Stability of Sampled-Data PDE-ODE Systems and Application to Observer Design and Output-Feedback Stabilization

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Abstract: Small-gain stability analysis is developed for a class of sampled-data systems consisting of a parabolic-type PDE and a linear ODE connected in series with a sampler-ZOH block. This stability analysis is shown to be useful in designing sampled-data exponentially-convergent observers and exponentially stabilizing output-feedback controllers for some classes of ODE-PDE cascades. This work generalizes existing results in the literature in various directions.

Keywords: sampled-data systems, ODE-PDE cascades, observer design, output-feedback control.

1. INTRODUCTION

Sampled-data observer design has been a hot topic, especially over the last decade, see e.g. (Ahmed-Ali et al. 2016a, 2017) and reference list therein. As a matter of fact, most existing studies have been devoted to (finite-dimensional) nonlinear systems modeled by ODEs only, see e.g. (Ahmed-Ali et al., 2016b). Quite a few studies have so far been reported on sampled-output observers for (infinite-dimensional) systems involving PDEs. In (Fridman and Blighovsk, 2012) a ZOH sampled innovation observer has been proposed and analyzed using Lyapunov-Krasovskii functional leading to sufficient conditions for exponential stability in the form of LMIs. In (Ahmed-Ali et al., 2016a), an exponentially convergent sampled-output observer has been designed for linear ODE-PDE cascades, using the backstepping technique.

In this paper, the results of (Ahmed-Ali et al., 2016a) are extended in different directions. First, the (stand-alone) technical stability result is extended to a larger class of (PDE-ZOH-ODE cascade) systems where the PDE includes additional reaction terms and allows more general boundary conditions. Exponential stability of this larger class is established using the small-gain method. This stability result is then shown to be useful in analyzing exponential convergence of sampled-output observers, obtained by the backstepping design method of (Krstic, 2009), for a larger class of ODE-PDE cascades. The new class allows for additional reaction and convection terms in the PDE and for much more general boundary conditions. Finally, it is shown that the new technical stability result (for PDE-ZOH-ODE cascades) is also beneficial for sampled output-feedback stabilization of a class of ODE-PDE cascades.

The paper is organized as follows: in Section 2, a new stand-alone stability result is established; a first observer problem is formulated and dealt with in Section 3; an extension of this observer design is presented in Section 4, along with an output-feedback control design. Some concluding remarks are given at the end of the paper along with an appendix.

Notation. $H^2(0,1)$ denotes the Sobolev space of scalar functions $\eta:[0,1]\to\mathbf{R}$ with absolutely continuous $d\eta/d\varsigma\in L^2[0,1]$ and $d^2\eta/d\varsigma^2\in L^2[0,1]$. Given a function $w:[0,1]\times\mathbf{R}_+\to\mathbf{R};(x,t)\to w(x,t)$, $w[t]$ and $w_x[t]$ refer to the functions defined on $0\le x\le 1$ by $(w[t])(x)=w(x,t)$ and $(w_x[t])(x)=\partial w(x,t)/\partial x$.

2. STABILITY RESULT FOR PDE-ZOH-ODE CASCADE

In this section, we analyse the stability of the following class of sampled-data systems:

$$\dot X(t)=A_0X(t)+A_1X(t_k)+bw(0,t_k)+Gz(t),$$
$$\text{for all } t\in[t_k,t_{k+1}) \text{ a.e. and } k=0,1,2\ldots \quad (1)$$

$$w_t(x,t)=aw_{xx}(x,t)+kw(x,t),$$
$$\text{for } (x,t)\in(0,1)\times(0,+\infty), \quad (2)$$

with

$$w_x(0,t)=qw(0,t) \text{ and } w(1,t)=0, \text{ for all } t\ge 0 \quad (3)$$

and

$$X(0)=X_0 \text{ and } w(x,0)=(w[0])(x) \text{ for } x\in[0,1] \quad (4)$$

where (a,k,q) are real constants to be defined later; $X(t)\in\mathbf{R}^n$ denotes the state vector of the finite-dimensional subsystem (1), $w(x,t)\in\mathbf{R}$ is the state of the distributed parameter subsystem (2)-(3); $z(t)\in\mathbf{R}^m$ is an exogenous input assumed to be measurable and locally essentially bounded; $A_0,A_1\in\mathbf{R}^{n\times n}$, $G\in\mathbf{R}^{n\times m}$ and $b\in\mathbf{R}^n$ are constant matrices and vector; $\{t_k\}_{k=0}^{\infty}$ is a partition of $\mathbf{R}_+$.

Remark 1. a) Clearly, the linear subsystem represented by the parabolic PDE (2)-(3), is continuous-time autonomous, while the linear subsystem (1) is sampled-data (due to the term $A_1 X(t_k)$) and input-dependent.

b) The interconnection of the two subsystems, depicted by Fig.1, results in a sampled-data PDE-ZOH-ODE cascade, to be distinguished from the (continuous-time) PDE-ODE cascades studied in e.g. (Krstic, 2009).

c) The class of sampled-data PDE-ODE cascades defined by (1)-(3) is a generalization of that considered in (Ahmed-Ali et al., 2016a) where $a=1$ and $k=q=0$.

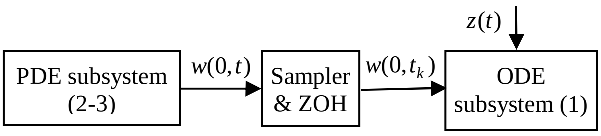

Fig. 1. PDE-ZOH-ODE cascade

The following lemma describes an essential stability estimate for the solution of the infinite-dimensional system (2)-(4).

Lemma 1. *Consider the system described by (2)-(4) with its parameters* (a,k,q) *satisfying the condition,*

$$a>0\,,\ q\ge 0 \text{ and } k<a\left(\frac{\pi}{2}+\theta\right)^2, \tag{5}$$

where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *is the unique solution of the equation* $\left(\frac{\pi}{2}+\theta\right)\tan(\theta)=q$ *in the interval* $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. *Then there exist constants* $\Theta>0$ *and* $\sigma>0$ *such that for every* $w[0]\in C^2([0,1];\mathbf{R})$ *with* $(w[0])(1)=(w_x[0])(0)-q(w[0])(0)=0$, *the initial value problem (2)-(3) has a unique solution* $w[t]\in C^2([0,1];\mathbf{R})$, *for all* $t\ge 0$, *which satisfies the following inequality:*

$$\|w[t]\|_\infty \le \Theta\exp(-\sigma t)\|w[0]\|_\infty,\ \textit{for all}\ \ t\ge 0 \tag{6}$$

See the proof in Appendix A. Next, let the finite-dimensional sampled-data subsystem (1) be rewritten as follows:

$$\dot{X}(t)=A_0X(t)+A_1X(t_k)+z_a(t) \tag{7}$$

with

$$z_a(t):=bw(0,t_k)+Gz(t)\ ,\ \text{for}\, t\in[t_k,t_{k+1})\ ,\ k=0,1,2\ldots \tag{8}$$

It has been shown in (Ahmed-Ali et al., 2016a) that the system (7) has the following Input-to-State Stability (ISS) property:

Lemma 2. *Consider the sampled-data system (7) where the matrix* A_0+A_1 *is Hurwitz. Let* $R,\lambda>0$ *be any real constants and* $\phi:\mathbf{R}_+\to\mathbf{R}_+$ *any continuous function satisfying, for all* $t\ge 0$ *:*

$$\left|\exp\left((A_0+A_1)t\right)\right|\le R\exp(-\lambda t)\ \text{ and }\ \left|\exp\left(A_0t\right)\right|\le\phi(t) \tag{9}$$

Also, let $T>0$ and $\sigma\in(0,\lambda)$ *be any constants satisfying:*

$$\lambda>\sigma+R|A_1|\exp(\sigma T)\left(|A_0|+|A_1|\right)\int_0^T\phi(s)ds \tag{10}$$

Then, there exist real constants $\overline{K},\bar{\gamma}>0$ *such that for every* $z_a\in L^\infty_{loc}\left(\mathbf{R}_+;\mathbf{R}^m\right)$, $X_0\in\mathbf{R}^n$, *and any* T*-diameter partition* $\{t_k\}_{k=0}^\infty$ of $\mathbf{R}_+$, *the unique solution of the initial value problem (7) with* $X(0)=X_0$ *exists for all* $t\ge 0$ *and satisfies the "following inequality, for all* $t\ge 0$ *:*

$$|X(t)|\le\overline{K}\exp(-\sigma t)|X_0|+\bar{\gamma}\sup_{0\le s\le t}\left(|z_a(s)|\exp(-\sigma(t-s))\right) \tag{11}$$

Combining the above two lemmas, we get the following ISS property of the original system (1)-(3).

Proposition 1. *Consider the sampled-data system of Lemma 1 and suppose further that the matrix* A_0+A_1 *is Hurwitz. Let* $R,\lambda>0$ *be any real constants and* $\phi:\mathbf{R}_+\to\mathbf{R}_+$ *be any continuous function satisfying (9) for all* $t\ge 0$. *Also, let* $T>0$ *be a real constant satisfying*

$$\lambda>R|A_1|\left(|A_0|+|A_1|\right)\int_0^T\phi(s)ds \tag{12}$$

Then, there exist real constants $\sigma,K,\gamma>0$ *such that for every* $z\in L^\infty_{loc}\left(\mathbf{R}_+;\mathbf{R}^m\right)$, $X_0\in\mathbf{R}^n$, $w[0]\in C^2([0,1];\mathbf{R})$ *with* $(w[0])(1)=(w_x[0])(0)-q(w[0])(0)=0$, *and any* T*-diameter partition* $\{t_k\}_{k=0}^\infty$ of $\mathbf{R}_+$, *the unique solution of the initial value problem (1)-(4) exists for all* $t\ge 0$ *and satisfies the following inequalities for all* $t\ge 0$ *:*

$$|X(t)|\le K\exp(-\sigma t)\left(|X_0|+\|w[0]\|_\infty\right)+\gamma\sup_{0\le s\le t}\left(|z(s)|\right) \tag{13}$$

Proof. Without loss of generality, we may assume that $\sigma>0$ is selected to be sufficiently small so that both (6) and (10) hold. By virtue of (6) we have, for all integers $k\ge 0$:

$$|w(0,t_k)|\le\Theta\exp\left(-\sigma t_k\right)\|w[0]\|_\infty \tag{14}$$

Using (14) and the fact that $\sup_{k\ge 0}\left(t_{k+1}-t_k\right)\le T$, it follows from (8) that:

$$\sup_{0\le s\le t}\left(|z_a(s)|\exp(\sigma s)\right)\le|b|\Theta\exp(\sigma T)\|w[0]\|_\infty + |G|\exp(\sigma t)\sup_{0\le s\le t}\left(|z(s)|\right) \tag{15}$$

Inequality (13), with appropriate constants $K,\gamma>0$, is a direct consequence of estimate (11) and inequality (15). The proof of Proposition 1 is complete. ◁

3. SAMPLED-DATA OBSERVER DESIGN FOR ODE-PDE CASCADES

3.1 Class of observed systems

In this section, we are interested in a class of continuous-time systems modelled by the following ODE-PDE cascade:

$$\dot{X}(t)=AX(t)+f(v(t))\,,\ \text{for }t\ge 0 \tag{16}$$

$$u_t(x,t)=au_{xx}(x,t)+ku(x,t)+g(x,v(t))\,,\ \text{for }(x,t)\in(0,1)\times(0,+\infty) \tag{17}$$

$$u_x(0,t)=qu(0,t)+p_0(v(t))\,,\ \text{for all }t\ge 0 \tag{18}$$

$$u(1,t)=cX(t)+p_1(v(t))\,,\ \text{for all }t\ge 0 \tag{19}$$

where $a>0$, $k\in\mathbf{R}$, $q\ge 0$ are real parameters satisfying (5); $A\in\mathbf{R}^{n\times n}$ and $c\in\mathbf{R}^{1\times n}$ are constant matrices; $f(.)$ is of class $C^1(\mathbf{R}^m;\mathbf{R}^n)$, g is of class $C^1([0,1]\times\mathbf{R}^m;\mathbf{R})$ and p_0, p_1 are of class $C^2(\mathbf{R}^m;\mathbf{R})$; the signal $X(t)\in\mathbf{R}^n$ denotes the state vector of the subsystem, described by (16) with initial condition $X(0)=X_0$, while $u(x,t)\in\mathbf{R}$ designates the state of the infinite-dimensional subsystem, described by the parabolic type PDE (17) with boundary conditions (18)-(19); $v(t)\in\mathbf{R}^m$ is any exogenous signal of class $C^2(\mathbf{R}_+;\mathbf{R}^m)$ that is accessible to measurements.

The pair (A,c) is observable and the whole system is observed through a ZOH sampling of the signal $u(0,t)$, i.e. the system output is:

$$y(t)=u(0,t_k)\ \text{, for all } t\in[t_k,t_{k+1}) \text{ and } k=0,1,2\ldots \quad (20)$$

where $\{t_k\}_{k=0}^{\infty}$ denotes the sampling time sequence, supposed to be a partition of $\mathbf{R}_+$ with diameter T. We seek an observer that provides accurate online estimates of both the (finite-dimensional) state vector $X(t)$ and the distributed state $u(x,t)$, $0\le x\le 1$, based the system input $v(t)$ and the output $y(t)$. The signal $u(1,t)$ is not accessible to measurements.

Global existence and uniqueness of solutions for the problem (16)-(17) follow from Theorem 2.1 in (Karafyllis and Krstic, 2016) (and the arguments in the proof of Lemma 1) for aby initial condition $X_0\in\mathbf{R}^n$, $u[0]\in C^2([0,1];\mathbf{R})$, any input $v\in C^2(\mathbf{R}_+;\mathbf{R}^m)$ with $(u_x[0])(0)=q(u[0])(0)+p_0(v(0))$, $(u[0])(1)=cX_0+p_1(v(0))$. If both functions p_0, p_1 are zero, then the input v can be of class $C^1(\mathbf{R}_+;\mathbf{R}^m)$.

3.2. Observer design and analysis

Consider the following backstepping transformation:

$$p(x,t)=u(x,t)-cM(x)X(t),\quad \text{for } (x,t)\in[0,1]\times[0,+\infty) \quad (21)$$

where $M(x)\in\mathbf{R}^{n\times n}$ is defined by the following ODE equation:

$$\frac{d^2M}{dx^2}(x)=a^{-1}M(x)(A-kI) \quad (22)$$

with initial condition

$$\frac{dM}{dx}(0)=qM(0) \quad (23)$$

The solution of the problem (22)-(23) is analytic and is expressed by the following globally convergent series

$$M(x)=M(0)\left((1+qx)I+\sum_{l=1}^{\infty}a^{-l}(A-kI)^l\left[\frac{x^{2l}}{(2l)!}+q\frac{x^{2l+1}}{(2l+1)!}\right]\right) \quad (24)$$

We next assume that there is a matrix $M(0)\in\mathbf{R}^{n\times n}$ such that

$$cM(1)=cM(0)\left((1+q)I+\sum_{l=1}^{\infty}\frac{(2l+1+q)}{a^l(2l+1)!}(A-kI)^l\right)=c \quad (25)$$

Now, using (22), (16) and (17), it follows that the new state $p(x,t)$ defined by (21) undergoes the following PDE, for all $(x,t)\in(0,1)\times(0,+\infty)$:

$$p_t(x,t)=ap_{xx}(x,t)+kp(x,t)+g(x,v(t))-cM(x)b(v(t)) \quad (26)$$

For convenience, the new system representation expressed in terms of the states $(X(t),p(x,t))$ is recapitulated here:

$$\dot{X}(t)=AX(t)+f(v(t)),\ \text{for all } t\ge 0 \quad (27)$$

$$p_t(x,t)=ap_{xx}(x,t)+kp(x,t)+g(x,v(t))-cM(x)f(v(t)),\quad \text{for all } (x,t)\in(0,1)\times(0,+\infty) \quad (28)$$

$$p_x(0,t)=qp(0,t)+p_0(v(t)),\ \text{for all } t\ge 0 \quad (29)$$

$$p(1,t)=p_1(v(t)),\quad \text{for all } t\ge 0 \quad (30)$$

$$u(x,t)=p(x,t)+cM(x)X(t) \quad (31)$$

where the boundary conditions (29)-(30) are immediately obtained from (21) using (18), (19), (23) and (25). A key feature of the new model is that the infinite-dimensional subsystem, here defined by (28)-(30), is decoupled from the finite-dimensional subsystem described by (27) (while coupling exists in the initial system representation (16)-(19)).

To get online estimates $\hat{X}(t)$ and $\hat{p}(x,t)$ of the unmeasurable states $X(t)$ and $p(x,t)$ of the system (27)-(31), the following sampled-output observer is considered:

$$\dot{\hat{X}}(t)=A\hat{X}(t)+f(v(t))-L(\hat{y}(t_k)-y(t_k)),\quad \text{for all } t\in[t_k,t_{k+1}) \text{ and } k=0,1,2\ldots \quad (32)$$

$$\hat{p}_t(x,t)=a\hat{p}_{xx}(x,t)+k\hat{p}(x,t)+g(x,v(t))-cM(x)f(v(t)),\quad \text{for all } (x,t)\in(0,1)\times(0,+\infty) \quad (33)$$

$$\hat{p}_x(0,t)-q\hat{p}(0,t)-p_0(v(t))=\hat{p}(1,t)-p_1(v(t))=0,\quad \text{for all } t\ge 0 \quad (34)$$

$$\hat{u}(x,t)=\hat{p}(x,t)+cM(x)\hat{X}(t),\ (x,t)\in[0,1]\times[0,+\infty) \quad (35)$$

with $\hat{y}(t_k)=\hat{u}(0,t_k)$, where $L\in\mathbf{R}^n$ is arbitrary vector such that $A-Lc$ is a Hurwitz matrix. The last requirement is not an issue since the pair (A,c) is observable. Clearly, the observer is a copy of the system (27)-(31) with a feedback innovation term in equation (32). To analyze this observer, the following state estimation errors are introduced:

$$\tilde{X}(t)=\hat{X}(t)-X(t),\ \tilde{p}(x,t)=\hat{p}(x,t)-p(x,t) \quad (36)$$

$$\tilde{u}(x,t)=\hat{u}(x,t)-u(x,t),\ \tilde{p}(x,t)=\hat{p}(x,t)-p(x,t) \quad (37)$$

Then, the following error system is readily obtained, using (27)-(31), (32)-(33) and definitions (36), (37):

$$\dot{\tilde{X}}(t)=A\tilde{X}(t)-Lc\tilde{X}(t_k)-L\tilde{p}(0,t_k),\quad \text{for all } t\in[t_k,t_{k+1}) \text{ and } k=0,1,2\ldots \quad (38)$$

$$\tilde{p}_t(x,t)=a\tilde{p}_{xx}(x,t)+k\tilde{p}(x,t),\quad \text{for all } (x,t)\in(0,1)\times(0,+\infty) \quad (39)$$

$$\tilde{p}_x(0,t)-q\tilde{p}(0,t)=\tilde{p}(1,t)=0,\quad \text{for all } t\ge 0, \quad (40)$$

$$\tilde{u}(x,t)=\tilde{p}(x,t)+cM(x)\tilde{X}(t),\quad \text{for all } (x,t)\in[0,1]\times[0,+\infty) \quad (41)$$

where the first equation is obtained using the fact that $\hat{y}(t_k)-y(t_k)=\tilde{u}(0,t_k)$ and equations (32) and (35). It is readily checked that the error system (38)-(41) fits the form of the sampled-data system (1)-(3) with (X,w) replaced by $(\tilde{X},\tilde{p})$, $A_0=A$, $A_1=-Lc$, $b=-L$ and $z(t)=0$. With these notations, $A+A_1=A-Lc$ is Hurwitz. It turns out that

Proposition 1 is applicable to the error system (38)-(41) leading to the following result:

Theorem 1. *Consider the class of systems defined by equations (16)-(19) with parameters* (a,k,q) *satisfying inequalities (5). Consider the observer defined (32)-(35), where the gain* $L\in \mathbf{R}^n$ *is selected so that the matrix* $A-Lc\in \mathbf{R}^{n\times n}$ *is Hurwitz and the matrix* $M(x)$ *is defined by (22)-(25). Then there exist real constants* $T,\rho,\sigma>0$ *such that, for any* T*-diameter partition* $\{t_k\}_{k=0}^{\infty}$ *and any* $v\in C^2(\mathbf{R}_+;\mathbf{R}^m)$, $X_0,\hat{X}_0\in \mathbf{R}^n$, $u[0],\hat{p}[0]\in C^2([0,1];\mathbf{R})$, *with* $(\hat{p}[0])(1)-p_1(v(0))=0$, $(\hat{p}_x[0])(0)-q(\hat{p}[0])(0)-p_0(v(0))=0$, $(u_x[0])(0)=q(u[0])(0)+p_0(v(0))$, and $(u[0])(1)=cX_0+p_1(v(0))$, *one has:*

(i) The initial value problem defined by (21)-(24) and (32)-(35) with initial conditions $X(0)=X_0$, $\hat{X}(0)=\hat{X}_0$, $u(x,0)=(u[0])(x)$, $\hat{p}(x,0)=(\hat{p}[0])(x)$ *for* $x\in[0,1]$, *has a unique solution;*

(ii) This unique solution satisfies for all $t\geq 0$ *:*

$$\left|\tilde{X}(t)\right|+\|\tilde{u}[t]\|_\infty \leq \rho\exp(-\sigma t)\left(\left|\hat{X}_0-X_0\right|+\|\tilde{p}[0]\|_\infty\right) \quad (42)$$

where $\tilde{X}(t)$, $\tilde{u}[t]\in C^2([0,1];\mathbf{R})$ *and* $\tilde{p}[t]\in C^2([0,1];\mathbf{R})$ *are defined by* (36)-(37), *for* $t\geq 0$.

Proof. Applying Proposition 1 to the system (38)-(39), it follows that for any $T>0$ sufficiently small so that (12) holds, there exist real constants $\sigma,K,\Theta,\gamma>0$ such that (13) and (6) hold, with (X,w) replaced by $(\tilde{X},\tilde{p})$ and $z(t)=0$. Accordingly, one has:

$$\|\tilde{p}[t]\|_\infty \leq \Theta\exp(-\sigma t)\|\tilde{p}[0]\|_\infty \quad (43)$$

$$\left|\tilde{X}(t)\right|\leq K\exp(-\sigma t)\left(\left|\tilde{X}(0)\right|+\|\tilde{p}[0]\|_\infty\right) \quad (44)$$

for all $t\geq 0$. Using (43)-(44), it follows from (41) that:

$$\|\tilde{u}[t]\|_\infty \leq \|\tilde{p}[t]\|_\infty+\max_{0\leq x\leq 1}\left(|cM(x)|\right)\left|\tilde{X}(t)\right|, \text{ for all } t\geq 0 \quad (45)$$

Combining (44) and (45) yields,

$$\left|\tilde{X}(t)\right|+\|\tilde{u}[t]\|_\infty \leq \rho\exp(-\sigma t)\left(\left|\tilde{X}(0)\right|+\|\tilde{p}[0]\|_\infty\right)$$

for some real constant $\rho>0$ and all $t\geq 0$, which proves (42). Theorem 1 is proved. ⊲

4. EXTENSIONS

4.1 Observer design extension

The observer design method of Section III will next be extended to the following wider class of systems:

$$\dot{X}(t)=AX(t)+f(v(t)), \text{ for } t\geq 0 \quad (46)$$

$$w_t(x,t)=aw_{xx}(x,t)+bw_x(x,t)+kw(x,t)+g(x,v(t)), \text{ for } (x,t)\in(0,1)\times(0,+\infty) \text{ a.e.} \quad (47)$$

$$w_x(0,t)=qw(0,t)+p_0(v(t)), \text{ for all } t\geq 0 \quad (48)$$

$$w(1,t)=cX(t)+p_1(v(t)), \text{ for all } t\geq 0 \quad (49)$$

Compared to the initial class of systems defined by (16)-(19), the new PDE equation (47) includes the convection term $bw_x(x,t)$ (with $b\in\mathbf{R}$); all other quantities remain unchanged with respect to (16)-(19). Here, we seek an exponentially convergent observer for the new system (46)-(49). To this end, we will introduce a state transformation and show that the resulting transformed system fits the model structure (16)-(19). Consider the transformation,

$$u(x,t)=e^{r(x-1)}w(x,t), \text{ for } (x,t)\in[0,1]\times[0,+\infty) \quad (50)$$

with

$$r=\frac{b}{2a} \quad (51)$$

Differentiating $u(x,t)$, one gets using (40):

$$u_t(x,t)=ae^{r(x-1)}w_{xx}(x,t)+be^{r(x-1)}w_x(x,t)+ke^{r(x-1)}w(x,t)+e^{r(x-1)}g(x,v(t)) \quad (52)$$

$$u_x(x,t)=re^{r(x-1)}w(x,t)+e^{r(x-1)}w_x(x,t) \quad (53)$$

$$u_{xx}(x,t)=r^2e^{r(x-1)}w(x,t)+2re^{r(x-1)}w_x(x,t)+e^{r(x-1)}w_{xx}(x,t) \quad (54)$$

for $(x,t)\in[0,1]\times[0,+\infty)$. It follows from (52)-(54) that:

$$u_t(x,t)=au_{xx}(x,t)+(k-\frac{b^2}{4a})u+e^{r(x-1)}g(x,v(t)) \text{ for } (x,t)\in(0,1)\times(0,+\infty) \text{ a.e.} \quad (55)$$

Similarly, the following boundary conditions are readily obtained from (50) and (53), using (48) and (49):

$$u(1,t)=w(1,t)=cX(t)+p_1(v(t)), \text{ for all } t\geq 0 \quad (56)$$

$$u_x(0,t)=(q+\frac{b}{2a})u(0,t)+e^{-r}p_0(v(t)), \text{ for all } t\geq 0 \quad (57)$$

The transformed system with states $(X(t),u(x,t))$ is modeled by equations (46), (55), (56) and (57). For convenience, this system model is rewritten;

$$\dot{X}(t)=AX(t)+f(v(t)), \text{ for } t\geq 0 \quad (58)$$

$$u_t(x,t)=au_{xx}(x,t)+(k-\frac{b^2}{4a})u+e^{r(x-1)}g(x,v(t)) \text{ for } (x,t)\in(0,1)\times(0,+\infty) \text{ a.e.} \quad (59)$$

$$u(1,t)=cX(t)+p_1(v(t)), \text{ for all } t\geq 0 \quad (60)$$

$$u_x(0,t)=(q+\frac{b}{2a})u(0,t)+e^{-r}p_0(v(t)), \text{ for } t\geq 0 \quad (61)$$

$$w(x,t)=e^{-r(x-1)}u(x,t) \quad (62)$$

Clearly, equations (58)-(61) fit the model structure (16)-(19) where the couple of parameters (k,q) are replaced by $(k-\frac{b^2}{4a},q+\frac{b}{2a})$ provided that the pair (A,c) is observable and (a,k,q) satisfy the conditions:

$$a>0,\ q+\frac{b}{2a}\geq 0, \text{ and } k-\frac{b^2}{4a}<a\left(\frac{\pi}{2}+\theta\right)^2 \quad (63)$$

where $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is the unique solution of the equation,

$$\left(\frac{\pi}{2}+\theta\right)\tan(\theta)=q+\frac{b}{2a} \quad (64)$$

Then, state estimates can be obtained by applying observer (32)-(35) with obvious notation adaptations. Specifically, the observer writes:

$$\dot{\hat{X}}(t) = A\hat{X}(t) + f(v(t)) - L(\hat{y}(t_k) - y(t_k)),$$
$$\text{for } t \in [t_k, t_{k+1}) \text{ and } k = 0,1,2\ldots \quad (65)$$

$$\hat{p}_t(x,t) = a\hat{p}_{xx}(x,t) + (k - \frac{b^2}{4a})\hat{p}(x,t) + e^{r(x-1)}g(x, v(t))$$
$$-cM(x)f(v(t)), \text{ for } (x,t) \in (0,1)\times(0,+\infty) \quad (66)$$

$$\hat{p}_x(0,t) - (q + \frac{b}{2a})\hat{p}(0,t) - p_0(v(t)) = 0, \text{ for } t \ge 0 \quad (67)$$

$$\hat{p}(1,t) - p_1(v(t)) = 0, \text{ for all } t \ge 0 \quad (68)$$

$$\hat{u}(x,t) = \hat{p}(x,t) + cM(x)\hat{X}(t),$$
$$\text{for } (x,t) \in [0,1]\times[0,+\infty) \quad (69)$$

$$\hat{w}(x,t) = e^{-r(x-1)}\hat{u}(x,t) \quad \text{with} \quad r = \frac{b}{2a} \quad (70)$$

where $M(x) \in \mathbf{R}^{n\times n}$ is as in (22)-(24) under assumption (25) with the parameters (k,q) replaced by $(k - \frac{b^2}{4a}, q + \frac{b}{2a})$. Then, Theorem 1 is applied yielding the following result.

Corollary 1. *Consider the system defined by equations (46)-(49) where the real parameters (a,k,q) satisfy inequalities (63)-(64). Consider the observer defined (65)-(70), where the gain $L \in \mathbf{R}^n$ is selected so that the matrix $A - Lc \in \mathbf{R}^{n\times n}$ is Hurwitz and the matrix $M(x)$ is defined by (22)-(24) under assumption (25) with (k,q) replaced by $(k - \frac{b^2}{4a}, q + \frac{b}{2a})$. Then there exist real constants $T, \rho, \sigma > 0$ such that, for any (T-diameter partition) $\{t_k\}_{k=0}^{\infty}$, $v \in C^2(\mathbf{R}_+; \mathbf{R}^m)$, any $X_0, \hat{X}_0 \in \mathbf{R}^n$, $w[0], \hat{p}[0] \in C^2([0,1];\mathbf{R})$, with $(\hat{p}[0])(1) - p_1(v(0)) = 0$, $(w_x[0])(0) = q(w[0])(0) + p_0(v(0))$, $(w[0])(1) = cX_0 + p_1(v(0))$, and $(\hat{p}_x[0])(0) - (q + \frac{b}{2a})(\hat{p}[0])(0) - p_0(v(0)) = 0$, one has:*

(i) The initial value problem defined by (46)-(49) and (65)-(70) with initial conditions $X(0) = X_0$, $\hat{X}(0) = \hat{X}_0$, $w(x,0) = (w[0])(x)$, $\hat{p}(x,0) = (\hat{p}[0])(x)$ for $x \in [0,1]$, has a unique solution;

(ii) This unique solution satisfies, for all $t \ge 0$:

$$\left|\tilde{X}(t)\right| + \left\|\tilde{w}[t]\right\|_\infty \le \rho \exp(-\sigma t)\left(\left|\hat{X}_0 - X_0\right| + \left\|\tilde{p}[0]\right\|_\infty\right)$$

where $\tilde{X}(t)$, $\tilde{w}[t] \in C^2([0,1];\mathbf{R})$ and $\tilde{p}[t] \in C^2([0,1];\mathbf{R})$ are defined by (36)-(37), *for $t \ge 0$.*

4.2 Output Feedback Stabilization

Consider the following system of the form (16)-(19):

$$\dot{X}(t) = AX(t) + Bv(t), \text{ for } t \ge 0 \quad (71)$$

$$u_t(x,t) = au_{xx}(x,t) + ku(x,t),$$
$$\text{for } (x,t) \in (0,1)\times(0,+\infty) \text{ a.e.} \quad (72)$$

$$u_x(0,t) = qu(0,t), \text{ for all } t \ge 0 \quad (73)$$

$$u(1,t) = cX(t), \text{ for all } t \ge 0 \quad (74)$$

where $B \in \mathbf{R}^{n\times m}$ is such that the pair the pair (A,B) is stabilizable and all other quantities are identical to (16)-(19). We seek the stabilization of the subsystem (71) based on the ZOH sampled system output defined by (20). To this end, we consider the output-feedback controller:

$$v(t) = -K\hat{X}(t) \quad (75)$$

where $K \in \mathbf{R}^{m\times n}$ is such that the matrix $A - BK$ is Hurwitz and $\hat{X}(t)$ is provided by the observer (32)-(35) letting there,

$$f(v) = Bv, g(.) = 0, p_0(.) = 0, p_1(.) = 0 \quad (76)$$

To analyse the closed-loop control system, it again proves to be useful representing the controlled system (71)-(74) in the coordinates $(X(t), p(x,t))$, with the second variable defined by (21). Then, substituting the right side of (75) to $v(t)$ in the obtained representation and in the observer (32)-(35), one gets the following closed-loop control system representation, expressed in terms of the states $X(t)$, $\tilde{X}(t)$, and $\tilde{p}(x,t)$:

$$\dot{X}(t) = (A - BK)X(t) - BK\tilde{X}(t), \text{ for } t \ge 0, \quad (77)$$

$$\dot{\tilde{X}}(t) = A\tilde{X}(t) - Lc\tilde{X}(t_k) - L\tilde{p}(0,t_k),$$
$$\text{for all } t \in [t_k, t_{k+1}) \text{ and } k = 0,1,2\ldots \quad (78)$$

$$\tilde{p}_t(x,t) = a\tilde{p}_{xx}(x,t) + k\tilde{p}(x,t),$$
$$\text{for all } (x,t) \in (0,1)\times(0,+\infty) \quad (79)$$

$$\tilde{p}_x(0,t) - q\tilde{p}(0,t) = \tilde{p}(1,t) = 0, \quad \text{for all } t \ge 0, \quad (80)$$

$$\tilde{u}(x,t) = \tilde{p}(x,t) + cM(x)\tilde{X}(t),$$
$$\text{for all } (x,t) \in [0,1]\times[0,+\infty) \quad (81)$$

The estimation error system (78)-(81) is obtained from (38)-(41) using (76). Then, Theorem 1 applies to the former which entails the exponential convergence of $\tilde{X}(t)$ to the origin. Since the matrix $A - BK$ is Hurwitz, it follows from (77) that $X(t)$ also converges to the origin exponentially. The performance of the output-feedback controller is described by the following corollary of Theorem 1.

Corollary 2. *Consider the class of systems (71)-(74) with parameters (a,k,q) which satisfy inequalities (5). Consider the output-feedback controller defined by the control law (75), with the gain K is such that the matrix $A - BK$ is Hurwitz, and the observer (32)-(35) with the gain $L \in \mathbf{R}^n$ and the matrix $M(x)$ are as in Theorem 1. Then, there exist real constants $T, \rho, \sigma > 0$ such that, for any T-diameter partition $\{t_k\}_{k=0}^{\infty}$, any $X_0, \hat{X}_0 \in \mathbf{R}^n$, $u[0], \hat{p}[0] \in C^2([0,1];\mathbf{R})$, with $(\hat{p}[0])(1) = 0$, $(u_x[0])(0) = q(u[0])(0)$, $(u[0])(1) = cX_0$ and $(\hat{p}_x[0])(0) - q(\hat{p}[0])(0) = 0$, one has:*

(i) The initial value problem defined by (71)-(74), (32)-(35) and (75)-(76) with initial conditions $X(0) = X_0$, $\hat{X}(0) = \hat{X}_0$, $u(x,0) = (u[0])(x)$, $\hat{p}(x,0) = (\hat{p}[0])(x)$ for $x \in [0,1]$, has a unique solution;

(ii) This unique solution satisfies, for all $t \ge 0$:

$$\left|X(t)\right|+\left|\tilde{X}(t)\right|+\left\|\tilde{u}[t]\right\|_\infty$$
$$\leq \rho\exp(-\sigma t)\left(\left|X_0\right|+\left|\hat{X}_0-X_0\right|+\left\|\tilde{p}[0]\right\|_\infty\right) \qquad (42)$$

with $X(t), \tilde{X}(t)$, $\tilde{u}[t]\in C^2([0,1];\mathbf{R})$ *and* $\tilde{p}[t]\in C^2([0,1];\mathbf{R})$.

5. CONCLUSION

The stability result of Proposition 1, which is an extension of previous results on sampled-data PDE-ZOH-ODE cascades, has been shown to be useful in design and analysis of exponentially convergent sampled-output observers (Theorem 1 and Corollary 1) and sampled output-feedback stabilizing controllers (Corollary 2). Further investigations are underway to extend the present stability result (of Proposition 1) and its application in observer and control design to nonlinear systems.

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Appendix A. PROOF OF LEMMA 1

Let $\left\{\phi_l(x)\in C^2([0,1];\mathbf{R})\right\}_{l=0}^\infty$ and $\mu_0<\mu_1<\mu_2<\ldots$ denote respectively the eigenfunctions and the eigenvalues of the Sturm-Liouville operator $-d^2/dx^2$ defined on the set

$$\Omega=\left\{ f\in H^2(0,1): f(1)=\frac{df}{dx}(0)-qf(0)=0 \right\}.$$

These eigenfunctions and eigenvalues are explicitly given by the following formulas, for $n=0,1,2,...$ and $z\in[0,1]$:

$$\phi_n(z)=\sqrt{2}\sqrt{\frac{q^2+\mu_n}{q^2+q+\mu_n}}\sin\left((1-z)\sqrt{\mu_n}\right) \qquad (A1)$$

$$\mu_n=\frac{1}{4}\left((2n+1)\pi+2\theta_n\right)^2 \text{ with } \theta_n\in\left[0,\frac{\pi}{2}\right) \qquad (A2)$$

where θ_n is the unique solution of the equation $\left(n\pi+\frac{\pi}{2}+\theta_n\right)\tan(\theta_n)=q$ in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ (recall that $q\geq 0$). It turns out that $\left\{\phi_l(x)\in C^2([0,1];\mathbf{R})\right\}_{l=0}^\infty$ and $\lambda_l=a\mu_l-k$ for $l=0,1,2,...$ are the eigenfunctions and the eigenvalues of the Sturm-Liouville operator $-ad^2/dx^2-k$ defined on $\Omega=\left\{ f\in H^2(0,1): f(1)=\frac{df}{dx}(0)-qf(0)=0 \right\}$.

Inequality (5) guarantees that $\lambda_0>0$. Moreover, formulas (A1) and (A2), together with the fact that $q\geq 0$, imply that $\max_{0\leq z\leq 1}\left(\left|\phi_n(z)\right|\right)\leq 1$ and $\lambda_n\geq an^2\pi^2-k$ for all $n=0,1,2,...$. It turns out that Assumptions (H1), (H2), (H3) in (Karafyllis and Krstic, 2016) all hold for the Sturm-Liouville operator $-ad^2/dx^2-k$ defined on Ω. Then, Theorem 2.1 in (Karafyllis and Krstic, 2016), entails global existence and uniqueness of solutions of the problem (2)-(3), for any $w[0]\in C^2([0,1];\mathbf{R})$ with $(w_x[0])(0)=q(w[0])(0)$, $(w[0])(1)=0$.

On the other hand, let $p,\gamma>0$ be constants with $p+\gamma<1$. Clearly, the function

$$\eta(z)=\sqrt{2}\sqrt{\frac{q^2+\mu_0}{q^2+q+\mu_0}}\sin\left((1-p-\gamma z)\sqrt{\mu_0}\right) \qquad (A3)$$

is positive on [0,1] and satisfies the differential equation $a\eta''(z)+k\eta(z)=-\sigma\eta(z)$ on [0,1] with $\sigma:=\lambda_0\gamma^2-k(1-\gamma^2)$. It is readily seen that, by selecting $\gamma>0$ to be sufficiently close to 1, one can guarantee that $\sigma>0$. Finally, notice that

$$\eta'(0)-q\eta(0)=$$
$$=-\sqrt{2}\sqrt{\frac{q^2+\mu_0}{q^2+q+\mu_0}}\left(\gamma\sqrt{\mu_0}\cos\left((1-p)\sqrt{\mu_0}\right)+q\sin\left((1-p)\sqrt{\mu_0}\right)\right)$$

Using (A2), it follows that the right side of the above equality equals $-\sqrt{2}\sqrt{\frac{q^2+\mu_0}{q^2+q+\mu_0}}(1-\gamma)\left(\frac{\pi}{2}+\theta_0\right)\sin(\theta_0)<0$ for $p=0$. By continuity, the inequality $\eta'(0)-q\eta(0)<0$ still holds for sufficiently small $p>0$. Therefore, Assumption (H4) in (Karafyllis and Krstic, 2016) holds. As we have already shown that Assumptions (H1), (H2) and (H3), it follows from Theorem 2.2 in (Karafyllis and Krstic, 2016) that the following inequality holds for all $t\geq 0$:

$$\max_{0\leq x\leq 1}\left(\frac{\left|w(x,t)\right|}{\eta(x)}\right)\leq\exp(-\sigma t)\max_{0\leq x\leq 1}\left(\frac{\left|w(x,0)\right|}{\eta(x)}\right)$$

The above inequality, together with definition (A3), implies the existence of a constant $\Theta>0$ such that (6) holds for all $t\geq 0$. This ends the proof of Lemma 1 $\lhd$