previous proposition, one can extend immediately the simple but crucial Proposition 4.1 in [11] to derive the next result.

Proposition 3.14. *The bundle $\pi_{Q,M} : Q \to M$ is a principal* $\mathrm{Aff}(n)$*-bundle with the left action* $\mu : \mathrm{Aff}(n) \times Q \to Q$*;*

$$\mu((\hat{y}, C), (x, \hat{x}; A)) = (x, C\hat{x} + \hat{y}; C \circ A).$$

The action μ preserves $\mathscr{D}_R$, i.e. for any $q \in Q$ and $B \in \mathrm{Aff}(n)$, we have $(\mu_B)_* \mathscr{D}_R|_q = \mathscr{D}_R|_{\mu_B(q)}$ where $\mu_B : Q \to Q$; $q \mapsto \mu(B, q)$. Moreover, for any $q = (x, \hat{x}; A) \in Q$, there exists a unique subgroup $\mathcal{H}_q^{\nabla}$ of $\mathrm{Aff}(n)$, called the affine holonomy group of (M, ∇) verifying

$$\mu(\mathcal{H}_q^{\nabla} \times \{q\}) = \mathcal{O}_{\mathscr{D}_R}(q) \cap \pi_{Q,M}^{-1}(x).$$

If $q' = (x, \hat{x}'; A') \in Q$ belongs to the same $\pi_{Q,M}$-fiber as q, then $\mathcal{H}_q^{\nabla}$ and $\mathcal{H}_{q'}^{\nabla}$ are conjugate in $\mathrm{Aff}(n)$ *and all conjugacy classes of $\mathcal{H}_q^{\nabla}$ are of the form $\mathcal{H}_{q'}^{\nabla}$. This conjugacy class is denoted by $\mathcal{H}^{\nabla}$ and its projection in $GL(n)$ is equal to H^{∇} the holonomy group of the affine connection ∇.*

Proof. Let $q = (x, \hat{x}; A) \in Q$ and $B = (\hat{y}, C) \in \mathrm{Aff}(n)$. Since $C \circ A$ is in $\mathrm{GL}(n)$, then $\mu(B, q) \in Q$. In order to prove that μ is transitive and proper, we can follow the same steps of the proof of Proposition 4.1 in [11] due to Proposition 3.13. □

3.2.2 Affine Holonomy Group of Δ

Consider now a smooth completely controllable distribution Δ on (M, ∇). We will determine the sub-distribution of $\mathscr{D}_R$ by restriction to Δ instead of considering the whole tangent space of M.

Definition 3.15. The rolling distribution Δ_R on Δ is the smooth sub-distribution of $\mathscr{D}_R$ defined on $(x, \hat{x}; A) \in Q$ by

$$\Delta_R|_{(x,\hat{x};A)} = \mathscr{L}_R(\Delta|_x)|_{(x,\hat{x};A)}. \tag{8}$$

Since Δ is completely controllable, we use Proposition 3.11 to obtain the next corollary.

Corollary 3.16. *For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ and any a.c. Δ-admissible curve $\gamma : [0,1] \to M$ starting at x_0, there exists a unique a.c. Δ_R-admissible curve $q_{\Delta_R}(\gamma, q_0) : [0, T] \to Q$ where $0 < T \le 1$.*

Since we can easily restrict the proof of Proposition 3.14 (cf. [11]) on Δ_R, we get the next proposition.

Corollary 3.17. *The action μ mentioned in Proposition 3.14 preserves the distribution Δ_R. Moreover, for every $q \in Q$, there exists a unique* algebraic *subgroup $\mathcal{H}_{\Delta_R|q}^{\nabla}$ of $\mathcal{H}_q^{\nabla}$, called the affine holonomy group of Δ_R, such that*

$$\mu(\mathcal{H}_{\Delta_R|q}^{\nabla} \times \{q\}) = \mathcal{O}_{\Delta_R}(q) \cap \pi_{Q,M}^{-1}(x),$$

where $x = \pi_{Q,M}(q)$ and $\mathcal{O}_{\Delta_R}(q)$ is the Δ_R-orbit at q.

As before, one gets the following: if $q' = (x, \hat{x}'; A') \in Q$ belongs to the same $\pi_{Q,M}$-fiber as q, then $\mathcal{H}^{\nabla}_{\Delta_R|q}$ and $\mathcal{H}^{\nabla}_{\Delta_R|q'}$ are conjugate in $\mathrm{Aff}(n)$ and all conjugacy classes of $\mathcal{H}^{\nabla}_{\Delta_R|q}$ are of the form $\mathcal{H}^{\nabla}_{\Delta_R|q'}$. This conjugacy class is denoted by $\mathcal{H}^{\nabla}_{\Delta_R}$ and its projection in $GL(n)$ is a subgroup of H^{∇} which is equal to the Δ-horizontal holonomy group associated with Δ and the affine connection ∇.

Definition 3.18. We denote by $\mathcal{O}^{loop}_{\Delta_R}(q_0)$ the set of the end points of the rolling development curves with initial conditions any point $q_0 = (x_0, \hat{x}_0; A_0)$ and any a.c. Δ-admissible loop at x_0, i.e., for $q_0 = (x_0, \hat{x}_0; A_0) \in Q$,

$$\mathcal{O}^{loop}_{\Delta_R}(q_0) = \{q_{\Delta_R}(\gamma, q_0)(1) \mid \gamma : [0,1] \to M, \text{ a.c. } \Delta\text{-admissible loop at } x_0\}.$$

If we fix a point q_0 of $Q = Q(M, \mathbb{R}^n)$ where the initial contact point on M is equal to x_0 and that on $\mathbb{R}^n$ is the origin, then we may consider the rolling development of M along a loop based at x_0. Then, one obtains a control problem whose state space is the fiber $\pi^{-1}_{Q,M}(x_0)$ and the reachable set is in the fiber $\pi^{-1}_{Q,M}(x_0)$ (for more details, cf. [11]). Then, $\mathcal{O}^{loop}_{\Delta_R}(q_0)$ is trivially in bijection with $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x_0)$ and so $\mu(\mathcal{H}^{\nabla}_{\Delta_R|q_0} \times \{q_0\}) \simeq \mathcal{O}^{loop}_{\Delta_R}(q_0)$.

Proposition 3.19. *For any $q_0 = (x_0, \hat{x}_0; A_0) \in Q$ the restriction of $\pi_{Q,M} : Q \to M$ into the orbit $\mathcal{O}_{\Delta_R}(q_0)$ is a submersion onto M.*

Proof. Clearly it is enough to show that $(\pi_{Q,M})_* T_{q_0}\mathcal{O}_{\Delta_R}(q_0) = T_{x_0}M$. Also recall that by the assumption of complete controllability of Δ we have $M = \mathcal{O}_{\Delta}(x_0)$.

Write $E^{x,t}(u)$ and $\tilde{E}^{q,t}(u)$ for the end-point maps of Δ and Δ_R starting from $x \in M$ and $q \in Q$, respectively. One easily sees that E and $\tilde{E}$ are related by

$$\pi_{Q,M} \circ \tilde{E}^{q,t} = E^{x,t}, \tag{9}$$

for any $q = (x, \hat{x}; A) \in Q$ and t where defined. We also denote by k the rank of Δ (i.e. the rank of Δ_R).

Let $\overline{u} \in L^2([0,1], \mathbb{R}^k)$ be any o-regular control of $E^{x_0,1}$ which belongs to the domain of definition of $\tilde{E}^{q_0,1}$. The existence of such an $\overline{u}$ is guaranteed by an application of Proposition 5.7 given in the appendix and Proposition 3.11, as in this case $(\hat{M}, \hat{g}) = \mathbb{R}^n$ is complete.

Let then $X \in T_{x_0}M$ be arbitrary, and notice that $T_{x_0}\mathcal{O}_{\Delta}(x_0) = T_{x_0}M$. By o-regularity of $\overline{u}$ with respect to $E^{x_0,1}$, there exists a C^1-map $u : I \to L^2([0,1], \mathbb{R}^k)$, where I is an open neighbourhood of 0, such that $u(0) = \overline{u}$ and $h(t,s) := E^{x_0,t}(u(s))$, $(t,s) \in [0,1] \times I$, satisfy $\frac{\partial}{\partial s} h(1,s)|_{s=0} = X$. Indeed, let $G : I \to \mathcal{O}_{\Delta}(x_0)$ be any smooth curve such that $\dot{G}(0) = X$. The o-regularity of $\overline{u}$ means that $D_u E^{x_0,1}$, i.e. the differential of $E^{x_0,1}$ at u, is surjective linear map from $L^2([0,1], \mathbb{R}^k)$ onto $T_{E^{x_0,1}(u)}\mathcal{O}_{\Delta}(x_0)$ when $u = \overline{u}$, and hence for all u close to $\bar{u}$ in $L^2([0,1], \mathbb{R}^k)$. One next defines $P(u)$ as the Moore-Penrose inverse of $D_u E^{x_0,1}$ and one considers the Cauchy problem $\frac{du(s)}{ds} = P(u(s))\frac{dG(s)}{ds}$, $u(0) = \bar{u}$. Then [7, Proposition 2] asserts that the maximal solution $u(\cdot)$ of the Cauchy problem is well-defined on a non empty interval centered at zero, which concludes the argument of the claim (after shrinking I if necessary).

Write $\tilde{h}(t,s) = \tilde{E}^{q_0,t}(u(s))$ for $(t,s) \in [0,1] \times I$. For each $i = 1, \dots, d$ and $s \in I$, the maps $t \mapsto h(t,s)$ and $t \mapsto \tilde{h}(t,s)$ are absolutely continuous and Δ- and Δ_R-admissible curves, respectively, and $h(t,s) = \pi_{Q,M}(\tilde{h}(t,s))$ by (9). In particular, $\frac{\partial}{\partial s}\tilde{h}(1,s)|_{s=0}$ is a vector in $T_{q_0}\mathcal{O}_{\Delta_R}(q_0)$ and

$$(\pi_{Q,M})_*\Big(\frac{\partial}{\partial s}\tilde{h}(1,s)|_{s=0}\Big) = \frac{\partial}{\partial s}h(1,s)|_{s=0} = X,$$

which shows that $X \in (\pi_{Q,M})_*(T_{q_0}\mathcal{O}_{\Delta_R}(q_0))$. Because X was arbitrary tangent vector of M at x_0, we conclude that $T_{x_0}M \subset (\pi_{Q,M})_*(T_{q_0}\mathcal{O}_{\Delta_R}(q_0))$.

The opposite inclusion $(\pi_{Q,M})_*(T_{q_0}\mathcal{O}_{\Delta_R}(q_0)) \subset T_{x_0}M$ being trivially true, this completes the proof. ☐

Remark 3.20. Here is an alternative proof in the case that the distribution Δ satisfies LARC on a connected manifold M i.e. $Lie_x(\Delta) = T_xM$ for all $x \in M$.

Given vector fields $Y_1, \dots, Y_r$ and a subset $J = \{i_1, \dots, i_l\}$ of $\{1, \dots, r\}$ we write Y_J for the iterated bracket $[Y_{i_1}, [Y_{i_2}, \dots [Y_{i_{l-1}}, Y_{i_l}] \dots]$ of length l. Given $X \in T_{x_0}M = T_{x_0}\mathcal{O}_\Delta(x_0)$, there are, by the assumption, vector fields $Y_1, \dots, Y_r$ tangent to Δ, subsets $J_1, \dots, J_t$ of $\{1, \dots, r\}$ and numbers $a_1, \dots, a_t$ such that $X = \sum_{s=1}^t a_s Y_{J_s}|_{x_0}$. The lifts $\mathscr{L}_R(Y_i)$, $i = 1, \dots, r$ are tangent to Δ_R and satisfy $(\pi_{Q,M})_*\mathscr{L}_R(Y_i) = Y_i$, hence if we write $\mathscr{L}_R(Y)_J$ for $[\mathscr{L}_R(Y_{i_1}), [\mathscr{L}_R(Y_{i_2}), \dots [\mathscr{L}_R(Y_{i_{l-1}}), \mathscr{L}_R(Y_{i_l})] \dots]$ when J is as above, we have that $\mathscr{L}_R(Y)_{J_s}$ is tangent to $\mathcal{O}_{\Delta_R}(q_0)$ for every $s = 1, \dots, t$

$$(\pi_{Q,M})_*|_{q_0}\Big(\sum_{s=1}^t a_s \mathscr{L}_R(Y)_{J_s}\Big) = \sum_{s=1}^t a_s Y_{J_s}|_{x_0} = X,$$

i.e. $X \in (\pi_{Q,M})_* T_{q_0}\mathcal{O}_{\Delta_R}(q_0)$. By arbitrariness of X in $T_{x_0}M$ we have the claimed submersivity of $\pi_{Q,M}$.

Classical results now apply to give the following.

Corollary 3.21. *In particular, for any $x \in M$ the fiber $\pi_{Q,M}^{-1}(x) \cap \mathcal{O}_{\Delta_R}(q_0)$ of $\mathcal{O}_{\Delta_R}(q_0)$ over x is either empty or a (closed) embedded submanifold of $\mathcal{O}_{\Delta_R}(q_0)$ of dimension $\delta = \dim \mathcal{O}_{\Delta_R}(q_0) - \dim M$.*

We arrive at the main result of this subsection.

Proposition 3.22. *Assume that Δ is a constant rank completely controllable distribution on (M, ∇) where M is a connected smooth manifold and ∇ an affine connection. Then, the Δ-horizontal holonomy group H_Δ^∇ and the affine holonomy group $\mathcal{H}_{\Delta_R}^\nabla$ of Δ_R as defined previously are Lie subgroups of* $\mathrm{Aff}(n)$.

Proof. It is enough to prove the claim for $\mathcal{H}_{\Delta_R}^\nabla$. We first argue that $\mathcal{H}_{\Delta_R|q_0}^\nabla$ is an algebraic subgroup of $\mathrm{Aff}(n)$. To this end, to any $p \in \pi_{Q,M}^{-1}(x_0)$ (i.e. p is an arbitrary element of the fiber of Q over x_0) we match a unique $(y_p, C_p) \in \mathrm{Aff}(n)$ such that $\mu((y_p, C_p), q_0) = p$. Recall that $\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$ is identified with $\mathcal{H}_{\Delta_R|q_0}^\nabla$ through this correspondence.

Then given $p_1, p_2 \in \mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$, there are Δ-admissible (piecewise smooth) loops $\gamma_1, \gamma_2 \in \Omega_M(x_0)$ in M based at x_0 such that $p_i = q_{\Delta_R}(\gamma_i, q_0)(1)$ for $i = 1, 2$. Letting $p = q_{\Delta_R}(\gamma_1 \cdot \gamma_2, q_0)(1)$ we have

$$\begin{aligned}
\mu((y_p, C_p), q_0) =& p = q_{\Delta_R}(\gamma_1 \cdot \gamma_2, q_0)(1) = q_{\Delta_R}\big(\gamma_1, q_{\Delta_R}(\gamma_2, q_0)\big)(1) = q_{\Delta_R}\big(\gamma_1, p_2\big)(1) \\
=& q_{\Delta_R}\big(\gamma_1, \mu((y_{p_2}, C_{p_2}), q_0)\big)(1) = \mu((y_{p_2}, C_{p_2}), q_{\Delta_R}(\gamma_1, q_0)(1)) \\
=& \mu((y_{p_2}, C_{p_2}), p_1) = \mu\big((y_{p_2}, C_{p_2}), \mu((y_{p_1}, C_{p_1}), q_0)\big) \\
=& \mu\big((y_{p_2}, C_{p_2})(y_{p_1}, C_{p_1}), q_0\big),
\end{aligned}$$

i.e. $(y_p, C_p) = (y_{p_2}, C_{p_2})(y_{p_1}, C_{p_1})$, because the action μ is free. Since $\gamma_1 \cdot \gamma_2$ is Δ-admissible loop, we have $p = q_{\Delta_R}(\gamma_1 \cdot \gamma_2, q_0)(1) \in \mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$ i.e. $(y_p, C_p) \in \mathcal{H}^{\nabla}_{\Delta_R|q_0}$, and therefore $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ is indeed an algebraic subgroup of $\mathrm{Aff}(n)$ as claimed.

In other words we have shown that if $m : \mathrm{Aff}(n) \times \mathrm{Aff}(n) \to \mathrm{Aff}(n)$ is the smooth group multiplication operation on $\mathrm{Aff}(n)$, then

$$m(\mathcal{H}^{\nabla}_{\Delta_R|q_0} \times \mathcal{H}^{\nabla}_{\Delta_R|q_0}) \subset \mathcal{H}^{\nabla}_{\Delta_R|q_0}.$$

By the orbit theorem 5.2 as given in the appendix (see also [16]), we know that any smooth map $f : Z \to Q$ for any smooth manifold Z such that $f(Z) \subset \mathcal{O}_{\Delta_R}(q_0)$ is smooth as a map $f : Z \to \mathcal{O}_{\Delta_R}(q_0)$. In other words, $\mathcal{O}_{\Delta_R}(q_0)$ is an *initial submanifold* of M (cf. [16]).

By Corollary 3.21 $\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$ is a smooth embedded submanifold of $\mathcal{O}_{\Delta_R}(q_0)$, hence an initial submanifold of Q. Since $\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0) \subset \pi_{Q,M}^{-1}(x_0)$ and $\pi_{Q,M}^{-1}(x_0)$ is diffeomorphic to $\mathrm{Aff}(n)$ using the action μ, we have that $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ is a smooth immersed submanifold of $\mathrm{Aff}(n)$ as well. Now the group multiplication m restricted to $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$ which we write as m' is a smooth map $m' : \mathcal{H}^{\nabla}_{\Delta_R|q_0} \times \mathcal{H}^{\nabla}_{\Delta_R|q_0} \to \mathrm{Aff}(n)$ whose image is a subset of $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$. Pulling this map back by the action μ on Q we obtain a smooth map $M : (\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)) \times (\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)) \to Q$ whose image is contained in $\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$. As mentioned above, $\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$ is an initial submanifold of Q, hence M is smooth as a map into $\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$. This then is reflected, by applying the action μ once more, in the fact that m' is smooth as a map into $\mathcal{H}^{\nabla}_{\Delta_R|q_0}$. Thus the latter space is a Lie-subgroup of $\mathrm{Aff}(n)$. □

Remark 3.23. The situation described in Remark 3.5 with the rolling formalism can be treated as the rolling system without spinning nor slipping of two oriented connected Riemannian manifolds (M, g) and $(\mathbb{R}^n, s_n)$, where s_n is the Euclidean metric on $\mathbb{R}^n$. Thus, the state space $Q(M, \mathbb{R}^n)$ is a principal $\mathrm{SE}(n)$-bundle (cf [9], [10] and [11] for more details).

3.3 Integrability of Δ_R

A natural question arises in the framework of horizontal holonomy, namely under which conditions the horizontal holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ is trivial. More generally, we pose this question on the level of Lie algebra, which translates on the group level to asking when $\mathcal{H}^{\nabla}_{\Delta_R}$ is discrete in its underlying Lie group topology. As $\mathcal{H}^{\nabla}_{\Delta_R}$ is identified with a fiber $\mathcal{O}_{\Delta_R}(q_0) \cap \pi_{Q,M}^{-1}(x_0)$ of the orbit $\mathcal{O}_{\Delta_R}(q_0)$, we see that answering question comes down to studying when the distribution Δ_R itself is involutive.

In the case where $\Delta = TM$, it is known that the answer is that $\mathcal{H}^{\nabla}_{\Delta_R}$ is discrete if and only if (M, ∇) has vanishing curvature and torsion. This justifies the following definition.

Definition 3.24. We say that the triple (M, ∇) is Δ-*horizontally flat* is Δ_R is involutive.

By (4), we see that for any vector field X, Y tangent to Δ we have for any $q = (x, \hat{x}; A) \in Q$,

$$[\mathscr{L}_R(X), \mathscr{L}_R(Y)]|_q = \mathscr{L}_R([X,Y])|_q - \mathscr{L}_{NS}(T^{\nabla}(X,Y))|_q + \nu(AR^{\nabla}(X,Y))|_q$$

where, as before, T^{∇} and R^{∇} are the torsion and the curvature of ∇, respectively.

This formula immediately implies a simple characterization of the involutivity of Δ_R.

Proposition 3.25. *The manifold with connection (M, ∇) is Δ-horizontally flat if and only if Δ is involutive and for all $x \in M$ and $X, Y \in \Delta|_x$,*

$$T^{\nabla}(X,Y) = 0, \quad R^{\nabla}(X,Y) = 0$$

For the rest of this subsection, we assume that M is a Riemannian manifold with metric g and that ∇ is the associated Levi-Civita connection.

Let $\Delta^{\perp}$ be the g-orthogonal complement of Δ, and let $P : TM \to \Delta$ be $P^{\perp} : TM \to \Delta^{\perp}$ be the orthogonal projections onto Δ and $\Delta^{\perp}$, respectively. Define the fundamental II form of Δ by

$$\mathrm{II}(X,Y) = P^{\perp}(\nabla_X Y), \quad \forall X, Y \in \Delta|_x,\ x \in M.$$

When $\xi \in \Delta^{\perp}|_x$ is given, one defines the shape operator $S_{\xi} : \Delta|_x \to \Delta|_x$ of Δ with respect to ξ to be given by

$$g(S_{\xi}(X), Y) = -g(\xi, \mathrm{II}(X,Y)), \quad \forall X, Y \in \Delta|_x.$$

In the case where Δ is involutive II is symmetric, and we define the *induced Δ-connection D* and *induced $\Delta^{\perp}$-connection $D^{\perp}$* by

$$\begin{aligned} D_X Y &= P(\nabla_X Y), \\ D^{\perp}_X \xi &= P^{\perp}(\nabla_X \xi), \end{aligned}$$

for $X \in \Delta|_x$, for any vector field Y tangent to Δ and for any vector field ξ tangent to $\Delta^{\perp}$. Furthermore, if one defines for $X, Y, Z \in \Delta|_x$, $\xi \in \Delta^{\perp}|_x$, where $x \in M$,

$$\begin{aligned} R^D(X,Y)Z &= D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z, \\ R^{\perp}(X,Y)\xi &= D^{\perp}_X D^{\perp}_Y \xi - D^{\perp}_Y D^{\perp}_X \xi - D^{\perp}_{[X,Y]} \xi, \end{aligned}$$

then the following result holds.

Corollary 3.26. *The Riemannian manifold* (M,g) *is* Δ*-horizontally flat if and only if* Δ *is involutive and for all* $X,Y,U,V \in \Delta|_y$, $\xi,\eta \in \Delta^{\perp}|_y$ *and* $y \in M$,

$$g(R^D(X,Y)U,V) = g\big(\mathrm{II}(X,U),\mathrm{II}(Y,V)\big) - g\big(\mathrm{II}(X,V),\mathrm{II}(Y,U)\big), \tag{10}$$

$$g((\nabla_X \mathrm{II})(Y,U),\xi) = g((\nabla_Y \mathrm{II})(X,U),\xi), \tag{11}$$

$$g(R^{\perp}(X,Y)\xi,\eta) = g(S_\xi(X),S_\eta(Y)) - g(S_\eta(X),S_\xi(Y)). \tag{12}$$

Proof. Indeed, if L is a leaf of Δ, h is the metric on L induced by g, then D is exactly the Levi-Civita connection of h, and II restricted to L is the second fundamental form of (L,h) in (M,g). The result follows from these observations combined ([4, Theorem 1.72]) with the Gauss, Codazzi-Mainardi and Ricci equations, which are (10), (11) and (12), respectively. □

4 Case Study: Holonomy of Free Step-two Homogeneous Carnot Group

The goal of this section is to provide an example of a triple (M,∇,Δ) such that Δ verifies the LARC (and thus is completely controllable) and H^{∇}_{Δ} is a Lie group strictly included in H^{∇}. After giving the required definitions to treat the example, we first compute H^{∇} and then H^{∇}_{Δ} using the rolling formalism.

4.1 Definitions

The affine manifold (M,∇) we consider is the free step-two homogeneous Carnot group $\mathbb{G}$ endowed with a Riemannian metric and its Levi-Civita connection. To describe it, we will use the definitions of Jacobian basis, homogeneous group and Carnot group of Chapters 1 and 2 of [5].

For m positive integer greater than or equal to 2, set $m+n$ where $n := m(m-1)/2$ and $\mathcal{I} := \{(h,k) \mid 1 \le k < h \le m\}$ of cardinal n. Let $S^{(h,k)}$ be the $m\times m$ real skew-symmetric matrix whose entries are -1 in the position (h,k), $+1$ in the position (k,h) and 0 elsewhere. On $\mathbb{R}^{m+n}$ where an arbitrary point is written (v,γ) with $v \in \mathbb{R}^m$, and $\gamma \in \mathbb{R}^n$, define the group law $\star$ by setting

$$(v,\gamma)\star(v',\gamma') = \begin{pmatrix} v_i + v'_i, \quad i = 1,\dots,m \\ \gamma_{h,k} + \gamma'_{h,k} + \frac{1}{2}(v_h v'_k - v_k v'_h), \quad (h,k)\in\mathcal{I} \end{pmatrix}. \tag{13}$$

Then it is easy to verify that $\mathbb{G} := (\mathbb{R}^{m+n},\star)$ is a Lie group, more precisely a free step-two homogeneous Carnot group of m generators. Indeed, a trivial computation shows that the dilation δ_λ given by

$$\delta_\lambda : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}; \quad \delta_\lambda(v,\gamma) = (\lambda v, \lambda^2\gamma), \tag{14}$$

is an automorphism of $\mathbb{G}$ for every $\lambda > 0$. On the other hand, the (Jacobian) basis of the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is given by X_h, $\Gamma_{h,k}$ where

$$X_h = \frac{\partial}{\partial v_h} + \frac{1}{2}\sum_{1\le j<i\le m}\Big(\ \textstyle\sum_{l=1}^m S_{h,l}^{(i,j)} v_l\ \Big)\big(\frac{\partial}{\partial \gamma_{i,j}}\big),$$

$$= \begin{cases} \frac{\partial}{\partial v_1} + \frac{1}{2}\sum_{1<i\le m} v_i \frac{\partial}{\partial \gamma_{i,1}} & \text{if } h=1, \\ \frac{\partial}{\partial v_h} + \frac{1}{2}\sum_{h<i\le m} v_i \frac{\partial}{\partial \gamma_{i,h}} - \frac{1}{2}\sum_{1\le j<h} v_j \frac{\partial}{\partial \gamma_{h,j}} & \text{if } 1<h<m, \\ \frac{\partial}{\partial v_m} - \frac{1}{2}\sum_{1\le j<m} v_j \frac{\partial}{\partial \gamma_{m,j}} & \text{if } h=m, \end{cases}$$

$$\Gamma_{h,k} = \frac{\partial}{\partial \gamma_{h,k}}, \qquad (h,k)\in\mathcal{I}.$$

while the Lie brackets on $\mathbb{G} = (\mathbb{R}^N, \star)$ are given by

$$[X_h, X_k] = \sum_{1\le j<i\le m} S_{h,k}^{(i,j)} \frac{\partial}{\partial \gamma_{i,j}} = \frac{\partial}{\partial \gamma_{h,k}} = \Gamma_{h,k}$$
$$[X_h, \Gamma_{i,j}] = 0,\ [\Gamma_{h,k}, \Gamma_{i,j}] = 0.$$

Then,

$$\mathrm{rank}(Lie\{X_1,\dots,X_m\}) = \dim(\mathrm{span}\{\frac{\partial}{\partial v_1},\dots,\frac{\partial}{\partial v_m}, (\Gamma_{h,k})_{(h,k)\in\mathcal{I}}\}) = N = \dim\mathfrak{g}.$$

Therefore, we can conclude that $\mathbb{G}$ is a homogeneous Carnot group of step 2 and m generators $X_1,\dots,X_m$. The Lie algebra $\mathfrak{g}$ is equal to $V_1 \oplus V_2$, where $V_1 = \mathrm{span}\{X_1,\dots,X_m\}$ and $V_2 = \mathrm{span}\{\Gamma_{h,k},\ (h,k)\in\mathcal{I}\}$.

Moreover, $(\mathbb{G}, g)$ is an analytic manifold where the metric g, with respect to the previous basis, is given by

$$\begin{aligned} & g(X_i, X_j) = \delta_{i,j}, && \text{if } i,j\in\{1,\dots,m\}, \\ & g(X_i, \Gamma_{h,k}) = 0, && \text{if } i\in\{1,\dots,m\} \text{ and } (h,k)\in\mathcal{I}, \\ & g(\Gamma_{h,k}, \Gamma_{i,j}) = \delta_{h,i}\delta_{k,j}, && \text{if } (i,j),(h,k)\in\mathcal{I}. \end{aligned} \tag{15}$$

In the sequel of this article, we find useful to introduce the following notation of vector fields instead of $\Gamma_{h,k}$, for $h,k\in\{1,\dots,m\}$, in order to facilitate computations by avoiding the confusion between the two cases $k<h$ and $h<k$.

Definition 4.1. For every $h,k\in\{1,\dots,m\}$, we define,

$$\Omega_{h,k} = \begin{cases} \Gamma_{h,k} & \text{if } h>k, \\ -\Gamma_{k,h} & \text{if } h<k, \\ 0 & \text{if } h=k. \end{cases} \tag{16}$$

By the above definition, the Lie bracket $[X_h, X_k]$ is equal to $\Omega_{h,k}$, for any $h, k \in \{1, ..., m\}$. Furthermore, let ∇^g be the Levi-Civita connection associated to the Riemannian metric in (15).

Lemma 4.2. *For $h, k, l, s, t \in \{1, \dots, m\}$, we have the following covariant derivatives on $(\mathbb{G}, g)$,*

$$\nabla^g_{X_h} X_k = \frac{1}{2}\Omega_{h,k}, \qquad \nabla^g_{\Omega_{h,k}} \Omega_{s,t} = 0,$$
$$\nabla^g_{X_l} \Omega_{h,k} = \frac{1}{2}(\delta_{kl} X_h - \delta_{hl} X_k), \qquad \nabla^g_{\Omega_{h,k}} X_l = \frac{1}{2}(\delta_{kl} X_h - \delta_{hl} X_k).$$

Proof. Let us denote by $\nabla^g_X Y$ the covariant differential of a vector field Y in the direction of another vector field X on $\mathbb{G}$. It is equal to

$$\nabla^g_X Y = \sum_{h=1}^{m} \alpha_h(X,Y) X_h + \sum_{1 \le k < h \le m} \beta_{(h,k)}(X,Y) \Omega_{h,k}. \tag{17}$$

On the other hand, by Koszul's formula (cf. [28]), we have

$$2g(\nabla^g_X Y, Z) = g([X,Y], Z) - g([X,Z], Y) - g([Y,Z], X). \tag{18}$$

Combining (17) and (18), we easily find the coefficients $\alpha_h(X,Y)$ and $\beta_{(h,k)}(X,Y)$ and hence we obtain the claim. ☐

4.2 Riemannian Holonomy Group of $(\mathbb{G}, g)$

The main of this subsection is to prove the following theorem.

Theorem 4.3. *Let $(\mathbb{G}, \nabla^g)$ be a free step-two homogeneous Carnot group of dimension N endowed with the Levi-Civita connection ∇^g given in Lemma 4.2. Then, $(\mathbb{G}, \nabla^g)$ has full holonomy group $H^{\nabla^g} = \mathrm{SO}(m+n)$.*

To this end, we compute the Riemannian tensor curvature R and as well as part of its covariant derivation of $(\mathbb{G}, \nabla^g)$.

Lemma 4.4. *For any $h, k, l, i, j \in \{1, \dots, m\}$, the Riemannian curvature tensor R of $(\mathbb{G}, \nabla^g))$ is given by the following skew-symmetric matrices,*

$$R(X_h, X_k) = \frac{3}{4}(X_h \wedge X_k) + \frac{1}{4}\sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j}, \tag{19}$$

$$R(X_l, \Omega_{h,k}) = \frac{1}{4}(X_h \wedge \Omega_{k,l} + X_k \wedge \Omega_{l,h}), \tag{20}$$

$$R(\Omega_{i,j}, \Omega_{h,k}) = \frac{1}{4}(\delta_{ik} X_h \wedge X_j + \delta_{jk} X_i \wedge X_h + \delta_{ih} X_j \wedge X_k + \delta_{jh} X_k \wedge X_i). \tag{21}$$

Proof. From Lemma 4.2 and the intrinsic definition of R,

$$R(X,Y)Z = \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z, \qquad \forall X,\ Y,\ Z \in T_x\mathbb{G},$$

we get, for any $h,k,l,i,j \in \{1,\dots,m\}$,

$$R(X_h,X_k)X_l = \tfrac{3}{4}(\delta_{hl}X_k - \delta_{kl}X_h),$$

$$R(X_h,X_k)\Omega_{i,j} = \tfrac{1}{4}(\delta_{ih}\Omega_{k,j} + \delta_{jh}\Omega_{i,k} - \delta_{ik}\Omega_{h,j} - \delta_{jk}\Omega_{i,h}).$$

Similarly, for any $h,k,l,i,j,t \in \{1,\dots,m\}$, $R(X_l,\Omega_{h,k})$ is given by

$$R(X_l,\Omega_{h,k})X_t = \tfrac{1}{4}(\delta_{th}\Omega_{k,l} - \delta_{tk}\Omega_{h,l}),$$

$$R(X_l,\Omega_{h,k})\Omega_{i,j} = \tfrac{1}{4}\big((\delta_{jk}\delta_{il} - \delta_{jl}\delta_{ki})X_h + (\delta_{jl}\delta_{hi} - \delta_{il}\delta_{jh})X_k\big).$$

Finally, for any $i,j,h,k,l \in \{1,\dots,m\}$, $R(\Omega_{i,j},\Omega_{h,k})$ is given by

$$\begin{aligned} R(\Omega_{i,j},\Omega_{h,k})X_l = \frac{1}{4}\Big(&(\delta_{lk}\delta_{jh} - \delta_{hl}\delta_{jk})X_i + (\delta_{hl}\delta_{ik} - \delta_{lk}\delta_{ih})X_j \\ &+ (\delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik})X_h + (\delta_{lj}\delta_{ih} - \delta_{hj}\delta_{il})X_k\Big), \end{aligned}$$

$$R(\Omega_{i,j},\Omega_{h,k})\Omega_{s,t} = 0, \qquad \forall s,t \in \{1,\dots,m\}.$$

Collecting the above equalities, we get Eq. (19), Eq. (20) and Eq. (21). □

Using the definition of the covariant derivative of tensors, which is,

$$(\nabla^g_Z R(X,Y))(W) = \nabla^g_Z(R(X,Y)W) - R(X,Y)\nabla^g_Z W, \qquad \forall X,\ Y,\ Z,\ W \in T_x\mathbb{G},$$

we deduce the following lemma.

Lemma 4.5. *The covariant derivatives of R in the direction of a vector fields X_t on $\mathbb{G}$, for $t \in \{1,\dots,m\}$, are*

$$\nabla^g_{X_t} R(X_h,X_k) = -R(X_t,\Omega_{h,k}) + \frac{1}{8}\sum_{j=1}^{m}\big(\delta_{kt}X_j \wedge \Omega_{h,j} - \delta_{ht}X_j \wedge \Omega_{k,j}\big),$$

$$\nabla^g_{X_t} R(X_l,\Omega_{h,k}) = \frac{1}{8}\big(\Omega_{t,h}\wedge\Omega_{k,l} + \Omega_{t,k}\wedge\Omega_{l,h} + 2\delta_{lt}X_h\wedge X_k + \delta_{ht}X_k\wedge X_l - \delta_{kt}X_h\wedge X_l\big),$$

$$\nabla^g_{X_t} R(\Omega_{i,j},\Omega_{h,k}) = \frac{1}{8}\big(\delta_{ik}R(X_j,\Omega_{h,t}) + \delta_{jk}R(X_h,\Omega_{i,t}) + \delta_{ih}R(X_k,\Omega_{j,t}) + \delta_{jh}R(X_i,\Omega_{k,t})\big),$$

where h,k,l,i,j are any integers in $\{1,\dots,m\}$.

Similarly, the covariant derivatives of R in the direction of a vector fields $\Omega_{s,t}$ on $\mathbb{G}$, for every $s,t \in \{1,\dots,m\}$, are

$$\nabla^g_{\Omega_{s,t}} R(X_h,X_k) = \frac{3}{8}\big(\delta_{th}X_s\wedge X_k - \delta_{sh}X_t\wedge X_k + \delta_{tk}X_h\wedge X_s - \delta_{sk}X_h\wedge X_t\big),$$

$$\nabla^g_{\Omega_{s,t}} R(X_l,\Omega_{h,k}) = \frac{1}{8}\big(\delta_{th}X_s\wedge\Omega_{k,l} - \delta_{sh}X_t\wedge\Omega_{k,l} + \delta_{tk}X_s\wedge\Omega_{l,h} - \delta_{sk}X_t\wedge\Omega_{l,h}\big),$$

$$\begin{aligned} \nabla^g_{\Omega_{s,t}} R(\Omega_{i,j},\Omega_{h,k}) = \frac{1}{8}\Big(&(\delta_{ih}\delta_{tk} - \delta_{ik}\delta_{th})X_j\wedge X_s + (\delta_{ik}\delta_{jt} - \delta_{jk}\delta_{ti})X_h\wedge X_s \\ &+ (\delta_{jh}\delta_{ti} - \delta_{ih}\delta_{tj})X_k\wedge X_s + (\delta_{jk}\delta_{th} - \delta_{jh}\delta_{tk})X_i\wedge X_s \\ &- (\delta_{ik}\delta_{js} - \delta_{jk}\delta_{is})X_h\wedge X_t - (\delta_{jk}\delta_{sh} - \delta_{jh}\delta_{sk})X_i\wedge X_t \\ &- (\delta_{jh}\delta_{si} - \delta_{ih}\delta_{sj})X_k\wedge X_t - (\delta_{ih}\delta_{sk} - \delta_{ik}\delta_{sh})X_j\wedge X_t\Big), \end{aligned}$$

where h,k,l,i,j are any integers in $\{1,\dots,m\}$.

We next deduce from the two previous lemma the main computational result of the section.

Proposition 4.6. *Fix some $q_0\in Q(\mathbb{G},\mathbb{R}^N)$ and let $q=(x,\hat{x};A)\in\mathcal{O}_{\mathscr{D}_R}(q_0)$, then $\mathrm{SO}(T_xM)\subset\mathcal{O}_{\mathscr{D}_R}(q_0)$.*

Proof. Fix some $q_0=(x_0,\hat{x}_0;A_0)\in Q$, for any $h,k,i,j\in\{1,\dots,m\}$ such that $i\neq j$ and $k\neq h$, the first order Lie brackets on $\mathcal{O}_{\mathscr{D}_R}(q_0)$ are

$$\begin{aligned}
[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]|_q &= \mathscr{L}_R(\Omega_{h,k})|_q+\nu(AR(X_h,X_k))|_q\\
&= \mathscr{L}_R(\Omega_{h,k})|_q+\frac{3}{4}\nu(A(X_h\wedge X_k))|_q+\frac{1}{4}\nu\big(A(\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j})\big)|_q,\\
[\mathscr{L}_R(\Omega_{i,j}),\mathscr{L}_R(\Omega_{h,k})]|_q &= \frac{1}{4}\nu(A(\delta_{ik}X_h\wedge X_j+\delta_{jk}X_i\wedge X_h+\delta_{ih}X_j\wedge X_k+\delta_{jh}X_k\wedge X_i))|_q,\\
[\mathscr{L}_R(X_i),\mathscr{L}_R(\Omega_{h,k})]|_q &= \frac{1}{4}\nu(A(X_h\wedge\Omega_{k,i}+X_k\wedge\Omega_{i,h}))|_q.
\end{aligned}$$

By taking $i=k$ in the bracket $[\mathscr{L}_R(\Omega_{i,j}),\mathscr{L}_R(\Omega_{h,k})]|_q$, we get that, for any $h,j\in\{1,\dots,m\}$, $\nu(A(X_h\wedge X_j))|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. In addition, from the first and the last brackets of the above Lie brackets, we obtain that $\nu\big(A(\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j})\big)|_q$ and $\nu(A(X_h\wedge\Omega_{k,i}+X_k\wedge\Omega_{i,h}))|_q$ are tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$, for any $h,k,i,j\in\{1,\dots,m\}$. Thus, we can compute the next bracket in $T_q\mathcal{O}_{\mathscr{D}_R}(q_0)$, for $q\in\mathcal{O}_{\mathscr{D}_R}(q_0)$,

$$\begin{aligned}
[\mathscr{L}_R(X_i),\nu((\cdot)(X_h\wedge X_k))]|_q &= \delta_{ki}\mathscr{L}_{NS}(AX_h)|_q-\delta_{hi}\mathscr{L}_{NS}(AX_k)|_q\\
&- \tfrac{1}{2}\nu(A(X_h\wedge\Omega_{k,i}+X_k\wedge\Omega_{i,h}))|_q.
\end{aligned}$$

Using $[\mathscr{L}_R(X_i),\mathscr{L}_R(\Omega_{h,k})]|_q$ and then putting $i=h$ in the last Lie bracket, we obtain that $\mathscr{L}_{NS}(AX_k)|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$, for all $k\in\{1,\dots,m\}$. In addition, we have

$$\begin{aligned}
&[\mathscr{L}_R(\Omega_{t,s}),\nu((\cdot)(X_h\wedge\Omega_{k,i}+X_k\wedge\Omega_{i,h}))]|_q\\
= \;&(\delta_{tk}\delta_{ls}-\delta_{tl}\delta_{sk})\mathscr{L}_{NS}(AX_h)|_q+(\delta_{ht}\delta_{ls}-\delta_{tl}\delta_{hs})\mathscr{L}_{NS}(AX_k)|_q\\
+ \;&\tfrac{1}{2}\big(\delta_{sh}\nu(A(X_t\wedge\Omega_{k,l}))|_q+\delta_{sk}\nu(A(X_t\wedge\Omega_{l,h}))|_q\\
&-\delta_{th}\nu(A(X_s\wedge\Omega_{k,l}))|_q+\delta_{tk}\nu(A(X_s\wedge\Omega_{l,h}))|_q\big).
\end{aligned}$$

Since $\mathscr{L}_{NS}(AX_k)|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$, for all $k\in\{1,\dots,m\}$, then $\nu(A(X_t\wedge\Omega_{h,k}))|_q$ is also tangent for any distinct integers $h,k,t\in\{1,\dots,m\}$. The last Lie bracket to compute is

$$\begin{aligned}
[\mathscr{L}_{NS}(X_t),\nu\big(A(X_l\wedge\Omega_{h,k})\big)]|_q = \frac{1}{2}\big(&\delta_{tk}\nu(A(X_l\wedge X_h))|_q-\delta_{th}\nu(A(X_l\wedge X_k))|_q\\
&+\nu(A(\Omega_{t,l}\wedge\Omega_{h,k}))|_q\big).
\end{aligned}$$

Therefore, for every $h,k,t,l \in \{1,\dots,m\}$, $\nu(A(\Omega_{t,l}\wedge\Omega_{h,k}))|_q$ is tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. Hence, for all $q\in\mathcal{O}_{\mathscr{D}_R}(q_0)$ the following vector fields

$$\nu(A(X_h\wedge X_k))|_q,\ \ \nu(A(X_t\wedge\Omega_{h,k}))|_q,\ \ \nu(A(\Omega_{t,l}\wedge\Omega_{h,k}))|_q,$$

are tangent to $\mathcal{O}_{\mathscr{D}_R}(q_0)$. This completes the proof because we have that $\nu(AB)|_q\in T_q\mathcal{O}_{\mathscr{D}_R}(q_0)$ if and only if $B\in\mathfrak{so}(T_x\mathbb{G})$ for $q=(x,\hat{x};A)\in\mathcal{O}_{\mathscr{D}_R}(q_0)$. ☐

We return to prove the main theorem in the beginning of the current subsection.

Proof of Theorem 4.3. . As the vertical bundle of Q is included in the tangent space of $\mathcal{O}_{\mathscr{D}_R}(q_0)$ by Proposition 4.6, then the rolling problem $(\Sigma)_R$ is completely controllable (see Corollary 5.21 in [9]). According to Theorem 4.3 in [11], the holonomy group of $\mathbb{G}$ is equal to $\mathrm{SO}(m+n)$. Note that one could have used as well the main result in [25] stating that the tangent space of the holonomy group at every point $x\in M$ contains the evaluations at x of the curvature tensor and its covariant derivatives at any order. ☐

4.3 Horizontal Holonomy Group of $(\mathbb{G},g)$

We define the distribution $\Delta:=\mathrm{span}\{X_1,\dots,X_m\}$ on $\mathbb{G}$ and $q_0=(x_0,\hat{x}_0;A_0)\in Q$. Note that is of cnstant rank m. We will first compute a basis of $T_q\mathcal{O}_{\Delta_R}(q_0)$ for any $q\in\mathcal{O}_{\Delta_R}(q_0)$ and then determine the holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ of rolling of $(\mathbb{G},g)$ against $(\mathbb{R}^N,s_N)$, where s_N is the Euclidean metric on $\mathbb{R}^N$.

4.3.1 The Tangent Space of $\mathcal{O}_{\Delta_R}(q_0)$

Proposition 4.7. *For any $q_0\in Q$, the tangent space of $\mathcal{O}_{\Delta_R}(q_0)$ is generated by the following linearly independent vector fields:*

$$\mathscr{L}_{NS}(X_h)|_q,\,\mathscr{L}_{NS}(AX_h)|_q,\,\mathscr{L}_{NS}(A\Omega_{h,k})|_q,\,\mathscr{L}_{NS}(\Omega_{h,k})|_q+\frac{1}{2}\nu(A(X_h\wedge X_k))|_q,\tag{22}$$
$$\nu\big(A(\sum_{j=1}^m X_j\wedge\Omega_{h,j})\big)|_q,\ \nu\big(A(X_h\wedge X_k+\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j})\big)|_q.$$

Proof. Recall that all the data of the problem are analytic. Then, by the analytic version of the orbit theorem of Nagano-Sussmann (cf. [2]), the orbit $\mathcal{O}_{\Delta_R}(q_0)$ is an immersed analytic submanifold in the state space Q and $T_q\mathcal{O}_{\Delta_R}(q_0)=Lie_q(\Delta_R)$. Therefore, we are left to determine vector fields spanning $Lie_q(\Delta_R)$, i.e., to compute enough iterated Lie brackets of Δ.

For any $h,k,l,s,t,p\in\{1,\dots,m\}$, we have

$$\begin{aligned}&[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]|_q=\mathscr{L}_R(\Omega_{h,k})|_q+\nu(AR(X_h,X_k))|_q\\ =\ &\mathscr{L}_R(\Omega_{h,k})|_q+\tfrac{3}{4}\nu(A(X_h\wedge X_k))|_q+\tfrac{1}{4}\nu\big(A(\textstyle\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j})\big)|_q.\end{aligned}\tag{23}$$

$$
\begin{aligned}
& [\mathscr{L}_R(X_l),[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]]|_q \\
= \; & \delta_{kl}\left(\tfrac{3}{4}\mathscr{L}_{NS}(AX_h)|_q+\tfrac{1}{8}\nu(A\textstyle\sum_{j=1}^m X_j\wedge\Omega_{h,j}))|_q\right) \\
- \; & \delta_{hl}\left(\tfrac{3}{4}\mathscr{L}_{NS}(AX_k)|_q+\tfrac{1}{8}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{k,j}))|_q\right).
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(X_t),[\mathscr{L}_R(X_l),[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]]]|_q \\
= \; & \delta_{kl}\left(\tfrac{1}{2}\mathscr{L}_{NS}(A\Omega_{t,h})|_q+\tfrac{1}{16}\nu(A(X_t\wedge X_h))|_q+\tfrac{1}{16}\nu(A(\textstyle\sum_{j=1}^m \Omega_{t,j}\wedge\Omega_{h,j}))|_q\right) \\
- \; & \delta_{hl}\left(\tfrac{1}{2}\mathscr{L}_{NS}(A\Omega_{t,k})|_q+\tfrac{1}{16}\nu(A(X_t\wedge X_k))|_q+\tfrac{1}{16}\nu(A(\textstyle\sum_{j=1}^m \Omega_{t,j}\wedge\Omega_{k,j}))|_q\right).
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(X_s),[\mathscr{L}_R(X_t),[\mathscr{L}_R(X_l),[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]]]]|_q \\
= \; & \delta_{kl}\Bigg(\delta_{hs}\left(\tfrac{3}{8}\mathscr{L}_{NS}(AX_t)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{t,j}))\right) \\
& \quad -\delta_{ts}\left(\tfrac{3}{8}\mathscr{L}_{NS}(AX_h)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{h,j}))\right)\Bigg) \\
- \; & \delta_{hl}\Bigg(\delta_{ks}\left(\tfrac{3}{8}\mathscr{L}_{NS}(AX_t)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{t,j}))\right) \\
& \quad -\delta_{ts}\left(\tfrac{3}{8}\mathscr{L}_{NS}(AX_k)|_q+\tfrac{1}{32}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{k,j}))\right)\Bigg).
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
& [\mathscr{L}_R(X_p),[\mathscr{L}_R(X_s),[\mathscr{L}_R(X_t),[\mathscr{L}_R(X_l),[\mathscr{L}_R(X_h),\mathscr{L}_R(X_k)]]]]]|_q \\
= \; & \delta_{kl}\Bigg(\delta_{hs}\left(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,t})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_t))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{t,j}))|_q\right) \\
& \quad -\delta_{ts}\left(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,h})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_h))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{h,j}))|_q\right)\Bigg) \\
- \; & \delta_{hl}\Bigg(\delta_{ks}\left(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,t})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_t))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{t,j}))|_q\right) \\
& \quad -\delta_{ts}\left(\tfrac{7}{32}\mathscr{L}_{NS}(A\Omega_{p,k})|_q+\tfrac{1}{64}\nu(A(X_p\wedge X_k))|_q+\tfrac{1}{64}\nu(A(\textstyle\sum_{j=1}^m \Omega_{p,j}\wedge\Omega_{k,j}))|_q\right)\Bigg).
\end{aligned}
\tag{27}
$$

First we should remark that, by iteration, the commutators

$$[\mathscr{L}_R(X_{\alpha_r}),\dots[\mathscr{L}_R(X_{\alpha_2}),\mathscr{L}_R(X_{\alpha_1})]\dots],$$

where $\alpha_i\in\{1,\dots,m\}$ and $r\geq 3$ are written either as the vectors in (24) and (26), or as those in (25) and (27). Therefore, one only has to prove that (23), (24), (25), (26) and (27) are linear combinations of the vector fields in (22) and then show that the Lie algebra generated by these vector fields is involutive. Indeed, fix some $h,k\in\{1,\dots,m\}$ such that $k\neq h$ and take $p=s=t=l=k$ in the above Lie brackets. Calculate $\frac{1}{4}(24)+(26)$ and $\frac{1}{2}(24)+(26)$, we get that

$$\mathscr{L}_{NS}(AX_h)|_q,\quad \nu(A(\sum_{j=1}^m X_j\wedge\Omega_{h,j}))|_q$$

are vectors in $Lie(\Delta_R)$. On the other hand, $\frac{1}{4}(25)+(27)$ and $\frac{7}{16}(25)+(27)$ imply that

$$\mathscr{L}_{NS}(A\Omega_{h,k})|_q,\quad \nu(A(X_h\wedge X_k+\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j}))|_q$$

belong also to $Lie(\Delta_R)$. The last two vectors with (23) give us another vector in $Lie(\Delta_R)$ which is

$$\mathscr{L}_{NS}(\Omega_{h,k})|_q+\frac{1}{2}\nu(A(X_h\wedge X_k))|_q.$$

To show that the vector fields given in Eq. (22) form a basis for $Lie(\Delta_R)$, it remains to compute their first order Lie brackets to see that they define an involutive Lie algebra.

We have,

$$\begin{aligned}&[\nu((\cdot)(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{h,j})),\nu((\cdot)(\sum_{j=1}^m X_j\wedge\Omega_{k,j}))]|_q\\ =\;&\nu(A(X_h\wedge X_k+\textstyle\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j}))|_q,\end{aligned}\tag{28}$$

$$\begin{aligned}&[\nu((\cdot)(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{l,j})),\nu((\cdot)(X_h\wedge X_k+\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j}))]|_q\\ =\;&\delta_{kl}\nu(A(\textstyle\sum_{j=1}^m X_j\wedge\Omega_{h,j}))|_q-\delta_{hl}\nu(A(\sum_{j=1}^m X_j\wedge\Omega_{k,j}))|_q,\end{aligned}\tag{29}$$

and,

$$\begin{aligned}&[\nu((\cdot)(X_l\wedge X_t+\textstyle\sum_{j=1}^m\Omega_{l,j}\wedge\Omega_{t,j})),\nu((\cdot)(X_h\wedge X_k+\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j}))]|_q\\ =\;&\delta_{kl}\nu(A(X_h\wedge X_t+\textstyle\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{t,j}))|_q+\delta_{hl}\nu(A(X_t\wedge X_k+\sum_{j=1}^m\Omega_{t,j}\wedge\Omega_{k,j}))|_q\\ +\;&\delta_{tk}\nu(A(X_l\wedge X_h+\textstyle\sum_{j=1}^m\Omega_{l,j}\wedge\Omega_{h,j}))|_q+\delta_{ht}\nu(A(X_k\wedge X_l+\sum_{j=1}^m\Omega_{k,j}\wedge\Omega_{l,j}))|_q.\end{aligned}\tag{30}$$

Moreover, the Lie brackets between $\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu(A(X_h\wedge X_k))$ and the remaining vectors of (22) are

$$
\begin{aligned}
&[\mathscr{L}_{NS}(X_l),\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu((\cdot)(X_h\wedge X_k))]|_q=0,\\
&[\mathscr{L}_{NS}((\cdot)X_l),\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu((\cdot)(X_h\wedge X_k))]|_q=0,\\
&[\mathscr{L}_{R}((\cdot)\Omega_{i,j}),\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu((\cdot)(X_h\wedge X_k))]|_q=0,\\
&[\mathscr{L}_{NS}((\cdot)\Omega_{l,t}),\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu((\cdot)(X_h\wedge X_k))]|_q=0,\\
&[\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu((\cdot)(X_h\wedge X_k)),\mathscr{L}_{NS}(\Omega_{l,t})+\frac{1}{2}\nu((\cdot)(X_l\wedge X_t))]|_q=0,\\
&[\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu((\cdot)(X_h\wedge X_k)),\nu((\cdot)(X_l\wedge X_t+\sum_{j=1}^m\Omega_{l,j}\wedge\Omega_{t,j}))]|_q=0,\\
&[\mathscr{L}_{NS}(\Omega_{h,k})+\frac{1}{2}\nu((\cdot)(X_h\wedge X_k)),\nu((\cdot)(\sum_{j=1}^m X_j\wedge\Omega_{l,j}))]|_q=0.
\end{aligned}
$$

Then, the vector fields

$$
\begin{gathered}
\mathscr{L}_{NS}(X_h)|_q,\ \mathscr{L}_{NS}(AX_h)|_q,\ \mathscr{L}_{NS}(A\Omega_{h,k})|_q,\ \mathscr{L}_{NS}(\Omega_{h,k})|_q+\frac{1}{2}\nu(A(X_h\wedge X_k))|_q,\\
\nu\big(A(\sum_{j=1}^m X_j\wedge\Omega_{h,j})\big)|_q,\ \nu\big(A(X_h\wedge X_k+\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j})\big)|_q,
\end{gathered}
$$

form an involutive distribution and any vector fields in $Lie(\Delta_R)$ is a linear combination of them. It remains to check that they are linearly independent. It is clearly enough to do that for the family of vector fields $\nu\big(A(\sum_{j=1}^m X_j\wedge\Omega_{h,j})\big)|_q$. Suppose there exists $(\alpha_h)_{1\le h\le m}$, such that $\sum_{h=1}^m\alpha_h\sum_{j=1}^m X_j\wedge\Omega_{h,j}=0$. Then $\sum_{j=1}^m X_j\wedge(\sum_{h=1,h\neq j}^m\alpha_h\Omega_{h,j})=0\ \forall j\in\{1,\dots,m\}$. Hence, $\sum_{h=1,h\neq j}^m\alpha_h\Omega_{h,j}=0$ for every $j,h\in\{1,\dots,m\}$, so $\alpha_h=0$ for every $h\in\{1,\dots,m\}$. Therefore,

$$
\begin{gathered}
\mathscr{L}_{NS}(X_h)|_q,\ \mathscr{L}_{NS}(AX_h)|_q,\ \mathscr{L}_{NS}(A\Omega_{h,k})|_q,\ \mathscr{L}_{NS}(\Omega_{h,k})|_q+\frac{1}{2}\nu(A(X_h\wedge X_k))|_q,\\
\nu\big(A(\sum_{j=1}^m X_j\wedge\Omega_{h,j})\big)|_q,\ \nu\big(A(X_h\wedge X_k+\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j})\big)|_q,
\end{gathered}
$$

is a global basis of $Lie_q(\Delta_R)$ and hence the dimension of $Lie_q(\Delta_R)$ is constant and equal to $3N$. We deduce that $\dim\mathcal{O}_{\Delta_R}(q_0)=3N$ and the tangent space of $\mathcal{O}_{\Delta_R}(q_0)$ is generated by the vectors in (22). □

Remark 4.8. According to this proposition, $\mathscr{L}_{NS}(AX_h)|_q$ and $\mathscr{L}_{NS}(A\Omega_{h,k})|_q$ are tangent to $\mathcal{O}_{\Delta_R}(q_0)$. This implies that $\pi_{Q,\mathbb{R}^N}(\mathcal{O}_{\Delta_R})=\mathbb{R}^{m+n}$ which means that all the translations along $\mathbb{R}^{m+n}$ are included in the tangent space of the orbit. Furthermore, the families of vector fields $\nu\big(A(\sum_{j=1}^m X_j\wedge\Omega_{h,j})\big)|_q$ and $\nu\big(A(X_h\wedge X_k+\sum_{j=1}^m\Omega_{h,j}\wedge\Omega_{k,j})\big)|_q$ form an involutive vertical distribution.

4.3.2 Determination of $\mathcal{H}^{\nabla}_{\Delta_R|q}$

The main result of the subsection is given next.

Proposition 4.9. *(i) The affine Δ-horizontal holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ is a Lie subgroup of* $\mathrm{SE}(m+n)$ *of dimension* $2(m+n)$.

(ii) The Δ-horizontal holonomy group H^{∇}_{Δ} is a Lie subgroup of $\mathrm{SO}(m+n)$ *of dimension* $m+n$. *Moreover, the connected component of the identity $(H^{\nabla}_{\Delta})_0$ of H^{∇}_{Δ} is compact.*

Proof. As an immediate adaptation of Proposition 3.22 to the case where one is dealing with principal $\mathrm{SE}(n)$-bundles, one gets that the affine holonomy group $\mathcal{H}^{\nabla}_{\Delta_R}$ is a Lie subgroup of $\mathrm{SE}(m+n)$. Notice then that, if $\Pi : \mathrm{SE}(m+n) \to \mathrm{SO}(m+n)$ is the projection onto the $\mathrm{SO}(m+n)$ factor of $\mathrm{SE}(m+n)$, one has, by definition $H^{\nabla} = \Pi(\mathcal{H}^{\nabla})$ and $H^{\nabla}_{\Delta} = \Pi(\mathcal{H}^{\nabla}_{\Delta_R})$. This shows that the Δ-horizontal holonomy group H^{∇}_{Δ} is a Lie subgroup of $\mathrm{SO}(m+n)$.

We next prove that for every $q' \in Q$, a basis of $\mathrm{Lie}(\mathcal{H}^{\nabla}_{\Delta_R|q'})$ the Lie-algebra of $\mathcal{H}^{\nabla}_{\Delta_R|q'}$ is given by the evaluation at q' of the vector fields whose values at $q = (x, \hat{x}, A) \in Q$ are

$$\mathscr{L}_{NS}(AX_h)|_q, \mathscr{L}_{NS}(A\Omega_{h,k})|_q, \nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q, \nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q, \tag{31}$$

and hence this Lie algebra has dimension $2(m+n) = m(m+1)$. To see that, consider some element $V \in \mathrm{Lie}(\mathcal{H}^{\nabla}_{\Delta_R|q'})$ as a linear subspace of $T_{q'}\mathcal{O}_{\Delta_R}(q_0)$. Then V is a linear combination of the vector fields described in Eq. (22) evaluated at q' and V projects to a zero-vector in TM. By an obvious computation, one deduces that V is a linear combination of the vector fields given in Eq. 31. Conversely, it is clear that the vector fields given in Eq. (31) generate a distribution whose integral manifolds lie in $\mathcal{O}_{\Delta_R}(q_0) \cap \pi^{-1}_{Q,M}(x')$ where $x' = \pi_{Q,M}(q')$. This proves that $\mathrm{Lie}(\mathcal{H}^{\nabla}_{\Delta_R|q'})$ the Lie-algebra of $\mathcal{H}^{\nabla}_{\Delta_R|q'}$ has dimension $2(m+n) = m(m+1)$. One could also check that the distribution generated by the vector fields in Eq. (31) is involutive.

By a similar reasoning, a basis $\mathrm{Lie}(H^{\nabla}_{\Delta|q'})$ of the Lie-algebra of $H^{\nabla}_{\Delta|q'}$ is given by the (evaluations at q' of) vector fields (see also (32) below)

$$\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)|_q,\ \nu\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)|_q,$$

and hence this Lie algebra has dimension $m+n = m(m+1)/2$.

It remains to prove the last claim in (ii). For $1 \le h \le m$, let $A_h \in \mathfrak{so}(m+n)$ corresponding to the vertical vector $\nu\big(A(\sum_{j=1}^{m} X_j \wedge \Omega_{h,j})\big)$ and, for $(h,k) \in \mathcal{I}$, let $B_{h,k} \in \mathfrak{so}(m+n)$ corresponding to the vertical vector $\big(A(X_h \wedge X_k + \sum_{j=1}^{m} \Omega_{h,j} \wedge \Omega_{k,j})\big)$. We extend the notations for the $B_{h,k}$ to for any $1 \le h,k \le m$ by setting $B_{h,k} = -B_{k,h}$. The basis of Lie algebra $L := \mathrm{Lie}(H^{\nabla}_{\Delta})$ of H^{∇}_{Δ} is given by the matrices A_h, $1 \le h \le m$ and $B_{h,k}$, $(h,k) \in \mathcal{I}$ and thanks to Eqs. (28), (29) and (30), one has

$$[A_i, A_j] = B_{i,j},\ [A_i, B_{h,k}] = \delta_{ki}A_h - \delta_{hi}A_k,\ [B_{l,t}, B_{h,k}] = \delta_{kl}B_{h,t} + \delta_{hl}B_{t,k} + \delta_{tk}B_{l,h} + \delta_{ht}B_{k,l}. \tag{32}$$

We prove in Subsection 5.2 that L is a compact semisimple Lie algebra. Since the connected component of identity $\left(H_{\Delta}^{\nabla}\right)_0$ of H_{Δ}^{∇} is a connected Lie with Lie algebra L, it follows from Weyl's theorem (cf. [26, Theorem 26.1]) that $\left(H_{\Delta}^{\nabla}\right)_0$ is compact in $\mathrm{SO}(m+n)$ and of dimension $m+n$. $\square$

Since $m+n \le (m+n)(m+n-1)/2$ (resp. $2(m+n) \le (m+n+1)(m+n)/2$) for all $m \ge 2$ and equality holds if and only if $m=2$ we have obtain our last result.

Corollary 4.10. *In the set up of Proposition 4.9, the inclusions* $\mathcal{H}_{\Delta_R}^{\nabla} \subset \mathrm{SE}(m+n)$ *and* $H_{\Delta}^{\nabla} \subset \mathrm{SO}(m+n)$ *are strict if and only if* $m \ge 3$.

5 Appendix

5.1 *o-regular* controls

We first generalize the usual definition of regular control and then provide a result about existence of such controls. Let M be an n-dimensional smooth manifold, $\mathcal{F}$ a (possibly infinite) family of smooth vector fields on M, and let $\Delta_{\mathcal{F}}$ be the smooth *singular* distribution (cf. [16]) spanned by $\mathcal{F}$, i.e.

$$\Delta_{\mathcal{F}}|_p = \mathrm{span}\{X|_p \mid X \in \mathcal{F}\} \subset T_pM, \quad p \in M.$$

We use the word "singular" (to emphasize the fact that the rank (dimension) of $\Delta_{\mathcal{F}}$ might vary from point to point. One can, in fact, prove given any such family $\mathcal{F}$, there is a *finite* subfamily $\mathcal{F}_0 = \{X_1, \dots, X_m\}$ such that $\Delta_{\mathcal{F}} = \Delta_{\mathcal{F}_0}$, and $m \le n(n+1)$ (see [12, 30], or [27] when $\Delta_{\mathcal{F}}$ has constant rank). Moreover, by span S we mean $\mathbb{R}$-linear span of a set S.

Definition 5.1. An absolutely continuous (a.c.) curve $\gamma : [0,T] \to M$ is horizontal with respect to $\mathcal{F}$ if there is a finite subfamily $\{X_1, \dots, X_m\}$ of $\mathcal{F}$ and $u = (u_1, \dots, u_d) \in L^1([0,T], \mathbb{R}^m)$, $m \in \mathbb{N}$ (here m might depend on the curve γ in question), such that for almost every $t \in [0,T]$,

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i(t) X_i|_{\gamma(t)}.$$

The *orbit* $\mathcal{O}_{\mathcal{F}}(p)$ of $\mathcal{F}$ through $p \in M$ is the set of all points of M reached by $\mathcal{F}$-horizontal paths γ with $\gamma(0) = p$.

If Δ is a smooth distribution of *constant rank* k on M, and if $\mathcal{F} = \mathcal{F}_{\Delta}$ is the set of smooth vector fields tangent to Δ, then it is easy to see that $\Delta = \Delta_{\mathcal{F}}$, and that an a.c. curve is Δ-horizontal if and only if it is $\mathcal{F}$-horizontal. Therefore, in this case the concept of orbit coincides with the notion we have used previously in the paper, and one can without ambiguity denote it by $\mathcal{O}_{\Delta}(p)$ instead of $\mathcal{O}_{\mathcal{F}}(p)$.

For a smooth vector field X write $\Phi_X : D \to M$ for its flow, where $D = D_X$ is an open connected subset of $\mathbb{R} \times M$ containing $\{0\} \times M$. We also use the notation $(\Phi_X)_t(x) = (\Phi_X)^x(t) = \Phi_X(t,x)$ when $(x,t) \in D$.

The orbit of a family $\mathcal{F}$ of vector fields has the following properties (cf. [16], [18]).

Theorem 5.2 (Orbit Theorem). *1. The orbit $\mathcal{O}_{\mathcal{F}}(p)$ is an immersed submanifold of M.*

2. *Any continuous (resp. smooth) map $f : Z \to M$, where Z is a smooth manifold, such that $f(Z) \subset \mathcal{O}_{\mathcal{F}}(p)$ is continuous (resp. smooth) as a map $f : Z \to \mathcal{O}_{\mathcal{F}}(p)$.*

3. *If one writes $G_{\mathcal{F}}$ for the set of all locally defined diffeomorphisms of M of the form $(\Phi_{X_r})^{t_1} \circ \cdots \circ (\Phi_{X_d})^{t_d}$ for $X_1, \ldots, X_d \in \mathcal{F}$ and $t_1, \ldots, t_d \in \mathbb{R}$ for which this map is defined, then*

$$\mathcal{O}_{\mathcal{F}}(p) = \{\varphi(p) \mid \varphi \in G_{\mathcal{F}}\}$$
$$T\mathcal{O}_{\mathcal{F}}(p) = \mathrm{span}\{\varphi_*(X) \mid \varphi \in G_{\mathcal{F}},\ X \in \mathcal{F}\},$$

wherever the expressions $\varphi(p)$ and $\varphi_*(X)$ are defined.

As a consequence of Case 3. of the theorem, one sees that $L^1([0,T], \mathbb{R}^m)$ in Definition 5.1 can be replaced by $L^2([0,T], \mathbb{R}^m)$, which for the rest paper will be the appropriate *space of controls* for our needs.

Following [27] we define the concepts of the end-point mapping and that of a regular (L^2-)control.

Definition 5.3. For every $p \in M$, any time $T > 0$, and any smooth finite family of vector fields $\mathcal{F} = \{X_1, \ldots, X_m\}$ on M, there exists a maximal open subset $U_{\mathcal{F}}^{p,T} \subset L^2([0,T], \mathbb{R}^m)$ such that for every $u = (u_1, \ldots, u_m) \in U_p^T$, there exists a unique absolutely continuous solution $\gamma_u : [0,T] \to M$ to the Cauchy problem

$$\dot{\gamma_u}(t) = \sum_{i=1}^{m} u_i(t) X_i(\gamma_u(t)), \quad \gamma_u(0) = p. \tag{33}$$

The end-point map $E_{\mathcal{F}}^{p,T}$ associated to $\mathcal{F}$ at p in time T is defined as the mapping

$$E_{\mathcal{F}}^{p,T} : U_{\mathcal{F}}^{p,T} \to M, \quad E_{\mathcal{F}}^{p,T}(u) = \gamma_u(T).$$

By [27, Proposition 1.8] we have the following.

Proposition 5.4. *With $p, T, \mathcal{F}$ as above, the end point map $E_{\mathcal{F}}^{p,T} : U_{\mathcal{F}}^{p,T} \to M$ is C^1-smooth.*

This proposition allows us to give the following definition.

Definition 5.5. A control $u \in U_{\mathcal{F}}^{p,T}$ is said to be *o-regular* with respect to p in time T if the rank of $D_u E_{\mathcal{F}}^{p,T} : L^2([0,T], \mathbb{R}^m) \to T_{E_{\mathcal{F}}^{p,T}(u)} M$, the differential of $E_{\mathcal{F}}^{p,T}(u)$ at u, is equal to $\dim \mathcal{O}_{\mathcal{F}}(p)$. Here, "o-regular" stands for orbitally regular.

Remark 5.6. A control u is usually said to be regular (with respect to p in time T) if the rank of $E_{\mathcal{F}}^{p,T}(u)$ is equal to the dimension n of the ambient manifold M (cf [27, Section 1.3]), implying in particular that the orbit $\mathcal{O}_{\mathcal{F}}(p)$ is open in M and thus is n-dimensional. If the distribution generated by $\mathcal{F}$ verifies the LARC, it can be proved that any pair of points in M can be joined by the trajectory tangent to this distribution and corresponding to a regular control, cf. [3]. In this paper, we have extended this definition without assuming controllability.

The main purpose of this appendix is to generalize the result of [3] to the case where the distribution Δ is not necessarily bracket-generating. Indeed, we have the following result.

Proposition 5.7. *Let M be an n-dimensional smooth manifold, $\mathcal{F} = \{X_1, \dots, X_m\}$, $m \in \mathbb{N}$, a smooth* finite *family of vector fields on M. Then, for every $p \in M$ and time $T > 0$, and every $q \in \mathcal{O}_{\mathcal{F}}(p)$, there exists a* o-regular *control with respect to p in time T such that the unique solution γ_u to the Cauchy problem* (33) *such that $\gamma_u(T) = q$.*

Remark 5.8. By the proof of Proposition 1.12 in [27] (see also [16]), the conclusion is immediate if $T_q\mathcal{O}_{\mathcal{F}}(p)$ is equal for every $q \in \mathcal{O}_{\mathcal{F}}(p)$ to $\mathrm{Lie}_q(\mathcal{F})$, the evaluation at q of the Lie algebra generated by $\mathcal{F}$. In fact, in this case a stronger result holds, namely the set of regular controls is dense in $U_{\mathcal{F}}^{q,T}$ for every $q \in \mathcal{O}_{\mathcal{F}}(p)$ and $T > 0$. As a consequence, any control $u_0 \in E_{\mathcal{F}}^{p,T}$ admits an o-regular control u arbitrarily close (in L^2) to u_0 such that $E_{\mathcal{F}}^{p,T}(u) = E_{\mathcal{F}}^{p,T}(u_0)$.

Proof. Fix $q_0 \in \mathcal{O}_{\mathcal{F}}(p)$ and $(Z_1^0, \dots, Z_d^0)$ a basis of $T_{q_0}\mathcal{O}_{\mathcal{F}}(p)$. According to Theorem 5.2, there exists $\varphi_1 \in G_{\mathcal{F}}$ and $Y_1 \in \mathcal{F}$ with $q_1 := \varphi_1^{-1}(q_0)$ such that $Z^0 := (\tilde{Z}_1^0, Z_2^0, \dots, Z_d^0)$, where $\tilde{Z}_1^0 = (\varphi_1)_* Y_1|_{q_0}$, forms a basis of $\mathcal{O}_{\mathcal{F}}(p)$ at q_0. The basis Z^0 is the pushforward of a basis $Z^1 = (Z_1^1, \dots, Z_d^1)$ of $T_{q_1}\mathcal{O}_{\mathcal{F}}(p)$ by φ_1 and obviously $Z_1^1 = Y_1$. We proceed inductively (using Theorem 5.2) with this construction for $1 \le l \le d$ so that the basis $Z^{l-1} = (Z_1^{l-1}, \dots, Z_{l-1}^{l-1}, \tilde{Z}_l^{l-1}, Z_{l+1}^{l-1}, \dots, Z_d^{l-1})$ of $T_{q_{l-1}}\mathcal{O}_{\mathcal{F}}(p)$ is the pushforward of a basis $Z^l = (Z_1^l, \dots, Z_d^l)$ with $q_l := \varphi_l^{-1}(q_{l-1})$, $Y_l := Z_l^l \in \mathcal{F}$ and $\tilde{Z}_l^{l-1} = (\varphi_l)_*(Z_l^l)$. Finally consider $\varphi_{d+1} \in G$ so that $\varphi_{d+1}(p) = q_n$ and set $\psi = \varphi_1 \circ \varphi_2 \circ \dots \circ \varphi_{d+1}$. One has that $\psi(p) = q_0$ and there exists $T > 0$ and $u \in L^2([0,T], \mathbb{R}^m)$ such that the unique solution γ_u to the Cauchy problem $\dot{\gamma}_u(t) = \sum_{i=1}^m u_i(t) X_i|_{\gamma_u(t)}$, $x(0) = p$ verifies $\gamma_u(T) = q_0$. Then the flow of diffeomorphisms $\psi^u(t,q)$ corresponding to the time-varying vector field $q \mapsto \sum_{i=1}^m u_i(t) X_i|_q$ verifies $\psi = \psi^u(T,0)$ and $\psi^u(t,p) = \gamma_u(t)$ where one has, for $0 \le s \le t \le T$, $\frac{\partial \psi^u(t,q)}{\partial t} = \sum_{i=1}^m u_i(t) X_i|_{\psi^u(t,q)}$ together with the initial condition $\psi^u(0,q) = q$ for every $q \in M$. With the above notations, it is clear that, for every $1 \le l \le d$,

$$\left(d_q\psi^u(T,p)(d_q\psi^u(t_l,p))^{-1} Y_l\right)_{l=1,\dots,d} =: (\tilde{Z}_1, \dots, \tilde{Z}_d)$$

forms a basis of $T_{q_0}\mathcal{O}_{\mathcal{F}}(p)$, where $d_q\psi^u(t,\cdot)$ denotes the differential of $\psi(t,q)$ with respect to the q variable.

Recall that the differential of the end-point map at u is the linear map $D_u E_{\mathcal{F}}^{p,T} : L^2([0,T], \mathbb{R}^m) \to T_{q_0}\mathcal{O}_{\mathcal{F}}(p)$ given by

$$D_u E_{\mathcal{F}}^{p,T}(v) = d_q\psi^u(T,p) \int_0^T (d_q\psi^u(t,p))^{-1} X^v(t, \gamma_u(t)) dt, \tag{34}$$

where $X^v(t,x) = \sum_{i=1}^m v_i(t) X_i|_x$ for almost every $t \in [0,T]$ and every $x \in M$. We further complete the notations as follows. Let $0 = t_0 < t_1 < \dots < t_{d+1} := T$ the sequence of times where $\gamma_u(t_l) = q_{d+1-l}$ with the convention that $p = q_{d+1}$ and thus $\psi^u(T,p)\psi^u(t_l,p)^{-1}(q_l) = q_0$, for $0 \le l \le d+1$. Moreover, one has $Y_l = \sum_{i=1}^m y_{il} X_i|_{q_l}$ for $1 \le l \le d$ and some real numbers (y_{il}).

For every $\varepsilon > 0$ small enough and $1 \le l \le d$, consider the sequence (v_ε^l) of functions in $L^2([0,T], \mathbb{R}^m)$ defined by $v_\varepsilon^l(t) = \frac{1}{\varepsilon}(y_{il})_{1 \le i \le k}$ if $t_l - \varepsilon \le t \le t_l$ and zero otherwise. It is a matter of standard computations (as performed in [27, Proposition 1.10] to prove that, for every $1 \le l \le d$,

$D_u E_{\mathcal{F}}^{p,T}(v_\varepsilon^l)$ tends to $d_q\psi^u(T,p)(d_q\psi^u(t_l,p))^{-1}Y_l = \tilde{Z}_l$ as ε tends to zero. Since the the range of $D_u E_{\mathcal{F}}^{p,T}$ is closed, we deduce that it contains $\tilde{Z}_l$ for every $1 \le l \le d$.

We have therefore proved that u is o-regular at p in time T in the sense of Definition 5.3. □

Remark 5.9. In contrast to what was discussed in Remark 5.8, we highlight the fact that in general case where the (finite) family $\mathcal{F}$ of vector fields does not satisfy (everywhere on the orbit) the Hörmander condition $\mathrm{Lie}_q\mathcal{F} = T_q\mathcal{O}_{\mathcal{F}}(p)$, for a given control $u_0 \in U_{\mathcal{F}}^{p,T}$ the o-regular controls u (in the sense of Definition 5.5) such that $E_{\mathcal{F}}^{p,T}(u_0) = E_{\mathcal{F}}^{p,T}(u)$ might lie far away from u_0 in L^2-sense.

As the standard example, consider on $M = \mathbb{R}^2$, with coordinates (x,y), the vector fields (cf. [16], p.12) $X = \frac{\partial}{\partial x}$ and $Y = \phi(x)\frac{\partial}{\partial y}$ where $\phi : \mathbb{R} \to \mathbb{R}$ is smooth such that $\phi(x) = 0$ if $x \le 0$ and $\phi(x) > 0$ for $x > 0$. Let $\mathcal{F} = \{X, Y\}$ and $\mathbb{R}^2_- = \{(x,y) \mid x < 0\}$

It is clear that for any point $p_0 = (x_0, y_0)$ with $x_0 < 0$, any $T > 0$ and any control u_0 such that $E_{\mathcal{F}}^{p_0,t}(u_0) \in \mathbb{R}^2_-$ for all $t \in [0,T]$, there is an L^2-neighbourhood of u_0 such that $E_{\mathcal{F}}^{p_0,T}$ is not regular at any of its points.

A regular control u steering p_0 to $q_0 = E_{\mathcal{F}}^{p_0,T}(u_0)$ in time T (i.e. $E_{\mathcal{F}}^{p_0,T}(u) = q$), which exists thanks to Proposition 5.7, must have the property that $E_{\mathcal{F}}^{p_0,T_0}(u) \notin \mathbb{R}^2_-$ for some $0 < T_0 \le T$. Therefore, if we write $\gamma_u(t) = (x_u(t), y_u(t)) = E^{p_0,t}(u)$ and $u = (u_1, u_2)$, one has

$$|x_0| \le |x_u(T_0) - x_0| = \Big|\int_0^{T_0} u_1(s)ds\Big| \le \sqrt{T_0}\,\|u_1\|_{L^2([0,T])} \le \sqrt{T}\,\|u\|_{L^2([0,T],\mathbb{R}^2)}.$$

If for example one took $u_0 = 0$, hence $q_0 = p_0$, the above inequality would prove, as was claimed above, that a regular control u steering p_0 to q_0 in time T cannot be near u_0 in L^2-sense.

5.2 Semisimplicity of the Lie algebra L

In this paragraph, we prove that the Lie algebra L of H_Δ^∇ whose generators are given in Eq. (32) is compact semisimple. In order to so, according to the proof of Proposition 26.3 in [26], it is enough (and necessary) to show that L is compact and has trivial center. We also recall that a Lie algebra $\mathfrak{g}$ is called compact ([26, Definition 26.2]) if there is a positive definite inner product k on $\mathfrak{g}$ which satisfies

$$k([x,y],z) + k(y,[x,z]) = 0, \quad \forall x,y,z \in \mathfrak{g}. \tag{35}$$

It follows immediately that any Lie-subalgebra $\mathfrak{h}$ of a compact Lie-algebra $\mathfrak{g}$ is also compact, and therefore, L as a Lie-subalgebra of the compact $\mathfrak{so}(n+m)$ is compact.

It remains to show that the center L is trivial. Consider C in the center of L, i.e., $[C,X] = 0$ for every $X \in L$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be respectively the $\mathbb{R}$-linear span of the $(A_h)_{1\le h\le m}$ and the span of the $(B_{h,k})_{(h,k)\in\mathcal{I}}$. Note that $\mathfrak{b} = \mathfrak{so}(m)$ (up to an isomorphism of Lie algebras) and L is the direct sum of $\mathfrak{a}$ and $\mathfrak{b}$. Thus we can write $C = A + B$ with unique $A \in \mathfrak{a}$ and $B \in \mathfrak{b}$.

For every $(h,k) \in \mathcal{I}$, one has $0 = [C, B_{h,k}] = [A, B_{h,k}] + [B, B_{h,k}]$. Thanks to the relations in Eq. (32), one also has that $[A, B_{h,k}] \in \mathfrak{a}$ and $[B, B_{h,k}] \in \mathfrak{b}$, and then, due to the direct sum property one concludes that, for every $(h,k) \in \mathcal{I}$,

$$[A, B_{h,k}] = [B, B_{h,k}] = 0.$$

Since $\mathfrak{b}$ is semisimple, its center reduces to zero and thus $B = 0$. We next set $A = \sum_{l=1}^{m} a_l A_l$ and use the relations $[A, A_h] = 0$ for $1 \leq h \leq m$. We get that, for $1 \leq h \leq m$,

$$0 = \sum_{l=1}^{m} a_l [A_l, A_h] = \sum_{l=1}^{m} a_l B_{l,h},$$

yielding at once that $a_l = 0$ for $1 \leq h \leq m$ because $m \geq 2$. (Indeed we need at least two distinct indices h as above.) Then $C = 0$ which concludes the proof of the claim.

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